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Detecting and Solving Hyperbolic Quadratic Eigenvalue Problems

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Outline

Two recent research themes:

- Cyclic reduction for solving Riccati eqns and nonlinear eqns.
- Identification and exploitation of structure in solving PEPs.

We show how cyclic reduction can be used to

- test for hyperbolicity,
- initiate a solution procedure that fully exploits the hyperbolicity.

Quadratic Eigenvalue Problem (QEP)

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A, B, C \in \mathbb{C}^{n\times n}$$

Q assumed **regular** ($\det Q(\lambda) \not\equiv 0$).

Find scalars λ and nonzero vectors x and y satisfying $Q(\lambda)x = 0$ and $y^* Q(\lambda) = 0$.

- $Q(\lambda)$ has $2n$ eigenvalues.
- For a given μ, the matrix $Q(\mu)$ has n eigenvalues.

A **solvent** of Q is a solution of

$$Q(X) = AX^2 + BX + C = 0.$$

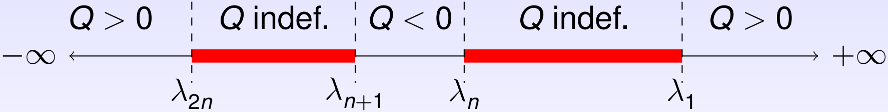
Hyperbolic Quadratics

$$Q(\lambda) = \lambda^2 A + \lambda B + C, \quad A, B, C \in \mathbb{C}^{n\times n}$$

is **hyperbolic** if A, B, C Hermitian, $A > 0$, and

$$(x^* Bx)^2 > 4(x^* Ax)(x^* Cx) \quad \text{for all nonzero } x \in \mathbb{C}^n.$$

Eigenvalues are all real and semisimple.



Q is hyperbolic iff $Q(\mu) < 0$ for some $\mu \in \mathbb{R}$.

Overdamped Quadratics

Q is **overdamped** if it is hyperbolic with $B > 0$, $C \geq 0$.

Theorem

The following statements are equivalent.

(a) *Q is overdamped.*

(b) *Q is hyperbolic with all e'vals real and nonpositive*

(c) $B > 0$, $C \geq 0$ *and* $B > \mu A + \mu^{-1} C$ *for some* $\mu > 0$.

From (b) if Q is hyperbolic then with $\theta > \lambda_{\max}$,

$$Q_\theta(\lambda) := Q(\lambda + \theta) = \lambda^2 A + \lambda(B + 2\theta A) + C + \theta B + \theta^2 A$$

is overdamped.

An Iteration Based on Cyclic Reduction

$$
\begin{aligned}
S_0 &= B, \quad A_0 = A, \quad B_0 = B, \quad C_0 = C, \\
S_{k+1} &= S_k - A_k B_k^{-1} C_k, \\
A_{k+1} &= A_k B_k^{-1} A_k, \\
B_{k+1} &= B_k - A_k B_k^{-1} C_k - C_k B_k^{-1} A_k, \\
C_{k+1} &= C_k B_k^{-1} C_k.
\end{aligned}
$$

Theorem (Guo & Lancaster, 2005)

If Q is overdamped,

(a) $A_k > 0$, $C_k \geq 0$, $B_k > 0$ *for all* $k \geq 0$.

(b) $\|A_k\| \, \|C_k\|$ ***converges quadratically*** *to zero with rate* $\frac{\lambda_n}{\lambda_{n+1}}$,

(c) S_k ***converges quadratically*** *to nonsingular* $\widehat{S}$ *with rate* $\frac{\lambda_n}{\lambda_{n+1}}$, *and* $S^{(1)} = -\widehat{S}^{-1} C$, $S^{(2)} = -A^{-1} \widehat{S}^*$.

