

DESINGULARIZATION OF EMBEDDED EXCELLENT SURFACES

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Summary. A gap in Hironaka's proof [5] and [6, Introduction] of desingularization of surfaces based on the study of Newton polyhedra may be filled. Indeed, in [5], only the case of a trivial residual extension is studied.

Introduction. In [5] the point at issue is reduced by Hironaka to the study of a singular point $x \in X$ isolated in its Samuel stratum. Then he makes the blowing-up $X' \rightarrow X$ with center x . The difficult case is when the directrix, i.e., the vector space of translations leaving stable the tangent cone to X at point x (as a scheme) is of dimension 2. Indeed, if the directrix is of dimension 1, the *near* points of the blowing-up with center x are rational, while if it is of dimension 0, there is none. We recall that a near point of X at x of a blowing-up $X' \rightarrow X$ is a point $x' \in X'$ verifying the following equality between the Hilbert-Samuel series of X and X' at x and x' : $H_x(X) = H_{x'}(X')/(1 - T)^d$, where d is the transcendence degree of the residue extension.

In the case of the directrix being of dimension 2, Hironaka proves that an invariant denoted $(\beta, \varepsilon, \alpha)$ strictly decreases for the lexicographical ordering at any *very near* rational point. Here we show that at any very near algebraic non rational point the invariant β strictly decreases. We recall that a near algebraic point is very near if its directrix has the same dimension as that of x .

We will not try to say more precisely here how the theorem below fits in with Hironaka's proof.

HYPOTHESES. Let R be a local regular excellent ring, \mathfrak{M} its maximal ideal, $k = R/\mathfrak{M}$, J an ideal of R . Let us suppose that $X = \text{Spec}(R/J)$ is of dimension 2 and that the closed point of X is isolated in its Samuel stratum. Then we can choose a regular system of parameters $(y_1, \dots, y_r, u_1, u_2)$ and a base $(f) = (f_1, \dots, f_m)$ of J such that:

(a) (f_1, \dots, f_m) is a 0-normalized (u) -standard base of J [6, (2-20) and (3-13)] with reference datum (y_1, \dots, y_r, D) [6, (2-20)], where D is an F -set [6, (2-1) and (2-2)] such that: $D \subset \mathcal{A}(y_1, \dots, y_r, u_1, u_2, f_1, \dots, f_m) =$

$\Delta(y, u, f)$ and $\partial D \cap \partial \Delta(y, u, f) = \emptyset$. We recall that $\Delta(y, u, f)$ is the Newton polyhedron defined by (y, u, f) . See the definition and the explicit construction in [6, (3-1)].

(b) Let $v(y, u, f)$ be the vertex of the smallest abscissa of $\Delta(y, u, f)$ and let $\beta(y, u, f)$ be its ordinate. The $(f) = (f_1, \dots, f_m)$ is not $v(y, u, f)$ -solvable with respect to (u, y) [6, (3-9)].

(c) The base (f) is $v(y, u, f)$ -normalized with respect to (u, y) [6, (3-11)].

Let $\beta(R, J) = \inf \beta(y, u, f)$ where (y, u, f) verifies (a), (b) and (c). We know [6, (3-15)] that we can choose (y, u, f) satisfying (a), (b) and (c) and:

(d) $\beta(y, u, f) = \beta(R, J)$.

(e) The base (f) is not w -solvable with respect to (u, y) , where w is the vertex of the smallest abscissa of the side of slope -1 of $\Delta(y, u, f)$.

(f) The initial form of (f) with respect to the side of slope -1 of $\Delta(y, u, f)$ is normalized [6, end of the proof of (3-10)] with respect to (u, y) .

REMARK. We set that $\beta(R, J) \neq 0$. Indeed there is no permissible smooth curve through x [3, (II-13)]. Therefore the abscissa of $v(y, u, f)$ is smaller than 1, which results in $\beta(R, J) \neq 0$.

THEOREM. *With these hypotheses, if we blow up the ideal \mathfrak{M} , then at any very near algebraic non rational point, the invariant $\beta = \beta(R, J)$ strictly decreases.*

The proof of the theorem consists of five lemmas in which we shall always consider an algebraic non rational very near point $x' \in X'$ of X at x . We will follow the same procedure as in [5].

