



## Research Article

Abdullah Alzahrani\* and Anwar Zeb

# Detectable sensation of a stochastic smoking model

<https://doi.org/10.1515/math-2020-0068>

received January 22, 2020; accepted July 26, 2020

**Abstract:** This paper is related to the stochastic smoking model for the purpose of creating the effects of smoking that are not observed in deterministic form. First, formulation of the stochastic model is presented. Then the sufficient conditions for extinction and persistence are determined. Furthermore, the threshold of the proposed stochastic model is discussed, when noises are small or large. Finally, the numerical simulations are shown graphically with the software MATLAB.

**Keywords:** stochastic smoking model, Itô's formula, extinction, persistence, numerical analysis

**MSC 2020:** 37H30, 26A18, 28Dxx, 34Cxx

## 1 Introduction

Epidemiology is used for understanding the spreading nature of infectious disease in a society. The epidemic models are mostly based on the similarity of the models but not necessary that all practical models possess all probable properties but somewhat include the mechanisms in the simplest probable mode so as to contain major constituents that influence disease spread. The prediction of real sensations in epidemic models must be carried out very carefully [1]. The first epidemic model was introduced by Kermack and McKendrick [2], in which they divided the whole population into three classes based on different aspects of spreading of disease. The work presented in [1,2] is the groundwork of epidemic models. First, two epidemic models were studied: one is SIS (susceptible; infected; susceptible) model and the other is SIR (susceptible; infected; recovered) model [3,4]. After that, many authors studied different epidemic models based on different types of diseases by including linear, bilinear, saturated incidence rates and similarly using delay differential equations, fractional differential equations and partial differential equations [1–6]. A major health concern is a social habit of smoking, which is common in people. Both preventable and premature type of deaths occur in the US and also worldwide due to smoking. According to a previous report, in a year over 4,40,000 deaths occur due to smoking-related disease in the US annually and 1,05,000 at UK [7]. The risk of heart attack is 70% more for smokers rather than nonsmokers. There is 78% chance of lung cancer and 24% of heart disease for smokers. Similarly, throat, mouth, stomach, cervix, breast and pancreas cancers are somehow related to smoking because one cigarette contains more than 4,000 chemicals. According to the third edition of cancer atlas jointly released by International Agency for Research on Cancer, American Cancer Society and Union for International Cancer Control on October 16, 2019, smoking causes more preventable cancer deaths than any other risk factors and in 2017 alone, 2.3 million people worldwide died of smoking, which accounts for

\* **Corresponding author: Abdullah Alzahrani**, Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia, e-mail: akalzahrani@kau.edu.sa

**Anwar Zeb:** Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Abbottabad, Khyber Pakhtunkhwa, 22060, Pakistan, e-mail: anwar@cuiatd.edu.pk

24% of all cancer deaths. On the other hand, based on the WHO global report on trends in the prevalence of tobacco use 2000–2025 [8,9], every year, more than 8 million people die of tobacco use, accounting for about half of its users. More than 7 million of them died of direct smoking, while about 1.2 million nonsmokers died from exposure to second-hand smoke. For this purpose, mathematicians are trying to present the nature of smoking and classify the populations with the help of modeling. From the first smoking model presented by Castillo-Garsow et al. [7], many researchers tried on different aspects to develop smoking models [7,10–14]. In recent years, the authors formulated different forms of smoking models with linear incidence rate [15,16], saturated incidence rate [17,18], square root-type incidence rate [12,19,20] and harmonic mean-type incidence rate [21]. The environmental white noise has great effect on the epidemic models but due to simplicity in deterministic models the authors ignore these terms. For more realistic models, several authors studied the stochastic models by introducing white noise [22–26]. The effects of environment in the AIDS model were studied by Dalal et al. [22] using the method of parameter perturbation. Tornatore et al. [23–25] studied the stochastic epidemic models with vaccination. In these studies, they proved the existence, uniqueness and positivity of the solution. A stochastic SIS epidemic model with vaccination is discussed by Zhu and Hu [26]. They obtained the condition of the disease extinction and persistence according to noise and threshold of the deterministic system. Similarly, several authors discussed the same conditions for stochastic models [27–35]. In this research, first formulation of a stochastic mathematical smoking model is presented. Then the sufficient conditions for extinction and persistence are determined. Furthermore, the threshold of the proposed stochastic model is discussed, when noises are small or large. Finally, numerical simulations are presented graphically using the software MATLAB.

The rest of the paper is organized as follows: Section 2 is related to the smoking model with random perturbation formulation. Section 3 is related to the unique positive solution of the proposed model. Furthermore, the exponential stability of the proposed model is investigated in Section 4. The persistent conditions are shown in Section 5. Finally, in Section 6 conclusion is given.