A Key Property of the Iteration

$$S_0 = B, \quad A_0 = A, \quad B_0 = B, \quad C_0 = C,$$
$$S_{k+1} = S_k - A_k B_k^{-1} C_k,$$
$$A_{k+1} = A_k B_k^{-1} A_k,$$
$$B_{k+1} = B_k - A_k B_k^{-1} C_k - C_k B_k^{-1} A_k,$$
$$C_{k+1} = C_k B_k^{-1} C_k.$$

Lemma

Let $\mu > 0$ and assume $A_k > 0$, $C_k \geq 0$. Then $B_k > \mu^{2^k} A_k + \mu^{-2^k} C_k$ if and only if $A_{k+1} > 0$, $C_{k+1} \geq 0$, and $B_{k+1} > \mu^{2^{k+1}} A_{k+1} + \mu^{-2^{k+1}} C_{k+1}$.

$Q(\lambda)$ overdamped $\Leftrightarrow$ $Q_k(\lambda) = \lambda^2 A_k + \lambda B_k + C_k$ overdamped $\forall k$.

Overdamping Test

$$
\begin{aligned}
\alpha_0 &= \sqrt{\|C\|/\|A\|}, \\
\widetilde{A}_0 &= \alpha_0 A, \quad B_0 = B, \quad \widetilde{C}_0 = \alpha_0^{-1} C, \\
A_{k+1} &= \widetilde{A}_k B_k^{-1} \widetilde{A}_k, \quad C_{k+1} = \widetilde{C}_k B_k^{-1} \widetilde{C}_k \\
B_{k+1} &= B_k - \widetilde{A}_k B_k^{-1} \widetilde{C}_k - \widetilde{C}_k B_k^{-1} \widetilde{A}_k, \\
\alpha_{k+1} &= \sqrt{\|C_{k+1}\|/\|A_{k+1}\|}, \\
\widetilde{A}_{k+1} &= \alpha_{k+1} A_{k+1}, \quad \widetilde{C}_{k+1} = \alpha_{k+1}^{-1} C_{k+1}.
\end{aligned}
$$

Theorem

$Q(\lambda)$ with $A, B > 0$ and $0 \neq C \geq 0$ is overdamped iff for some $m \geq 0$, $B_k > 0$ for $k = 1: m-1$, $B_m > \widetilde{A}_m + \widetilde{C}_m$.

$B_m > \widetilde{A}_m + \widetilde{C}_m \Leftrightarrow Q(-\mu_m) < 0$ with $\mu_m = \alpha_0 \alpha_1^{2^{-1}} \alpha_2^{2^{-2}} \cdots \alpha_m^{2^{-m}}$.

Damped Mass-Spring System

$$A = I_n,\ C = \text{tridiag}(-5, 15, -5),$$
$$B = \beta\ \text{tridiag}(-10, 30, -10) - 10 e_1 e_1^T - 10 e_n e_n^T,$$

$\beta > 0$ is a real parameter.

Number of iterations m to verify that $Q(\lambda)$ **is** overdamped:

β	1	0.62	0.61	0.52	0.5197	0.519616	0.51961525	0.5196152423
m	0	0	1	2	3	5	8	12

Number of iterations m to verify that $Q(\lambda)$ **is not** overdamped.

β	0.36	0.47	0.51	0.5196	0.519615	0.51961524	0.5196152422
m	1	2	4	8	11	15	17

Weakly Overdamped Quadratics

$Q(\lambda) = \lambda^2 A + \lambda B + C$ is **weakly overdamped** if A, B, and C are Hermitian, $A, B > 0$, $C \geq 0$ and

$$\gamma = \min_{\|x\|_2=1}[(x^*Bx)^2 - 4(x^*Ax)(x^*Cx)] \geq 0.$$

- If $\gamma = 0$, Q has $2n$ real e'vals $\lambda_1 \geq \cdots \geq \lambda_n = \lambda_{n+1} \geq \cdots \geq \lambda_{2n}$.
- Partial multiplicities of λ_n are at most 2 (2 is generic).
- For weakly overdamped Q, cyclic iteration converges **linearly with constant $1/2$** in the generic case.

