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DETECTING THE DISJOINT DISKS PROPERTY

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This paper explores conditions under which a metric space S satisfies the following Disjoint Disks Property: any two maps of the standard 2-cell B^2 into S can be approximated by maps having disjoint images. Among its many applications, it provides a proof that if Y is the cell-like image of an n-manifold ($n \geq 3$), then $Y \times E^2$ has the Disjoint Disks Property, which implies that $Y \times E^2$ is a manifold. It adds further evidence for the unifying force of this property by giving comparatively easy proofs for established facts about certain decomposition spaces that are manifolds.

The significance of the Disjoint Disks Property is made manifest by its role in a recently-proved fundamental result about cell-like decompositions, due to R. D. Edwards [17]: if an ANR X is the proper cell-like image of an n-manifold M ($n \geqq 5$) and satisfies the Disjoint Disks Property, then X is an n-manifold homeomorphic to M.

J. W. Cannon, who obtained a fairly strong partial result of this type [12], receives the credit for focusing attention on the Disjoint Disks Property and making plausible the claim that it should be the crucial additional feature forcing such an ANR to be a manifold [10], [11], [12]. Like so much of the subject of manifold decompositions, origins of this property can be traced to early work of R. H. Bing, in this case where he developed methods for determining whether certain cellular decompositions of Euclidean 3-space E^3 were shrinkable [5], [6].

Enhancing the significance of the Disjoint Disks Property is another fundamental result, announced by F. Quinn [28]: if X is a generalized n-manifold (namely, a finite dimensional ANR such that for all $x \in X$ $H_*(X, X - \{x\})$ coincides with $H_*(E^n, E^n - \{0\})$ and $n \geqq 5$, then there exists a cell-like proper map of an n-manifold M onto X. The combination of Edwards' and Quinn's work presents a basic characterization of topological manifolds: for $n \geqq 5$ a space X is an n-manifold iff it is a generalized n-manifold that satisfies the Disjoint Disks Property.

This paper aims toward applications of this characterization, primarily within decomposition theory itself. At the heart of most applications here is the well-known result [31] (see also [1] or the discussion in [10, p. 323]) that if an ANR X is the proper cell-like image of an n-manifold M (without boundary), then X is a generalized n-manifold. Indeed, even when X fails to be an ANR, it follows from [4] that $H_*(X, X - \{x\})$ always coincides with $H_*(E^n, E^n - \{0\})$.

Much of the space in this paper is devoted to such finite dimensional generalized manifolds. However, it begins (§ 2) with an investigation into the behavior of the Disjoint Disks Property in a somewhat abstract setting. The generality serves two purposes: first, it clarifies the rather minimal topological conditions on which production of this property is founded, and second, it permits application in the realm of Q-manifolds, where the property also appears to be important (see [16]). The section culminates with the result, currently being used rather widely throughout this subject, that the product of E^2 and any generalized n-manifold ($n \geqq 3$) has the Disjoint Disks Property.

Once the preliminary investigation is complete, the paper sets forth three distinct kinds of applications. The first (§ 3) details several pivotal facts about product decompositions. The technology developed in § 2 renders the proofs relatively short and easy, and receives further justification through subsequent reuse. The second (§ 4) provides a comparatively fast proof for a slightly improved version of the result by J. W. Cannon [9] that every nicely spherical generalized n-manifold ($n \geqq 5$) is a topological n-manifold. The final application (§ 5) gives mild conditions on a cell-like image X of an n-manifold $M (n \geqq 4)$ ensuring that $X \times E^1$ be homeomorphic to $M \times E^1$. As a consequence, a primary benefit of § 5 is its straightforward proof, directly from the Disjoint Disks Property, that the ghastly generalized n-manifold X ($n \geqq 4$) described by Cannon-Daverman [14], one of the more complicated such spaces extant, becomes a manifold upon crossing with E^1. Although they also establish this fact in [14], their argument is a good bit more intricate and it accomplishes far less.

Following a spreading practice, we say that a map $f: X \to Y$ is $1 - 1$ *over a subset* Z of Y if $f | f^{-1}(Z)$ is $1 - 1$.