## 2 Model formulation

In this section, a smoking model with random perturbation is formulated as follows:

$$\begin{aligned}\frac{dP(t)}{dt} &= \Lambda - \beta P(t)S(t) - \mu P(t) + \delta Q(t) - \rho P(t)S(t)dB(t), \\ \frac{dS(t)}{dt} &= \beta P(t)S(t) - (\gamma + \mu)S + \rho P(t)S(t)dB(t), \\ \frac{dQ(t)}{dt} &= \gamma S(t) - \mu Q(t) - \delta Q(t),\end{aligned}\tag{1}$$

and the description of parameters and variables is given in Table 1.

**Table 1:** Parameters and description

Notations	Description
$P(t)$	Susceptible (potential) smokers
$S(t)$	Series (chain) smoker class
$Q(t)$	Smokers who quit smoking
$\Lambda$	The membership rate to susceptible class by birth or migration
$\beta$	Rate at which the potential smokers move to chain smoker class
$\mu$	Natural death
$\gamma$	The quitting rate
$B(t)$	The standard Brownian motion, with $\rho^2 > 0$ and with intensity of white noise
$\delta$	The relapse rate

In deterministic form, model (1) is as follows:

$$\begin{aligned} \frac{dP(t)}{dt} &= \Lambda - \beta P(t)S(t) - \mu P(t) + \delta Q(t), \\ \frac{dS(t)}{dt} &= \beta P(t)S(t) - (\gamma + \mu)S, \\ \frac{dQ(t)}{dt} &= \gamma S(t) - \mu Q(t) - \delta Q(t), \end{aligned} \tag{2}$$

and

$$\frac{dN}{dt} = \Lambda - \mu N, \tag{3}$$

where  $N(t) = P(t) + S(t) + Q(t)$  represents the total population and  $N(0) = P(0) + S(0) + Q(0)$ . Eq. (3) has the exact solution:

$$N(t) = e^{-\mu t} \left[ N(0) + \frac{\Lambda}{\mu} e^{\mu t} \right]. \tag{4}$$

Also, we have

$$P(0) \geq 0, \quad S(0) \geq 0, \quad Q(0) \geq 0 \Rightarrow P(t) \geq 0, \quad S(t) \geq 0, \quad Q(t) \geq 0.$$

So the solution has the positivity property. For stability analysis of model (2), the reproductive number is

$$R_0 = \frac{\beta}{\gamma + \mu} N. \tag{5}$$

If  $R_0 < 1$ , then system (2) will be locally stable and will be unstable if  $R_0 \geq 1$ . Similarly, for  $\Lambda = 0$ , system (2) will be globally asymptotically stable (see [12]).

### 3 Existence and uniqueness of the positive solution

Here, first we make the following assumptions:

- Set  $R_+^d = \{\chi_i \in R^d, \chi_i > 0, 1 \leq d\}$ .
- Suppose a complete probability space  $(\Omega, \mathfrak{F}, \{\mathfrak{F}\}_{t \geq 0}, P)$  with filtration  $\{\mathfrak{F}\}_{t \geq 0}$  which satisfies the usual conditions.

Generally, consider a stochastic differential equation of  $n$ -dimensions as

$$dx(t) = F(y(t), t)dt + G(y(t), t)dB(t), \quad \text{for } t \geq t_0, \tag{6}$$

with initial value  $y(t_0) = y_0 \in R^d$ . By defining the differential operator  $L$  with Eq. (6)

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^d F_i(y, t) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^d [G^T(y, t)G(y, t)]_{ij} \frac{\partial^2}{\partial y_i \partial y_j}. \tag{7}$$

If operator  $L$  acts on a function  $V = (\mathbb{R}^d \times \tilde{\mathbb{R}}_+; \tilde{\mathbb{R}}_+)$ , then

$$LV(y, t) = V_t(y, t) + V_y(y, t)F(y, t) + \frac{1}{2} \text{trace}[G^T(y, t)V_{yy}(y, t)G(y, t)]. \tag{8}$$

**Theorem 3.1.** *There is a unique positive solution  $(P(t), S(t), Q(t))$  of system (1) for  $t \geq 0$  with  $(P(0), S(0), Q(0)) \in R_+^3$ , and solution will be left in  $R_+^3$ , with probability 1.*