In Lemma 1 we construct a regular system of parameters $(y'_1, \dots, y'_r, u_1, \Phi')$ at the point x' , with $y'_j = y_j/u_1$, $1 \leq j \leq r$. The original point here is the construction of Φ' .

In Lemma 2 we compute the ordinate $\beta(y', u_1, \Phi', f')$ of $v(y', u_1, \Phi', f')$, the point of the smallest abscissa of the new polyhedron $\Delta(y', u_1, \Phi', f')$ with $f'_i = f_i/u_1^{n(i)}$, $n(i) = v_{\mathfrak{M}}(f_i)$.

Lemma 3 ends the proof of the theorem if (f') is not $v(y', u_1, \Phi', f')$ -solvable with respect to (u_1, Φ', y') , [6, (3-9)].

In Lemma 4 we show that under the hypothesis denoted (**) we can choose the regular system of parameters (y, u_1, u_2) at x such that we get the hypotheses (a) (b) (c) (d) (e) (f) and that of Lemma 3.

The basis (f') may be $v(y', u_1, \Phi', f')$ -solvable with respect to $(u_1,$

Φ', y') and the hypothesis (**) may not be verified. Then in Lemma 5 we dissolve the point $v(y', u_1, \Phi', f')$. Then the coordinates of the vertex with the smallest abscissa of the new polyhedron are not both integer. So this new vertex is not solvable. This brings the proof home in this particular and last case.

Construction of the parameters. The directrix being of dimension 2, we have $\text{gr}_{\mathfrak{M}}(R/J)_{\text{red}} = k[U_1, U_2]$, where $U_i = \text{image of } u_i \text{ in } \mathfrak{M}/\mathfrak{M}^2$. So the reduced exceptional divisor E of $X' \rightarrow X$ is a projective line. A non rational point $x' \in E$ corresponds to an irreducible unitary homogeneous polynomial of $k[U_1, U_2]$:

$$\Phi(U_1, U_2) = U_1^d + \alpha_1 U_1^{d-1} U_2 + \cdots + \alpha_d U_2^d, \quad \alpha_d \neq 0, \quad d > 1.$$

Furthermore $k(X') \neq k(X)$, so x' belongs to the open affine set Z'_2 of the transform Z' of $Z = \text{Spec}(R)$, such that $Z'_2 = \text{Spec}(R[y'_1, \dots, y'_r, u'_2]/(y_j - u_1 y'_j, u_2 - u_1 u'_2))$, $1 \leq j \leq r$. The ideal of $E \cap Z'_2$ in Z'_2 is generated by (u_1, y'_1, \dots, y'_r) and $E \cap Z'_2 = \text{Spec}(k[u'_2])$. The ideal of x' in $E \cap Z'_2$ is generated by $\Phi(1, u'_2)$ and the residue field is $k(x') = k[1, u'_2]/\Phi(1, u'_2)$, which we denote by k' , [3, (II-1-4)].

LEMMA 1. *Let us choose a coefficient set \hat{k} of k in R . If $\alpha \in k$ we denote $\hat{\alpha} \in \hat{k}$ the coefficient whose image in R/\mathfrak{M} is α . If $F(U_1, U_2) = \sum F_{ij} U_1^i U_2^j \in k[U_1, U_2]$ is homogeneous of degree n , we denote $\hat{F} = \sum \hat{F}_{ij} u_1^i u_2^j \in R$. Then $F' = F/u_1^n$ belongs to $R' = O_{Z', x'}$. The image of F' in the function ring $k[u'_2]$ of $E \cap Z'_2$ is $F(1, u'_2)$. In fact $(y'_1, \dots, y'_r, u_1, \Phi')$ is a regular system of parameters of $R' = O_{Z', x'}$ and the image of Φ' in $O_{E, x'}$ is that of $\Phi(1, u'_2)$.*

PROOF. Clear from [3, (II-1-4)] and the construction of the parameters.

