Detecting a Tail Effect in Gravitational-Wave Experiments

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Future gravitational-wave experiments looking at inspiralling compact binaries could achieve the detection of a very small effect of phase modulation induced by the *tails* of gravitational waves. Once a binary signal has been identified, further analysis of data will provide a measure of the total mass-energy M of the binary, which enters as a factor in this tail effect, by means of optimal signal processing. The detection of the effect will then consist in showing the compatibility of the measured values of M and of the other parameters depending on the two masses of the binary. This illustrates the high potentiality of gravitational-wave experiments for testing general relativity.

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The first *direct* detection of gravitational radiation will probably take place in future gravitational-wave experiments such as LIGO and VIRGO. For the moment, the detection of gravitational radiation has only been indirect, thanks to the very precise timing observations of the binary pulsar 1913 + 16 [1]. Among the best candidate sources for a direct detection of gravitational radiation are binary systems of compact objects (neutron stars or black holes) in their late inspiralling stages of evolution [2]. The number of neutron-star coalescences is expected to be a few per year out to a distance of 100 Mpc [3] (with maybe a comparable number of black-hole coalescences), at which distance LIGO and VIRGO might observe the waves with a signal-to-noise ratio (SNR) ~ 10. Such a *première* will open a totally new field in astronomy, and will permit verification of some fundamental predictions of general relativity. Often quoted is the possibility of verifying that the waves are of pure helicity two, with no admixture of other spin states.

The purpose of our work (this Letter and the detailed account [4]) is to show, on the basis of a particular effect related to the so-called gravitational-wave *tail* effect, that the observations of inspiralling compact binaries will permit also verification of some aspects of the *nonlinear* structure of general relativity. This verification is made possible by the now recognized fact [5] that a very precise general relativity prediction is needed to reach full potential accuracy on the measurement of the binary's parameters.

The tail effect is essentially due to the propagation of gravitational radiation on the *curved* background space-time generated by its own source. More specifically, the tail of the radiation results, at lowest order, from the nonlinear interaction between the time-varying quadrupole moment of the source (which generates the linear radiation) and its monopole moment, or total mass-energy M (which generates the background). The tail radiation has the distinctive property ("nonlocality" in time) of depending on the source's dynamics at arbitrary remote instants in the past, anterior to the simply retarded time $t - r/c$. This reflects the fact that gravity propagates not only on the light cone (direct propagation with the speed of light c), but also *within* the light cone (averaged propagation with all velocities less than c). (See [6] for references on tails and related nonlinear effects.)

The detection of the tail effect (or of effects immediately related to it) in future gravitational-wave experiments will provide *direct* evidence that gravity propagates on a curved space-time—that generated by its own source. (Note that indirect evidence from the observations of the binary pulsar is probably out of reach [6].) This will represent an interesting test of the nonlinearity of general relativity in the "gravitodynamics" regime of the theory, involving rapidly varying and strong gravitational fields. This will also provide an independent measurement of the total mass-energy M of the source.

The tail effect arises at the so-called 1.5 post-Newtonian (1.5-PN) approximation in the radiation, i.e., at the relative order c^{-3} beyond the usual quadrupole radiation. Let us consider the radiation emitted by a general isolated source, at a large distance r from the source (neglecting terms that die out like $1/r^2$). More precisely, we denote by $h(t)$ that linear combination of the components of the wave which is directly felt by some detector [e.g., $h(t)$ is the relative variation of the arm's length of a laser interferometric detector]. Then the expression of $h(t)$, including all terms in the post-Newtonian expansion up to the order c^{-3}, can be written [7] as

$$h(t) = h_0(t) + \frac{2GM}{c^3}\int_{-\infty}^{t} dt'\left[\ln\left(\frac{t-t'}{2b}\right) + \frac{11}{12}\right]\frac{d^2h_0}{dt^2}(t') . \qquad (1)$$

This expression is valid for any slowly moving source, independent of the strength of its internal gravity. The first term $h_0(t)$ denotes the usual multipolar radiation

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up to the order c^{-3}, and can be referred to as the "front" of the wave. The second term in (1) is the tail, which is the first purely nonlinear contribution in the wave. The tail depends on two constants, namely, the total mass M of the source, and a gauge-dependent constant b having the dimension of time. The constant b can be chosen at will: it is defined by the relation $t = t_H - (2GM/c^3)\ln(r/cb)$, linking the time t used by the experimenters at distance r from the source and the time t_H of a harmonic coordinate system covering the source.