**Proof.** Since the coefficient of differential equations of system (1) is locally Lipschitz continuous for  $(P(0), S(0), Q(0)) \in \mathbb{R}_+^3$ , there is a unique local solution  $(P(t), S(t), Q(t))$  on  $t \in [0, \tau_e)$ , where  $\tau_e$  is time for noise caused by an explosion (see [6]). For demonstration, the solution is global, it is sufficient that  $\tau_e = \infty$  a.s. Suppose that  $k_0 \geq 0$  be sufficiently large that  $(P(0), S(0), Q(0)) \in \left[\frac{1}{k_0}, k_0\right]$ . For each integer  $k \geq k_0$ , define the stopping time

$$\tau_e = \inf \left\{ t \in [0, \tau_e) : \min(P(t), S(t), Q(t)) \leq \frac{1}{k_0} \text{ or } \max(P(t), S(t), Q(t)) \geq k \right\},$$

where we set  $\inf \phi(\text{empty set}) = \infty$  throughout the paper. For  $k \rightarrow \infty$ ,  $\tau_k$  is clearly increasing. Set  $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$  where  $\tau_\infty \leq \tau_e$ . If we can show that  $\tau_\infty = \infty$  a.s., then  $\tau_e = \infty$ . If false, then there is a pair of constants  $T > 0$  and  $\varepsilon \in (0, 1)$  such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

So there is an integer  $k_1 \geq k_0$ , which satisfies

$$P\{\tau_k \leq T\} \geq \varepsilon \text{ for all } k \geq k_1.$$

Define a  $C^2$ -function  $V : \mathbb{R}_+^3 \rightarrow \tilde{\mathbb{R}}_+$  by

$$V(P, S, Q) = \left( P - c - c \ln \frac{P}{c} \right) + (S - 1 - \ln S) + (Q - 1 - \ln Q). \tag{9}$$

By applying Itô's formula, we obtain that

$$\begin{aligned} dV(P, S, Q) &= \left( 1 - \frac{c}{P} \right) dP + \frac{1}{2P^2} (dP)^2 + \left( 1 - \frac{1}{S} \right) dS + \frac{1}{2S^2} (dS)^2 + \left( 1 - \frac{1}{Q} \right) dQ \\ &= LV dt + \rho(S - P) dB(t), \end{aligned} \tag{10}$$

where  $LV : \mathbb{R}_+^3 \rightarrow \tilde{\mathbb{R}}_+$  is defined by

$$\begin{aligned} LV &= \left( 1 - \frac{c}{P(t)} \right) (\Lambda - \beta P(t)S(t) - \mu P(t) + \delta Q(t)) + \frac{1}{2} \rho^2 S^2 \\ &\quad + \left( 1 - \frac{1}{S} \right) (\beta P(t)S(t) - (\gamma + \mu)S) + \frac{1}{2} \rho^2 P^2 + \left( 1 - \frac{1}{Q} \right) (\gamma S(t) - \mu Q(t) - \delta Q(t)) \\ &= \Lambda - \beta P(t)S(t) - \mu P(t) + \delta Q(t) - \frac{c\Lambda}{P(t)} + c\beta S(t) + c\mu - c\delta \frac{Q(t)}{P(t)} + \frac{1}{2} \rho^2 S^2 \\ &\quad + \beta P(t)S(t) - (\gamma + \mu)S(t) - \beta P(t) + (\gamma + \mu) + \frac{1}{2} \rho^2 P^2 + \gamma S(t) - \mu Q(t) - \delta Q(t) \\ &\quad - \gamma \frac{S(t)}{Q(t)} + \mu + \delta \leq \Lambda - (\gamma + \mu)S + c\beta S(t) + c\mu + \gamma + \mu + \mu + \delta + \frac{1}{2} \rho^2 S^2 + \frac{1}{2} \rho^2 P^2. \end{aligned}$$

By choosing  $c = \frac{\gamma + \mu}{\beta}$ , it follows

$$LV \leq \Lambda + c\mu + \gamma + \mu + \mu + \delta + \frac{1}{2} \rho^2 S^2 + \frac{1}{2} \rho^2 P^2 \triangleq B. \tag{12}$$

Further proof follows from Ji et al. [28]. □

### 4 Extinction

In this section, we investigate the condition for extinction of the spread of smoking. Here, we define

$$\langle y(t) \rangle = \frac{1}{t} \int_0^t y(s) ds \tag{13}$$

and

$$\tilde{\alpha} = \beta \left( \frac{\Lambda}{\mu} \right) \frac{1}{(y + \mu) + \frac{1}{2}\rho^2 \left( \frac{\Lambda}{\mu} \right)^2}. \tag{14}$$

A useful lemma concerned to this work is follows.

**Lemma 4.1.** [26] *Let  $M = \{M_t\}_{t \geq 0}$  be a real value, continuous, local martingale and vanishing at  $t = 0$ . Then*

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty$$

a.s. implies that

$$\lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0$$

and also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0.$$

**Theorem 4.1.** *Let  $(P(t), S(t), Q(t))$  be the solution of system (1) with initial value  $(P(0), S(0), Q(0)) \in \mathbb{R}_+^3$ . If*

1.  $\rho^2 > \max \left( \frac{\beta^2}{2(y + \delta + \mu + \alpha)}, \frac{\beta\mu}{\Lambda} \right)$  or

2.  $\tilde{R} < 1$  and  $\rho^2 \leq \frac{\beta\mu}{\Lambda}$ .