Remarks and notations. Let L be the linear form $L(a_1, a_2) = (a_1 + a_2)/\delta$ such that $L = 1$ is an equation of the side of slope -1 of $\Delta(y, u, f)$. The initial form of f_i with respect to L [2, Definition 1] is denoted by $\text{in}_L(f_i) \in k[Y_1, \dots, Y_r, U_1, U_2]$. We recall that $v_L(y^B u^A) = |B| + L(A)$ in which B and A are multi-indices. Then we get with clear notations $\text{in}_L(f_i) = F_i(Y) + \sum Y^B P_{i,B}(U_1, U_2) \in k[Y_1, \dots, Y_r, U_1, U_2]$, $|B| < n(i) = \text{val}(f_i)$, where $P_{i,B}(U_1, U_2) \in k[U_1, U_2]$ is homogeneous and either $P_{i,B} = 0$ or $\deg(P_{i,B}) = (n(i) - |B|)$ with some $P_{i,B} \neq 0$ [6, (3-9-2)]. Let us write $P_{i,B}(U_1, U_2) = \Phi^{a(i,B)}(U_1, U_2) Q_{i,B}(U_1, U_2)$ with $a(i, B)$ maximal. Then we get $f_i = \hat{F}_i(y) + \sum y^B \hat{\Phi}^{a(i,B)}(u_1, u_2) \hat{Q}_{i,B}(u_1, u_2) + g_i$, $v_L(g_i) > n(i)$. We have denoted by $\hat{F}_i(y)$ the polynomial in y whose coefficients are in \hat{k} , their images in $k = R/\mathfrak{M}$ being the coefficients of $F_i(y)$. Let us con-

sider the blowing-up centered at \mathfrak{M} . We call $f'_i = f_i/u_1^{n(i)}$. The last computation implies:

$$f'_i = F_i(y') + \sum y'^B (u_1^{(n(i)-|B|)(\delta-1)} \Phi'^{a(i,B)} Q'_{i,B}) + g'_i, \quad v_{L'}(g'_i) > n(i),$$

with $L'(a, a') = a/(\delta - 1)$ and $v_{L'}(y'^B u_1^a \Phi'^{a'}) = |B| + a/(\delta - 1)$.

LEMMA 2. *With the notations as above, we get: $\beta(y', u_1, \Phi', f') = \inf\{a(i, B)/(n(i) - |B|); 1 \leq i \leq m, n(i) = v_{\mathfrak{M}}(f_i), 0 \leq |B| \leq n(i), Q_{i,B}(1, u'_2) \neq 0 (= Q_{i,B}(U_1, U_2) \neq 0)\}$.*

PROOF. Indeed, by definition, if $Q_{i,B} \neq 0$, then $Q_{i,B}(1, u'_2)$ is a unit in $R'/(y', u'_1) = k[u'_2]_{(\mathfrak{O}(1, u'_2))}$, so $Q'_{i,B}$ is a unit in R' . q.e.d.

LEMMA 3. *The base (f') is v' -normalized, $v' = v(y', u_1, \Phi', f')$ being the vertex of the smallest abscissa of $\Delta(y', u_1, \Phi', f')$. Furthermore if (f') is not v' -solvable (e.g., if $\delta - 1$ or $\beta(y', u_1, \Phi', f')$ is not an integer), we get the inequality:*

$$\beta(R', J') \leq \beta(y', u_1, \Phi', f') \leq \beta(R, J).$$

PROOF. We recall that x' is a very near point of X at x , so $\beta(R', J')$ is well defined. The property (f) says that the base (f') is v' -normalized. One of the two conditions in parenthesis taken for granted, the coordinates of v' are not both integers, so (f') is not v' -solvable [6, (3-9-2)]. We will end the proof after the next remark:

REMARK (3-1). Let L'' be the linear form $L''(a_1, a_2) = (a_1 + a_2)/\delta'$ such that $L''(a_1, a_2) = 1$ is the equation of the side of slope -1 of $\Delta(y', u_1, \Phi', f')$. Then $\delta' \geq 1$. Indeed, if $\delta' < 1$, we should get $v_{\mathfrak{M}}(f'_i) < n(i)$ for some i ($1 \leq i \leq m$) in which \mathfrak{M}' is the maximal ideal of R' . Hence $H_{x'}(X') < H_x(X)$, a contradiction because x' is very near to x .