The Fourier transform $\tilde{h}(\omega)$ of $h(t)$, where $\omega = 2\pi f$ denotes the angular frequency, can be computed in terms of the Fourier transform $\tilde{h}_0(\omega)$ of $h_0(t)$. The result is particularly simple, and reads [6] (see also [8,9]) as

$$\tilde{h}(\omega) = \tilde{h}_0(\omega)\left[1 + \frac{\pi GM}{c^3}|\omega|\right]e^{i\theta(\omega)}, \tag{2}$$

where the square brackets represent a tail-induced modulation of the *amplitude* of the wave front, and where

$$\theta(\omega) = \frac{2GM\omega}{c^3}\ln(2|\omega|b') \tag{3}$$

represents a tail-induced modulation of its *phase*. ($|\omega|$ is the absolute value of ω.) Note that (2) and (3) are valid only for *low* frequencies such that $GM\omega/c^3$ is of small post-Newtonian order c^{-3}. The constant b' in (3) is related to b and to Euler's constant $C = 0.577\ldots$ by $b' = b\exp(C - 11/12)$. In this Letter, we shall choose the value $b' = 1/(2\omega_s)$, where ω_s is the "seismic cutoff" frequency of a laser interferometric detector, below which the seismic noise prevents detection.

The tail-induced modulation of the amplitude in the square brackets of (2) implies a modification of the total amount of energy carried away by the radiation, and thus a modification, driven by radiation reaction, of the internal dynamics of the source. This results in a tail contribution in the equation governing the evolution of the orbital frequency of an inspiralling binary. It can then be shown that the modulation of the amplitude given in (2) is exactly canceled by this effect, and that an important tail contribution arises in the *phase* of the wave front itself [5,8,10]. We shall refer to the latter as the "radiation reaction" tail contribution in the phase, Considering as usual a "model" wave form where only the post-Newtonian corrections in the phase are taken into account, we can write the Fourier transform (2)–(3) of the wave generated by an inspiralling binary as

$$\tilde{h}(\omega) = \mathcal{H}\,\omega^{-7/6}e^{i\{\Psi(\omega)+\theta(\omega)\}} \tag{4}$$

(with $\omega > 0$), where $\theta(\omega)$ is given by (3), and where [10]

$$\Psi(\omega) = \omega t_c - \varphi_c - \frac{\pi}{4} + \frac{3}{4}\left(\frac{c^3}{4G\mathcal{M}\omega}\right)^{5/3}\left[1 + \left(\frac{3715}{756} + \frac{55}{9}\nu\right)x + (4\beta - 16\pi)x^{3/2} + \cdots\right]. \tag{5}$$

The constant $\mathcal{H}$ is inversely proportional to the distance of the source, and t_c, φ_c denote the instant of coalescence and a final constant phase. The phase $\Psi(\omega)$ involves a post-Newtonian expansion parametrized by $x = \{G(m_1 + m_2)\omega/(2c^3)\}^{2/3}$, where m_1 and m_2 are the two masses of the binary, depending at 1-PN level on the mass ratio $\nu = m_1m_2/(m_1 + m_2)^2$ and at 1.5-PN level on a particular combination β of the orbital angular momentum and of the spins of the binary. This expansion is endowed with a large multiplying factor of order c^5 (where c^{-5} is the order at which radiation reaction effects appear), involving the "chirp mass" $\mathcal{M} = (m_1m_2)^{3/5}/(m_1 + m_2)^{1/5}$. The radiation reaction tail contribution is the term with 16π in $\Psi(\omega)$.

The "direct" tail-induced modulation of the phase $\theta(\omega)$ given by (3) does not modify, at lowest order, the total energy carried away by the radiation, but implies or effect of shifting the positions of the different frequency components of the wave along its path. This phase modulation is required in order that wave packets propagate from the source to the detector with the "correct" group velocity (see [6]). As is clear from (4)–(5), $\theta(\omega)$ is smaller by a factor $\sim c^{-5}$ than the radiation reaction tail contribution in $\Psi(\omega)$. Here we shall assume that the post-Newtonian expansion in $\Psi(\omega)$ does *not* contain, at the 4-PN level, a term of the type $\nu x^4\ln x$ which would yield, when multiplied by the factor $\sim(G\mathcal{M}\omega/c^3)^{-5/3}$, a term of the same type as $\theta(\omega)$. This assumption is justified by the physical interpretation of $\theta(\omega)$ in [6], but it cannot be proved presently, neither by a wave-generation computation, which is limited to 2-PN level, nor even by a method of perturbation of the Schwarzschild background, which deals with the limit $\nu \to 0$.

The problem of the actual detection of $\theta(\omega)$, by the method of parameter estimation, is now considered. Matched filtering is the appropriate technique for extracting the binary signal out of the detector noise. It consists of correlating the output of the detector, containing the signal $h(t)$ superposed with some Gaussian noise, with a filter $q(t)$ whose Fourier transform is

$$\tilde{q}(\omega) = k\tilde{h}(\omega)/S_h(\omega). \tag{6}$$

In (6), $\tilde{h}(\omega)$ is the Fourier transform of the signal, $S_h(\omega)$ is the power spectral density of the noise, and k is an arbitrary real constant (see, e.g., [2] for a review).