Then

$$\limsup_{t \rightarrow \infty} \frac{\log S(t)}{t} \leq -(y + \mu) + \frac{\beta}{2\rho^2} < 0 \quad \text{a. s. if (1) holds;} \tag{15}$$

$$\limsup_{t \rightarrow \infty} \frac{\log S(t)}{t} \leq \beta \frac{\Lambda}{\mu} \left( 1 - \frac{1}{\tilde{\alpha}} \right) < 0 \quad \text{a. s. if (2) holds.} \tag{16}$$

In addition,

$$\lim_{t \rightarrow \infty} P(t) = \frac{\Lambda}{\mu} = P_0, \quad \lim_{t \rightarrow \infty} S(t) = 0, \quad \lim_{t \rightarrow \infty} Q(t) = 0, \quad \text{a. s.}$$

**Proof.** Taking integration of system (1) □

$$\frac{P(t) - P(0)}{t} = \Lambda - \beta \langle P(t)S(t) \rangle - \mu \langle P(t) \rangle + \delta \langle Q(t) \rangle - \rho P(t)S(t)dB(t),$$

$$\frac{S(t) - S(0)}{t} = \beta \langle P(t)S(t) \rangle - (y + \mu) \langle S(t) \rangle + \rho P(t)S(t)dB(t),$$

$$\frac{Q(t) - Q(0)}{t} = \gamma \langle S(t) \rangle - (\mu + \delta) \langle Q(t) \rangle,$$

$$\begin{aligned} \frac{P(t) - P(0)}{t} + \frac{S(t) - S(0)}{t} + \frac{\delta}{\mu + \delta} \frac{Q(t) - Q(0)}{t} &= \Lambda - \beta \langle P(t)S(t) \rangle - \mu \langle P(t) \rangle + \delta \langle Q(t) \rangle - \rho P(t)S(t)dB(t) \\ &\quad + \beta \langle P(t)S(t) \rangle - (y + \mu) \langle S(t) \rangle + \rho P(t)S(t)dB(t) + \gamma \langle S(t) \rangle - (\mu + \delta) \langle Q(t) \rangle \\ &= \Lambda - \mu \langle P(t) \rangle - \left( (y + \mu) - \frac{\delta\gamma}{\mu + \delta} \right) \langle S(t) \rangle \\ &= \Lambda - \mu \langle P(t) \rangle - \left( \frac{(y + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle, \end{aligned}$$

$$\langle P(t) \rangle = -\frac{1}{\mu} \left[ \frac{P(t) - P(0)}{t} + \frac{S(t) - S(0)}{t} + \frac{\delta}{\mu + \delta} \frac{Q(t) - Q(0)}{t} \right] + \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle.$$

By applying  $\lim_{t \rightarrow 0}$

$$\langle P(t) \rangle = \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle, \tag{17}$$

$$d \log S(t) = \left( \beta P - (\gamma + \mu) - \frac{1}{2} \rho^2 P^2 \right) dt + \rho P dB(t), \tag{18}$$

$$\frac{\log S(t) - \log S(0)}{t} = \beta \langle P(t) \rangle - (\gamma + \mu) - \frac{1}{2} \rho^2 \langle P(t)^2 \rangle + \frac{\rho}{t} \int_0^t P(r) dB(r) \tag{19}$$

$$\leq \beta \langle P(t) \rangle - (\gamma + \mu) - \frac{1}{2} \rho^2 \langle P(t) \rangle^2 + \frac{\rho}{t} \int_0^t P(r) dB(r). \tag{20}$$

By putting the value of  $\langle P(t) \rangle$  from Eq. (17)