If $1 < \delta'$, then $\text{in}_{\mathfrak{M}'}(f'_i) = F_i(Y') \in k[Y'] \subset \text{gr}_{\mathfrak{M}'}(R') = k[Y', U_1, \Psi]$, $1 \leq i \leq m$, in which $Y' = \text{in}_{\mathfrak{M}'}(y')$, $U_1 = \text{in}_{\mathfrak{M}'}(u_1)$ and $\Psi = \text{in}_{\mathfrak{M}'}(\Phi')$. Moreover, the property (a) is verified by (f') and the regular system of parameters (y', u_1, Φ') .

If $\delta' = 1$, x' being very near to x , we can dissolve, using [7, (1-10)], the points on the side of slope -1 of $\Delta(y', u_1, \Phi', f')$ [6, (3)]; furthermore if (f') is $v(y', u_1, \Phi', f')$ -normalized and not $v(y', u_1, \Phi', f')$ -solvable, then $v(y', u_1, \Phi', f')$ is not an element of the side of slope -1 of $\Delta(y', u_1, \Phi', f')$. So the property (a) will be true for the new equations and the new system of parameters. Then [6, (3-15)] shows that $v(y', u_1, \Phi', f')$ will be the vertex of the smallest abscissa of the new polyhedron and properties (b) and (c) will be true. Moreover in this

last case we shall get $\beta(R', J') \leq \beta(y', u_1, \Phi', f')$. Let us prove $\beta(y', u_1, \Phi', f') \leq \beta(R, J)$. For all (i, B) such that $Q_{i,B}(U_1, U_2) \neq 0$, we get $\beta(R, J) \geq (a(i, B)\deg \Phi + \deg Q_{i,B}(1, U_2))/(n(i) - |B|)$. Since $\deg \Phi \geq 2$ we get $\beta(R, J) \geq 2a(i, B)/(n(i) - |B|)$. Then (3-1) shows $\beta(R, J) \geq 2\beta(y', u_1, \Phi', f') \geq 2\beta(R', J')$. As $\beta(R, J) \neq 0$, we are done.

NOTATION (3-2). Henceforth we will write as a rule $f, Q_B, a(B), n$ etc. instead of $f_i, Q_{i,B}, a(i, B), n(i)$ etc.

DISSOLUTION OF v' (3-3). If (f') is v' -solvable, we get:

$$\text{in}_{v'}(f') = F(Y') + \sum Y'^B \Phi(1, u_2)^{\beta(n-|B|)} U_1^{n-|B|(\delta-1)} Q_B(1, u_2), \quad |B| < n,$$

where $\beta = \beta(y', u_1, \Phi', f')$, $\text{in}_{v'}(f') = F(Y' + \Psi^\beta U_1^{\beta-1} A_j) \in k'[Y', U_1, \Psi]$, with $k' = R'/\mathfrak{M}'$ and $A_j \in k'$ for $1 \leq j \leq r$, [6, (3-7) and (3-9)]. Let $A_j(U_1, U_2) \in k[U_1, U_2]$ be the homogeneous polynomial such that $\deg(A_j) < \deg(\Phi)$ and the image of $A_j(1, U_2)$ in $k' = k[U_2]/(\Phi(1, U_2))$ is A_j . Then we denote $z'_j = y'_j + \Phi'^\beta u_1^{\beta-1} A'_j \in R'$, $1 \leq j \leq r$, with $A'_j = \hat{A}_j/u_1^{\alpha(j)}$, $\alpha(j) = \deg A_j$, (the notation of Lemma 1).

REMARK (3-4). The following condition (**) implies (*), which in turn shows that $z_j = z'_j u_1$ is an element of R :

(*) For any j , $1 \leq j \leq r$, we have $\beta \deg \Phi + \deg A_j \leq \delta$ with $\beta = \beta(y', u_1, \Phi', f')$.

(**) There exists B and i with $1 \leq i \leq m$ such that $Q_{i,B} \neq 0$ and $a(i, B)\deg \Phi + \deg Q_{i,B}(1, U_2) \geq (\beta + 1)(n(i) - |B|)\deg \Phi$.