The SNR obtained after matched filtering is the best achievable with a linear filter. It reads as

$$\rho = \left(\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{|\tilde{h}(\omega)|^2}{S_h(\omega)} \right)^{1/2}. \qquad (7)$$

The matched filtering technique necessitates beforehand the knowledge of the signal. In practice, however, only the *form* of the signal is known (with some precision), and some unknown parameters, such as the two masses of the binary, are to be measured. This is accomplished by maximizing the correlation with a whole family of filters (6), corresponding to different values of the parameters. The parameters of the filter maximizing the correlation are the "measured" parameters attributed by the experiments to the real signal. These parameters do not exactly agree with the real signal parameters, since they depend on a particular realization of noise in the detector. However, their statistical distribution over a large number of realizations of noise in a large ensemble of detectors is Gaussian (for Gaussian noise and for high enough SNR) and centered on the signal parameters, with computable variances and correlation coefficients. This consideration assumes, of course, that the filters are accurately matched, by (6), on the signal. If this is not the case, the expectation values of the measured parameters will disagree with the real parameters. Thus, the inclusion of post-Newtonian correction terms in the filters permits a more accurate determination of the signal parameters. Note that a small higher-order term like $\theta(\omega)$ improves the value of the maximum SNR by a quantity of the order of the *square* of this term, which is in general negligible (see, e.g., the appendix of [4]). Thus, if one wants a high SNR but accepts a poor determination of the parameters, one can use a simpler filter including only lower-order effects.

The wave-front phase $\Psi(\omega)$, given by (5), is a function of the two masses m_1 and m_2 of the binary through the chirp mass $\mathcal{M}$ and the parameters ν, x. Let us choose $\mathcal{M}$ and the reduced mass $\mu = \mathcal{M}\nu^{2/5}$ as two independent mass parameters. Now the tail-induced phase $\theta(\omega)$ brings in the total mass $M = m_1 + m_2$ as a new parameter, which evidently is not independent of the parameters in the wave front since it is equal to $\mathcal{M}^{5/2}\mu^{-3/2}$. In order to detect $\theta(\omega)$, we shall, first, correlate the output of the detector with the family of filters (6) in which $\tilde{h}(\omega)$ is given by (4)–(5), and, second, maximize the correlation by varying the mass M in factor of $\theta(\omega)$ *independently* of the other parameters, i.e., $\mathcal{M}$ and μ. (Thus, we assume that, in the filters, the values of M and the combination $\mathcal{M}^{5/2}\mu^{-3/2}$ do not *a priori* agree.) In this way, the measurement of M will permit a *test* of the existence of $\theta(\omega)$ in the real signal. Indeed, if $\theta(\omega)$ exists, the best filter of the family, corresponding to the maximum correlation, will find a value of M which is compatible with the measured values of the other parameters (i.e., which is approximately equal to $\mathcal{M}^{5/2}\mu^{-3/2}$). On the contrary, if $\theta(\omega)$ does *not* exist, the best filter of the family will find a value of M which is compatible with zero, together with some realistic values of $\mathcal{M}$ and μ.

The test could be represented in the m_1, m_2 plane of the two masses of the binary, in a way somewhat similar to the test of the existence of gravitational radiation in the binary pulsar 1913 + 16, where the change in the orbital period $\dot{P}$ of the pulsar, the relativistic periastron shift $\dot{\omega}$ of the orbit, and the redshift-Doppler parameter γ are plotted in the m_p, m_c plane of the pulsar and its companion [1]. The test would consist of the intersection at *one* single point in the half-plane $m_1 \geq m_2$ (say) of three curves corresponding to the measurements of the total mass M which is in factor of the tail, and of the two parameters of the wave front $\mathcal{M}$ and μ (the three parameters $M, \mathcal{M}$, and μ being independently varied and measured in the filtering process). The curves would be surrounded by 1-σ error bars reflecting the uncertainties in the measurement. The intersection point would give, within these uncertainties, the values of the two separate masses m_1 and m_2 of the binary, as determined by general relativity.

We now investigate the level at which the above test could be implemented, i.e., the level at which it would be possible to detect $\theta(\omega)$. For this purpose, it is sufficient to find the level at which the parameter M in factor of $\theta(\omega)$ can be attributed, with some confidence, a *nonzero* value. Thus, we need to compute the 1-σ error bar, say, σ_M in the measurement of M, and to compare σ_M with the value of M itself. The 1-σ confidence level at which $\theta(\omega)$ can be detected is simply the level at which σ_M gets *smaller* than M. We shall compute σ_M in the case where we know *a priori* that the spins of the stars are negligible, and thus where five parameters are relevant to the construction of filters (6): four parameters $t_c, \varphi_c, \mathcal{M}$, and μ in the wave front, and the parameter M multiplying the tail contribution.