$$\begin{aligned} \frac{\log S(t) - \log S(0)}{t} &\leq \beta \left( \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle \right) - (\gamma + \mu) \\ &\quad - \frac{1}{2} \rho^2 \left( \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle \right)^2 + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= \beta \left( \frac{\Lambda}{\mu} - \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle \right) - (\gamma + \mu) \\ &\quad - \frac{1}{2} \rho^2 \left( \left( \frac{\Lambda}{\mu} \right)^2 - \left( \frac{\gamma + \mu}{\mu + \delta} \right)^2 \langle S(t) \rangle^2 \right) + 2 \frac{\Lambda}{\mu} \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= \frac{\beta \Lambda}{\mu} - (\gamma + \mu) - \frac{1}{2} \rho^2 \left( \frac{\Lambda}{\mu} \right)^2 - \left( \frac{\beta(\gamma + \mu)}{\mu + \delta} \right) \langle S(t) \rangle \\ &\quad + 2 \frac{\Lambda}{\mu} \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle - \frac{1}{2} \rho^2 \left( - \left( \frac{\gamma + \mu}{\mu + \delta} \right)^2 \langle S(t) \rangle^2 \right) + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= \frac{\beta \Lambda}{\mu} - \left\{ (\gamma + \mu) + \frac{1}{2} \rho^2 \left( \frac{\Lambda}{\mu} \right)^2 \right\} - \left( \frac{\beta(\gamma + \mu)}{\mu + \delta} \right) \langle S(t) \rangle \\ &\quad + 2 \frac{\Lambda}{\mu} \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle - \frac{1}{2} \rho^2 \left( - \left( \frac{\gamma + \mu}{\mu + \delta} \right)^2 \langle S(t) \rangle^2 \right) + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= \frac{\beta \Lambda}{\mu} \left[ 1 - \frac{\mu \left\{ (\gamma + \mu) + \frac{1}{2} \rho^2 \left( \frac{\Lambda}{\mu} \right)^2 \right\}}{\beta \Lambda} \right] - \left( \frac{\beta(\gamma + \mu)}{\mu + \delta} \right) \langle S(t) \rangle \\ &\quad + 2 \frac{\Lambda}{\mu} \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle - \frac{1}{2} \rho^2 \left( - \left( \frac{\gamma + \mu}{\mu + \delta} \right)^2 \langle S(t) \rangle^2 \right) + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= \frac{\beta \Lambda}{\mu} \left[ 1 - \frac{1}{\bar{R}} \right] - \left( \frac{\beta(\gamma + \mu)}{\mu + \delta} \right) \langle S(t) \rangle \\ &\quad + 2 \frac{\Lambda}{\mu} \left( \frac{\gamma + \mu}{\mu + \delta} \right) \langle S(t) \rangle - \frac{1}{2} \rho^2 \left( - \left( \frac{\gamma + \mu}{\mu + \delta} \right)^2 \langle S(t) \rangle^2 \right) + \frac{\rho}{t} \int_0^t P(r) dB(r). \end{aligned}$$

If condition (2) is satisfied, then

$$\limsup_{t \rightarrow \infty} \frac{\log S(t)}{t} \leq \beta \frac{\Lambda}{\mu} \left(1 - \frac{1}{\mathfrak{R}}\right) < 0, \tag{21}$$

conclusion (16) is proved. Next according to Eq. (20)

$$\begin{aligned} \frac{\log S(t) - \log S(0)}{t} &\leq \beta \langle P(t) \rangle - (\gamma + \mu) - \frac{1}{2} \rho^2 \langle P(t) \rangle^2 + \frac{\rho}{t} \int_0^t P(r) dB(r) \\ &= -\frac{1}{2} \rho^2 \left( \langle P(t) \rangle - \frac{\beta}{\rho^2} \right) + \frac{\beta}{2\rho^2} - (\gamma + \mu) + \frac{\rho}{t} \int_0^t P(r) dB(r). \end{aligned}$$

If condition (1) is satisfied, then

$$\frac{\log S(t)}{t} \leq \frac{\beta}{2\rho^2} - (\gamma + \mu) + \frac{\rho}{t} \int_0^t P(r) dB(r) + \frac{\log S(0)}{t}, \tag{22}$$

conclusion (15) is proved

$$\lim_{t \rightarrow \infty} \frac{\log S(t)}{t} \leq -(\gamma + \mu) + \frac{\beta}{2\rho^2} < 0 \quad \text{a.s.}$$

According to (15) and (16)

$$\lim_{t \rightarrow \infty} S(t) = 0. \tag{23}$$

Now from the third equation of system (1), it follows that

$$Q(t) = e^{-(\mu+\delta)t} \left[ Q(0) + \int_0^t \delta S(r) e^{(\mu+\delta)r} dr \right]. \tag{24}$$

By applying L'Hospital's rule with previous result, we have

$$\lim_{t \rightarrow \infty} Q(t) = 0. \tag{25}$$

As from Eq. (4), it follows that

$$\begin{aligned} N(t) &= e^{-\mu t} \left[ N(0) + \frac{\Lambda}{\mu} e^{\mu t} \right], \\ P(t) + S(t) + Q(t) &= \frac{\left[ P(0) + S(0) + Q(0) + \frac{\Lambda}{\mu} e^{\mu t} \right]}{e^{\mu t}}, \\ \lim_{t \rightarrow \infty} P(t) &= \lim_{t \rightarrow \infty} \left[ \frac{\left\{ P(0) + S(0) + Q(0) + \frac{\Lambda}{\mu} e^{\mu t} \right\}}{e^{\mu t}} - S(t) - Q(t) \right], \\ \lim_{t \rightarrow \infty} P(t) &= \frac{\Lambda}{\mu}. \end{aligned}$$

Hence, the proof is complete. □

## 5 Persistence

This section concern the persistence of system (1).