Indeed, for all (i, B) such that $Q_{i,B} \neq 0$, we get $a(i, B)\deg \Phi + \deg Q_{i,B}(U_1, U_2) = \delta(n(i) - |B|)$. By (**) we get the inequality $(\beta + 1) \times (n(i) - |B|)\deg \Phi \leq a(i, B)\deg \Phi + \deg Q_{i,B}(1, U_2) \leq \delta(n(i) - |B|)$. Then we use the inequality $(\beta + 1)\deg \Phi > \beta \deg \Phi + \deg A_j$, $1 \leq j \leq r$. q.e.d.

The condition (**) may be more plainly stated: let us denote by γ the ordinate of the vertex of the smallest abscissa of the side of slope -1 of $\Delta(y, u, f)$. Then:

$$\gamma = \sup\{a(i, B)\deg \Phi + \deg Q_{i,B}(1, U_2)/(n(i) - |B|); \\ 1 \leq i \leq m, \quad Q_{i,B} \neq 0\},$$

so (**) if and only if $\gamma \geq (\beta + 1)\deg \Phi$, where $\beta = \beta(y', u_1, \Phi', f') \neq \beta(R, J)$.

LEMMA 4. With the notations in (3-4), if the condition (**) is true, then $z_j = y_j + \hat{\Phi}^\beta u_1^{\beta-1} \hat{A}_j$ (notation of Lemma 1) is an element of R ($\alpha(j) = \deg A_j$). The system $(z_1, \dots, z_r, u_1, u_2)$ is a regular system of parameters of R . The vertex of the smallest abscissa of the side of

slope -1 of $\Delta(z, u, f)$ is the same as that of $\Delta(z, u, f)$. We have $\gamma - \beta(y', u_1, \Phi', f') < \gamma - \beta(z', u_1, \Phi', f')$. Furthermore the conditions (a) (b) (c) (d) (e) (f) are true for (z, u, f) .

PROOF. We have proved in (3-4) that z_j is an element of R . It is clear that (z, u) is a regular system of parameters of R . The condition (**) implies $\gamma > \beta \deg \Phi + \deg A_j(1, U_2)$, $1 \leq j \leq r$, so $\beta(z, u, f) = \beta(R, J)$ and (z, u, f) is $v(z, u, f)$ -normalized and not $v(z, u, f)$ -solvable [6, (3, 15)]. For the same reason the vertex of the smallest abscissa of the side of slope -1 of $\Delta(z, u, f)$ is the same as that of $\Delta(y, u, f)$. We have $\beta(z', u_1, \Phi', f') > \beta(y', u_1, \Phi', f')$, which proves the inequality. Let us denote $\rho = U_1^{\delta - \beta \deg \Phi + \alpha(j)} A_j(U_1, U_2)$. The expansion of $F_i(Y) = F_i(Z - \rho)$ is:

$$F_i(Z - \rho) = \text{in}_L(f_i, u, z) = F_i(Z) + \sum K_{i,C,D} Z^C \rho^D + \sum (Z - \rho)^B P_{i,B}(U_1, U_2), \\ K_{i,C,D} \in k, \quad |C| + |D| = n(i).$$

As (y, u, f) verifies the condition (a), none of the exponents of the monomials of F_i live in $E^r(F_1, \dots, F_{i-1})$ (notation of [6, (3-11)]). It is also the case for the exponents C of the first \sum . The condition (f) being true for (y, u, f) , we have the same for the second \sum .

COROLLARY (4-1). *There exists a regular system of parameters (y, u) such that the conditions (a) (b) (c) (d) (e) (f) are true for (y, u, f) and either the condition (**) is not true for (y, u, f) , or (f') is not $v(y', u_1, \Phi', f')$ -solvable.*

PROOF. Apply Lemma 4 and proceed by the descending induction on $\gamma - \beta(y', u_1, \Phi', f')$.