Some simple models of noise in the detector are used. We assume first that the spectral density $S_h(\omega)$ of the noise is infinite outside the bandwidth $[\omega_s, \omega_u]$, where ω_s is the seismic cutoff frequency to which b' in (3) has been related. Inside the bandwidth, we assume that the noise is either white,

$$S_h(\omega) = \text{const}, \qquad (8)$$

or colored in the sense appropriate for shot noise in the standard recycling configuration of a laser interferometric detector (see, e.g., [2]),

$$S_h(\omega) = \text{const}\,\omega_k[1 + (\omega/\omega_k)^2]. \qquad (9)$$

The frequency ω_k in (9) is the so-called "knee" frequency. We adopt here the value $\omega_k = 1.44\omega_s$ which maximizes the SNR (7), all other parameters being fixed.

The optimal filtering process is now simulated. A signal $\tilde{h}(\omega)$ given by (4)–(5), and depending on a known set of signal parameters, is added to a particular realization of simulated Gaussian noise with spectral density (8) or (9). The resulting noisy data are correlated with a lattice of filters (6) matched on the signal (4)–(5) and constructed using the method of [11,12]. By maximizing the correlation, we determine a first measured value of the parameter M, and by repeating the process for a large number of realizations of noise ($\sim$100), we obtain the whole statistical distribution of the measured values of M. The standard deviation σ_M of this distribution is then deduced. Finally, the computation is redone with other signal parameters and the variations of σ_M in terms of these parameters are obtained. It is convenient to express σ_M, for a given type of noise, as a function of the optimal SNR ρ of the signal, given by (7). The result of the computation, for both the white and colored noises (8)–(9) (where $\omega_s/2\pi = 100$ Hz and $\omega_u/2\pi = 2000$ Hz are used), is presented in Fig. 1.

As Fig. 1 shows, the precision σ_M in the measurement of M is a decreasing function of the SNR. This is to be expected since the more signal we have, the more accurate is the determination of M. Note also that, for a given SNR, σ_M is larger in the colored noise case than in the white noise case. This results from the fact that the colored noise (9) is relatively narrowband as compared to the white noise (8) (i.e., most of the signal power is extracted in a smaller range of frequencies). Finally it can be shown that for a given type of noise σ_M is fairly insensitive on both the value of μ and M. This means in particular that a lower SNR is needed to detect the tail from a binary with a comparatively higher total mass.

The minimal SNR required to detect $\theta(\omega)$ from a coalescing binary with total mass M is directly read from Fig. 1. All signals whose optimal SNR is such that

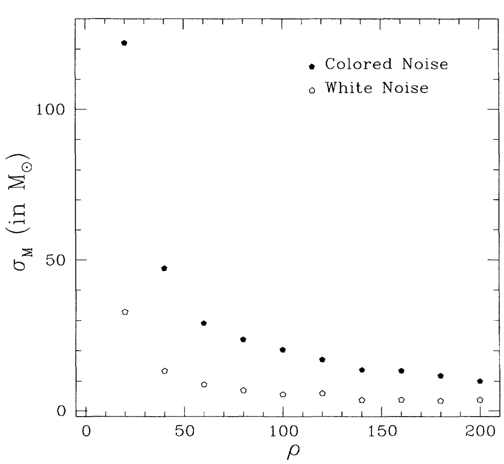

FIG. 1. The standard deviation σ_M is plotted against the SNR ρ (simulation with five independent parameters).

σ_M in Fig. 1 is smaller than M have sufficient strength for detection of $\theta(\omega)$. Note that, in practice, we shall compare the (anticipated) value of σ_M in Fig. 1 not with M, but with some *measured* value of M. However, this does not make any difference if the SNR is high enough. For a black-hole binary with $M = 20M_\odot$, we obtain a minimal SNR $\sim$35 in the white noise case, and $\sim$100 in the colored noise case. For a neutron-star binary with total mass $M = 2.8M_\odot$, we obtain $\sim$250 (white noise) and $\sim$750 (colored noise) [4]. Note that these values depend on the number of independent parameters which are used in the filtering process [13]. With six parameters including the spin-orbit parameter β the values are larger by a factor $\sim$4, but with four parameters excluding β and μ we get a factor $\sim$0.4 of improvement (see [4]).

Our conclusion is that the detection of the tail effect $\theta(\omega)$ could be achieved in future gravitational-wave experiments, at least in the case of black-hole coalescences. (Of course, this conclusion relies very much on the statistics of coalescence events [3] and on the anticipated sensitivity of future detectors.) Owing to the extreme smallness (4-PN relative order) of this effect, this shows more generally that coalescing compact binaries could permit many tests of the nonlinear structure of general relativity.

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