**Theorem 5.1.** *Suppose that  $\mu > \frac{\rho^2}{2}$ . Let  $(P(t), S(t), Q(t))$  be any solution of model (1) with initial conditions  $(P(0), S(0), Q(0)) \in R_+^3$ . If  $\hat{R}_0 > 1$ , then*

$$\lim_{t \rightarrow \infty} \langle P(t) \rangle = \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \langle S(t) \rangle = \frac{\Lambda}{\mu} - \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \beta - \frac{2\Lambda}{\mu} \right)}, \tag{26}$$

$$\lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \frac{\gamma + \mu}{\mu + \delta} \right) \left( \beta - \frac{2\Lambda}{\mu} \right)}, \tag{28}$$

$$\lim_{t \rightarrow \infty} \langle Q(t) \rangle = \frac{\gamma}{\gamma + \mu} \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \beta - \frac{2\Lambda}{\mu} \right)}. \tag{29}$$

**Proof.** As we know that

$$\frac{\log S(t)}{t} \leq \frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right] - \left( \frac{\gamma + \mu}{\mu + \delta} \right) \left( \beta - \frac{2\Lambda}{\mu} \right) \langle S(t) \rangle + \frac{\rho}{t} \int_0^t P(r) dB(r) + \frac{\log S(0)}{t}.$$

By applying the limit

$$\lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \frac{\gamma + \mu}{\mu + \delta} \right) \left( \beta - \frac{2\Lambda}{\mu} \right)}.$$

Using Eq. (17) we have

$$\lim_{t \rightarrow \infty} \langle P(t) \rangle = \frac{\Lambda}{\mu} - \frac{1}{\mu} \left( \frac{(\gamma + \mu)(\mu + \delta) - \gamma\delta}{\mu + \delta} \right) \lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{\Lambda}{\mu} - \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \beta - \frac{2\Lambda}{\mu} \right)}.$$

Further

$$\frac{Q(t) - Q(0)}{t} = \gamma \langle S(t) \rangle - (\mu + \delta) \langle Q(t) \rangle.$$

By applying the limit  $t \rightarrow \infty$ , we have

$$\lim_{t \rightarrow \infty} \langle Q(t) \rangle = \frac{\gamma}{\mu + \delta} \lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{\gamma}{\mu + \delta} \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \frac{\gamma + \mu}{\mu + \delta} \right) \left( \beta - \frac{2\Lambda}{\mu} \right)} = \frac{\gamma}{\gamma + \mu} \frac{\frac{\beta\Lambda}{\mu} \left[ 1 - \frac{1}{\hat{R}} \right]}{\left( \beta - \frac{2\Lambda}{\mu} \right)}.$$

Hence, the proof is complete. □

## 6 Numerical Simulation

For the illustration of our obtained results, we use the values of parameters and variables given in Table 2.

Now for the numerical simulation, we use Milstein’s higher order method [34]. The results obtained through this method are shown graphically in Figure 1 for both deterministic and stochastic forms.



Table 2: Values of variables and parameters for numerical solution

Variables and parameters	Values of variables and parameters
$P(t)$	59
$S(t)$	40
$Q(t)$	30
$\Lambda$	0.08
$\beta$	0.003
$\mu$	0.001
$\gamma$	0.02011
$\rho$	0.0075
$\delta$	0.004

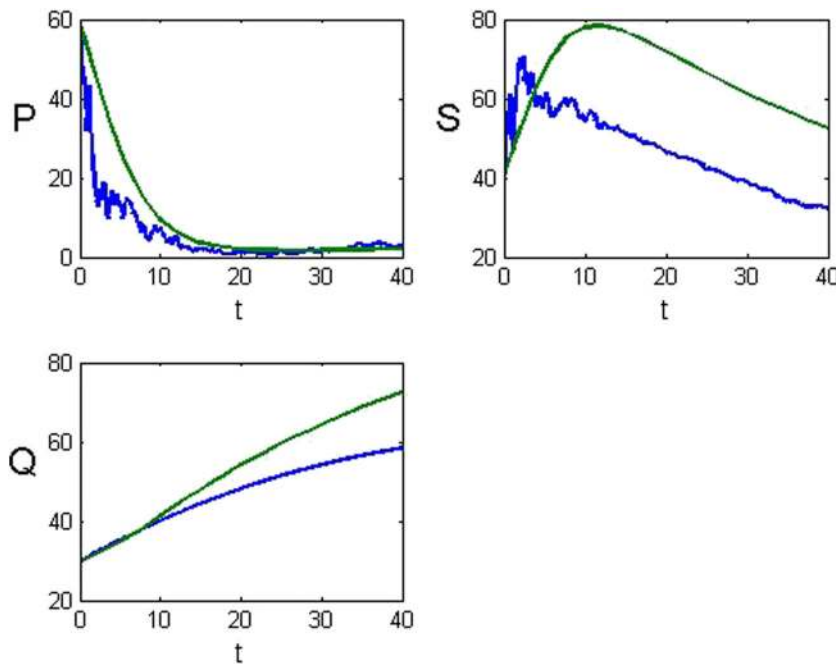


Figure 1: Graphs of (P) potential smokers using a deterministic method (green line) and from a stochastic solution (blue lines), (S) chain smokers using a deterministic method (green line) and from a stochastic solution (blue lines) and (Q) quit smokers using a deterministic method (green line) and from a stochastic solution (blue lines). The stability of stochastic graphs shows best expression than deterministic graphs.