LEMMA 5. *If the condition (**) is not true for (y, u, f) and if (f') is $v(y', u_1, \Phi', f')$ -solvable, then if we dissolve this vertex, we get a new regular system of parameters (z', u_1, Φ') of R' in which (z') is defined as above. Then (f') is $v(z', u_1, \Phi', f')$ -normalized and not $v(z', u_1, \Phi', f')$ -solvable. Furthermore, we have*

$$\beta(R', J') \leq \beta(z', u_1, \Phi', f') \leq \beta(R, J).$$

PROOF. We use the notations of Lemma 4. Let us denote $\mu_j = u_1^{\delta-1} \Phi'^{\beta} A'_j$, $1 \leq j \leq m$, with $A'_j = \hat{A}_j / u_1^{\alpha(j)}$, $\alpha(j) = \deg A_j$, (the notation of (3-3)). Then we may write

$$(1) \quad F(z') = F(y' + \mu) = F(y') = \sum_{|C| < n, |D|=n} (-1)^{|D|} K_{C,D} y'^C \mu^D + h,$$

where $v_L(h) > n$, L' being the linear form $L'(a_1, a_2) = a_1/(\delta - 1)$. We

notice that L' defines the same valuation with respect to (z', u_1, Φ') or to (y', u_1, Φ') . Then (1) implies

$$f' = F(z') + \sum_{|B| < n} y'^B (\Phi'^{\alpha(B)} Q'_B - \sum_{D, |B+D|=n} (-1)^{|D|} K_{B,D} u_1^D) + h',$$

where $v_{L'}(h') > n$. We can write the content in the parenthesis as

$$\Phi'^{(n-|B|)\beta} u_1^{(n-|B|)(\delta-1)} (\Phi'^{\alpha(B)-(n-|B|)\beta} Q'_B - \sum'' (-1)^{|D|} \hat{K}_{B,D} A'^D).$$

Let us denote: $b(B) = \alpha(B) - (n - |B|)\beta$. Then $\Phi'^{b(B)} Q'_B - \sum'' (-1)^{|D|} \hat{K}_{B,D} A'^D$ is an element of \mathfrak{M}' . So in $R'/(y', u_1) = k[U_2]/\Phi(1, U_2)$ we get $\Phi^{b(B)}(1, U_2) Q_B(1, U_2) - \sum_{D, |B+D|=n} (-1)^{|D|} \hat{K}_{B,D} A^D(1, U_2) = \Phi^{c(B)}(1, U_2) R_B(1, U_2)$. If $R_B(U_1, U_2) \neq 0$, then R_B is a homogeneous polynomial divisible neither by $\Phi(U_1, U_2)$ nor by U_1 .

The condition (**) is not true, so for any i , $1 \leq i \leq m$, and for any B with $|B| < n(i)$ and $Q_{i,B} \neq 0$, we have $b(i, B) \deg \Phi + \deg Q_{i,B}(1, U_2) < (n(i) - |B|) \deg \Phi$. Then (with the abbreviated notations) for any B , $|B| < n$, with $R_B \neq 0$, we have $c(B) < n - |B|$. Let us admit that $R_B \neq 0$ for some B (we will prove it later). Then

$$f' = \hat{F}(z') + \sum_{|B| < n} y'^B \Phi'^{(n-|B|)+c(B)} u_1^{(n-|B|)(\delta-1)} R'_B + h'',$$

with $v_{L'}(h'') > n$. That implies

$$(2) \quad f' = \hat{F}(z') + \sum_{|B| < n} (z' - \Phi'^{\beta} A' u_1^{\delta-1})^B \Phi'^{(n-|B|)+c(B)} u_1^{(n-|B|)(\delta-1)} R'_B + h''.$$