Alg. for detection and numerical solution

Let $Q(\lambda) = \lambda^2 A + \lambda B + C$ be Hermitian with $A > 0$.
Three steps:

- **Preprocessing**: form
  $Q_\theta(\lambda) \equiv Q(\lambda + \theta) = \lambda^2 A_\theta + \lambda B_\theta + C_\theta$
  with θ s.t. $B_\theta > 0$ and $C_\theta \geq 0$, or conclude Q is not hyperbolic and terminate alg.
- **Overdamping test**: check overdamping condition for Q_θ. If overdamped, $\mu \in \mathbb{R}$ s.t. $Q_\theta(\mu) < 0$ is computed; otherwise terminate alg.
- **Solution**: Q_θ converted to $2n \times 2n$ definite pencil $\lambda X + Y$ with $X > 0$ or $Y > 0$. Eigenpairs of Q obtained from eigendecomp. of $\lambda X + Y$ by exploiting definiteness of X or Y and block structure of X and Y.

Solving hyperbolic QEPs

Let μ s.t. $Q_\theta(\mu) = Q(\mu+\theta) < 0$. Hence with $\omega = \mu + \theta$

$$\begin{aligned} Q_\omega(\lambda) = Q(\lambda+\omega) &= \lambda^2 A + \lambda(B + 2\omega A) + C + \omega B + \omega^2 A \\ &= \lambda^2 A_\omega + \lambda B_\omega + C_\omega, \end{aligned}$$

with $C_\omega = Q(\omega) < 0$ and $A_\omega = A > 0$.

The pencils

$$\lambda \begin{bmatrix} A_\omega & 0 \\ 0 & -C_\omega \end{bmatrix} + \begin{bmatrix} B_\omega & C_\omega \\ C_\omega & 0 \end{bmatrix}, \quad \lambda \begin{bmatrix} 0 & A_\omega \\ A_\omega & B_\omega \end{bmatrix} + \begin{bmatrix} -A_\omega & 0 \\ 0 & C_\omega \end{bmatrix}$$

are both **Hermitian definite linearizations of Q_ω**.

Costs

- **Overdamping test**: (m, ℓ: # of iter., $m \le \ell$)
  - Guo & Lancaster alg.: $(19\ell + 16)/3n^3$ flops.
  - Guo, Higham & T. alg.: $20mn^3/3$ flops.
- **Overall eigensolution**:
  - QZ on linearized problem: $240n^3$ flops.
  - Guo & Lancaster alg.: $(19\ell/3 + 25)n^3$ flops.
  - Guo, Higham & T. alg.: $(20m/3 + 13)n^3$ flops.

Guo, Higham & T. alg. works entirely with symm. matrices and guarantees to produce real e'vals.

Numerical Experiments

- type 1: λ_k, $k = 1\colon 2n$, uniformly distributed in $[-100, -1]$.
- type 2: λ_k uniformly distributed in $[-100, -6]$ for $k = n+1\colon 2n$ and $[-5, -1]$ for $k = 1\colon n$.
- type 3: λ_k uniformly distributed in $[-100, 20]$.

(Min, average, max) $\#$ iterations to test overdamping and % of overdamped problems.

n	type 1		type 2		type 3	
5	(0, 2.4, 6)	100%	(0, 0.8, 3)	100%	(0, 2.4, 5)	25%
10	(0, 3.6, 10)	100%	(0, 0.5, 3)	100%	(2, 2.7, 4)	5%
50	(0, 4.2, 11)	100%	(0, 2.1, 4)	100%	(2, 2.1, 3)	0%
100	(3, 6.2, 10)	100%	(0, 2.6, 4)	100%	(2, 2.0, 2)	0%
250	(2, 6.0, 11)	100%	(2, 3.0, 4)	100%	(2, 2.0, 2)	0%
500	(3, 7.5, 11)	100%	(2, 3.0, 4)	100%	(2, 2.0, 2)	0%

Concluding Remarks

- Have devise an **efficient and reliable numerical test for hyperbolicity** or overdamping of a given Hermitian quadratic.

- Have build upon an affirmative test result an efficient alg. for solving the QEP which
  - **exploits hyperbolicity**,
  - **guarantees real computed eigenvalues** in floating point arithmetic.

For papers and Eprints,
`http://www.ma.man.ac.uk/~ftisseur/`

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