## 7 Conclusion

In this work, a formulation of a stochastic mathematical smoking model is presented. The sufficient conditions are determined for extinction and persistence. Furthermore, the threshold of the proposed stochastic model is discussed, when noises are small or large. Finally, numerical simulations are shown graphically with the software MATLAB. The conclusions obtained are as follows: (a) if  $R_0 < 1$ , then system (2) will be locally stable and will be unstable if  $R_0 \geq 1$ ; (b) if  $R_0 < 1$ , then system (2) will be locally stable and will be unstable if  $R_0 \geq 1$ ; (c) for  $\Lambda = 0$ , system (2) will be globally asymptotically stable; (d) smoking will be in control if  $\tilde{R} < 1$  and  $\rho^2 \leq \frac{\beta\mu}{\Lambda}$  means that white noise is not large; and (e) the value of  $\tilde{R} > 1$  will lead to the prevalence of smoking.

**Acknowledgments:** This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. D1439-159-130. The authors, therefore, gratefully acknowledge DSR for technical and financial support.

**Conflict of interest:** The authors declare that there is no conflict of interest regarding the publication of this paper.

**Authors contribution:** The authors equally contributed to the preparation of this manuscript.

**Funding:** This article was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. D1439-159-130.

**Availability of data and materials:** The authors confirm that the data supporting the findings of this study are available within the article cited therein.

## References

- [1] E. A. Smith and R. E. Malone, “*Everywhere the soldier will be*”: wartime tobacco promotion in the US military, *Am. J. Public Health* **99** (2009), no. 9, 1595–1602, DOI: 10.2105/AJPH.2008.152983.
- [2] W. O. Kermack and A. G. McKendrick, *Contributions to the mathematical theory of epidemics, part 1*, *Proc. Roy. Soc. Edin. Sec. A. Math.* **115** (1927), 700–721.
- [3] B. Shulgin, L. Stone, and Z. Agur, *Pulse vaccination strategy in the SIR epidemic model*, *Bull. Math. Biol.* **60** (1998), 1123–1148.
- [4] G. Zaman, Y. H. Kang, and I. H. Jung, *Stability analysis and optimal vaccination of an SIR epidemic model*, *BioSyst.* **93** (2008), 240–249.
- [5] A. A. Lashari and G. Zaman, *Global dynamics of vector-borne diseases with horizontal transmission in host population*, *Comp. Math. Appl.* **61c** (2011), 745–754.
- [6] H. W. Hethcote, *Qualitative analyses of communicable disease models*, *Math. Biosci.* **7** (1976), 335–356.
- [7] C. Castillo-Garsow, G. Jordan-Salivia, and A. Rodriguez Herrera, *Mathematical models for dynamics of tobacco use, recovery and relapse, Technical Report Series BU-1505-M*, Cornell University, 2000.
- [8] X. K. Zhang, Z. Z. Zhang, J. Y. Tong, and M. Dong, *Ergodicity of stochastic smoking model and parameter estimation*, *Adv. Difference Equ.* **2016** (2016), 274, DOI: 10.1186/s13662-016-0997-x.
- [9] WHO global report on trends in prevalence of tobacco use 2000-2025, third edition, Geneva: World Health Organization, 2019, <https://tech.sina.com.cn/roll/2019-12-20/doc-iihnzhfz7080832.shtml> [Accessed on February 21, 2020]
- [10] O. Sharomi and A. B. Gumel, *Curtailing smoking dynamics: A mathematical modeling approach*, *Appl. Math. Comput.* **195** (2008), 475–499.
- [11] G. Zaman, *Qualitative behavior of giving up smoking models*, *Bull. Malays. Sci. Soc.* **34** (2011), 403–415.
- [12] A. Zeb, G. Zaman, and S. Momani, *Square root dynamic of a giving up smoking model*, *Appl. Math. Model.* **37** (2013), no. 7, 5326–5334.
- [13] O. K. Ham, *Stages and processes of smoking cessation among adolescents*, *West J. Nurs. Res.* **29** (2007), 301–315.
- [14] Z. Alkudhari, S. Al-Sheikh, and S. Al-Tuwairqi, *Global dynamics of a mathematical model on smoking*, *ISRN Applied Mathematics* **2014** (2014), 847075, DOI: 10.1155/2014/847075.
- [15] G. A. K. van Voorn and B. W. Kooi, *Smoking epidemic eradication in a eco-epidemiological dynamical model*, *Ecol. Complex.* **14** (2013), 180–189.
- [16] H. F. Huo and C. C. Zhu, *Influence of relapse in a giving up smoking model*, *Abstr. Appl. Anal.* **2013** (2013), 525461, DOI: 10.1155/2013/525461.
- [17] A. Zeb, A. Bano, E. Alzahrani, and G. Zaman, *Dynamical analysis of cigarette smoking model with a saturated incidence rate*, *AIP Advances* **8** (2018), no. 4, 045317, DOI: 10.1063/1.5028157.
- [18] A. Labzai, O. Balatif, and M. Rachik, *Optimal control strategy for a discrete time smoking model with specific saturated incidence rate*, *Discrete Dyn. Nat. Soc.* **2018** (2018), 5949303, DOI: 10.1155/2018/5949303.
- [19] Q. Din, M. Ozair, T. Hussain, and U. Saeed, *Qualitative behavior of a smoking model*, *Adv. Difference Equ.* **2016** (2016), 96, DOI: 10.1186/s13662-016-0830-6.
- [20] Z. Z. Zhang, R. B. Wei, and W. J. Xia, *Dynamical analysis of a giving up smoking model with time delay*, *Adv. Difference Equ.* **2019** (2019), 505, DOI: 10.1186/s13662-019-2450-4.