We can write it as

$$(3) \quad f' = \hat{F}(z') + \sum_{|C| < n} z'^C S_C + h''.$$

By the condition (a), the exponents C in (1) do not live in $E^r(F_1, \dots, F_{i-1})$ [6, (3-11)], so by the condition (f), the exponents B in (2) do not live in $E^r(F_1, \dots, F_{i-1})$. It is also the case for exponents C in (3). Let us choose B so that $R_B \neq 0$ and B is maximal under this condition. Then $S_B = u_1^{(n-|B|)(\delta-1)} \Phi'^{c(B)+(n-|B|)\beta} R'_B$. Hence this exponent B verifies $\beta(y', u_1, \Phi', f') < \beta(z', u_1, \Phi', f') \leq (c(B) + (n - |B|)\beta)/(n - |B|) < \beta(y', u_1, \Phi', f') + 1$. As $\beta(y', u_1, \Phi', f')$ is an integer (Lemma 3), these inequalities prove that $\beta(z', u_1, \Phi', f')$ is not an integer. Hence (f') is not $v(z', u_1, \Phi', f')$ -solvable. Furthermore, (f') is $v(z', u_1, \Phi', f')$ -normalized, because the exponents C of (3) do not live in $E^r(F_1, \dots, F_{i-1})$. The point x' is very near at x , so (3-1) implies $\beta(R', J') \leq \beta(z', u_1, \Phi', f') \leq \beta(y', u_1, \Phi', f') + 1$. We know that $\beta(R, J) \geq \beta(y', u_1, \Phi', f') \deg \Phi$ and $\deg \Phi \geq 2$, hence $\beta(R', J') < \beta(y', u_1, \Phi', f') + 1 \leq \beta(R, J)/2 + 1$. If $\beta(R, J) \geq 2$ then $\beta(R', J') < \beta(R, J)$. If $\beta(R, J) < 2$, since $\beta(R, J) \geq \beta(y', u_1, \Phi', f') \times \deg \Phi$ and $\beta(y', u_1, \Phi', f')$ is an integer, then $\beta(y', u_1, \Phi', f') = 0$. Thus

$\beta(R', J') < 1$. Lemma 3 implies that δ is an integer and the condition (a) implies that $\delta > 1$. Since there is no permissible smooth curve through x , we get $\beta(R, J) \geq \delta - 1 \geq 1 > \beta(R', J')$.

Finally, we must prove that some $R_B \neq 0$. Let us suppose that $R_B = 0$ for all B . We get the equality

$$\Phi^{b(B)}(1, U_2)Q_B(1, U_2) = \sum_{D, (|B|+D|=n)} (-1)^{|D|} K_{B,D} A^D(1, U_2), \quad |B| < n.$$

Let us denote $\Gamma_j(U_1, U_2) = U_1^{\delta - \alpha(j) - \beta \deg \Phi} A_j(U_1, U_2)$, where $\alpha(j) = \deg A_j$ and $\beta = \beta(y', u_1, \Phi', f')$. Note that the exponent of U_1 may be negative. If we multiply each term of the last equality by $U_1^{(n-|B|)(\delta - \beta \deg \Phi)}$, we get $\Phi^{b(B)}(U_1, U_2)Q_B(U_1, U_2) = \sum_{D, |B|+D|=n} (-1)^{|D|} K_{B,D} A^D(U_1, U_2)$ for $|B| < n$. That implies $\text{in}_L(f) = F(Y + A(U_1, U_2)) \in k[Y, U_1, U_2]$. We will prove now that the condition (*) is true ($\delta - \deg A_j - \beta \deg \Phi \geq 0$ for all j , $1 \leq j \leq r$). Let us assume that $\delta - \deg A_j - \beta \deg \Phi$ is the smallest and negative for $j = 1$. As $\text{gr}_{\mathfrak{m}}(R/J)_{\text{red}} = k[U_1, U_2]$ we can choose an integer a and polynomials $P_i(Y) \in k[Y]$, where $P_i \neq 0$ for some i and $\deg P_i = a - n(i)$ if $P_i \neq 0$. A computation of the valuations with respect to U_1 leads to

$$\begin{aligned} a(\delta - \deg A_1 - \beta \deg \Phi) &\geq \inf(a - n(i))(\delta - \deg A_j - \beta \deg \Phi) \\ &> a(\delta - \deg A_1 - \beta \deg \Phi). \end{aligned}$$

This is a contradiction. Hence for all j , $1 \leq j \leq r$, we have $\delta - \deg A_j - \beta \deg \Phi \geq 0$ (*). Let us denote $z_j = y_j + \hat{f}_j(u_1, u_2)$ and let $Z_j = Y_j + \hat{f}_j(U_1, U_2)$ be the image of z_j in $\text{gr}_{\mathfrak{m}}(R) = k[Y, U_1, U_2]$. We get $F_i(Z) = \text{in}_L(f_i)$, which contradicts the condition (e).

We thus complete the proof of Lemma 5, hence that of the Theorem.

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