- [21] G. Rahman, R. P. Agarwal, and Q. Din, *Mathematical analysis of giving up smoking model via harmonic mean type incidence rate*, Appl. Math. Comput. **354** (2019), 128–148, DOI: 10.1016/j.amc.2019.01.053.
- [22] N. Dalal, D. Greenhalgh, and X. Mao, *A stochastic model for internal HIV dynamics*, J. Math. Anal. Appl. **341** (2008), no. 2, 1084–1101, DOI: 10.1016/j.jmaa.2007.11.005.
- [23] E. Tornatore, S. M. Buccellato, and P. Vetro, *On a stochastic disease model with vaccination*, Rend. Circ. Mat. Palermo **55** (2006), 223–240, DOI: 10.1007/BF02874704.
- [24] E. Tornatore, P. Vetro, and S. M. Buccellato, *SIVR epidemic model with stochastic perturbation*, Neural. Comput. Appl. **24** (2014), 309–315, DOI: 10.1007/s00521-012-1225-6.
- [25] E. Tornatore, S. M. Buccellato, and P. Vetro, *Stability of a stochastic SIR system*, Phys. A **354** (2005), 111–126, DOI: 10.1016/j.physa.2005.02.057.
- [26] L. Zhu and H. X. Hu, *A stochastic SIR epidemic model with density dependent birth rate*, Adv. Difference Equ. **2015** (2015), 330, DOI: 10.1186/s13662-015-0669-2.
- [27] Y. Zhao, D. Jiang, and D. O'Regan, *The extinction and persistence of the stochastic SIS epidemic model with vaccination*, Phys. A **392** (2013), no. 20, 4916–4927, DOI: 10.1016/j.physa.2013.06.009.
- [28] C. Ji, D. Jiang, and N. Shi, *The behavior of an SIR epidemic model with stochastic perturbation*, Stoch. Anal. Appl. **30** (2012), no. 5, 755–773, DOI: 10.1080/07362994.2012.684319.
- [29] A. Gray, D. Greenhalgh, L. Hu, X. Mao, and J. Pan, *A stochastic differential equation SIS epidemic model*, SIAM J. Appl. Math. **71** (2011), no. 3, 876–902, DOI: 10.1137/10081856X.
- [30] Y. Zhao and D. Jiang, *The threshold of a stochastic SIS epidemic model with vaccination*, Appl. Math. Comput. **243** (2014), 718–727, DOI: 10.1016/j.amc.2014.05.124.
- [31] Y. Zhao and D. Jiang, *The threshold of a stochastic SIRS epidemic model with saturated incidence*, Appl. Math. Lett. **34** (2014), 90–93, DOI: 10.1016/j.aml.2013.11.002.
- [32] C. Ji, D. Jiang, and N. Shi, *Multigroup SIR epidemic model with stochastic perturbation*, Phys. A **390** (2011), no. 10, 1747–1762, DOI: 10.1016/j.physa.2010.12.042.
- [33] X. Mao, *Stochastic Differential Equations and Applications*, Ellis Horwood, Chichester, 1997.
- [34] D. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Rev. **43** (2001), no. 3, 525–546, DOI: 10.1137/S0036144500378302.
- [35] A. Lahrouz and L. Omari, *Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence*, Stat. Probab. Lett. **83** (2013), no. 4, 960–968, DOI: 10.1016/j.spl.2012.12.021.