Let (S, ρ) denote a metric space. For $A \subset S$, the *diameter of* A is denoted as diam A and, of course, is defined as

$$\operatorname{diam} A = \sup \{\rho(a, a'): a, a' \in A\} .$$

For $\varepsilon > 0$, the *ε-neighborhood* $N_\varepsilon(A)$ is the set of points in S whose distance from A is less than ε. We say that A is *locally k-connected*, written LC^k, if for each $a \in A$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that every map of S^i into $N_\delta(a)$ extends to a map of B^{i+1} into $N_\varepsilon(a)$ ($i = 0, \cdots, k$). Similarly, we say that A is *locally k-co-connected*, written k-LCC, or, equivalently, that $S - A$ is k-LC at

A, if for each $a \in A$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that every map of S^k into $N_\delta(a) \cap (S - A)$ extends to a map of B^{k+1} into $N_\varepsilon(a) \cap (S - A)$.

Given two maps $f, g: X \to S$, we use $\rho(f, g)$ to denote $\sup \{\rho(f(x), g(x)): x \in X\}$.

All homology groups mentioned should be interpreted as singular homology groups with integer coefficients.

2. **Disjointness properties.** Instrumental to this discussion are two properties of a metric space S subordinate to the Disjoint Disks Property. First, S is said to have the Disjoint Arcs Property (DAP) if any two maps of B^1 to S can be approximated, arbitrarily closely, by maps having disjoint images. Second, S is said to have the Disjoint Arc-Disk Property (DADP) if, for all maps $f: B^1 \to S$ and $g: B^2 \to S$ and for each $\varepsilon > 0$, there exist maps $F: B^1 \to S$ and $G: B^2 \to S$ such that $\rho(F, f) < \varepsilon$, $\rho(G, g) < \varepsilon$ and $F(B^1) \cap G(B^2) = \varnothing$.

Throughout this section X will denote a locally compact, LC^1 metric space whose dimension, at each point, is at least two—in other words, no closed 1-dimensional subset of X contains a non-empty open set. In applications X ordinarily will represent the cell-like image of an n-manifold ($n > 2$) or, in somewhat more restricted situations, of a Q-manifold.

LEMMA 2.1. *Suppose X is a locally compact metric space such that, for some integer $r > 0$ and for every $x \in X$, $H_i(X, X - \{x\}) \cong 0$ ($i = 0, 1, \cdots, r$). Then for each k-dimensional closed subset A of S, where $k \leqq r$, $H_j(X, X - A) \cong 0$ whenever $j \in \{0, \cdots, r - k\}$.*

This result is based on a relatively routine Mayer-Victoris argument. For completeness, details are provided in the Appendix.

PROPOSITION 2.2. *Suppose that for each open subset U of X and for each 1-dimensional closed subset A of X, $H_1(U, U - A) \cong 0$. Then X has the* DAP.

Proof. The initial part of the argument shows that $X - A$ is 0-LC at each point $a \in A$. To see this, consider any neighborhood V of a and find a path-connected neighborhood U contained in V. From the long exact sequence of $(U, U - A)$, one has

$$0 \cong H_1(U, U - A) \longrightarrow \tilde{H}_0(U - A) \longrightarrow \tilde{H}_0(U) \cong 0 .$$

Hence, $\tilde{H}_0(U - A) \cong 0$, or, equivalently, $U - A$ is path-connected.

Next consider maps $f_1, f_2: B^1 \to X$ and $\varepsilon > 0$. Partition B^1 as

$$-1 = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_k = 1$$

so that each interval $[t_{i-1}, t_i]$ has small size (depending on f_1). Since Peano continua are arcwise connected, each $f_1([t_{i-1}, t_i])$ contains an arc from $f_1(t_{i-1})$ to $f_1(t_i)$, whenever $f_1(t_{i-1}) \neq f_1(t_i)$. Now it is a simple matter to produce a map $f_1'\colon B^1 \to X$ approximating f_1 such that $f_1' | [t_{i-1}, t_i]$ is either a constant mapping to $f_1(t_i)$ or an embedding defining an arc in $f_1([t_{i-1}, t_i])$. As a result, the image $A = f_1'(B^1)$ is 1-dimensional.

Cover A by pathwise connected open sets U_a, $a \in A$, each having diameter less than ε, and determine a new partition

$$-1 = s_0 < \cdots < s_{i-1} < s_i < \cdots < s_m = 1$$

of B^1 such that $f_2([s_{i-1}, s_i])$ is contained in some $U_i \in \{U_a | a \in A\}$ whenever it intersects A. Now define $f_2'(s_i)$ as $f_2(s_i)$ whenever $s_i \notin A$ and as an element of $(U_i \cap U_{i+1}) - A$ otherwise (here interpret U_0 as U_1 and U_{m+1} as U_m). Since $U_i - A$ is pathwise connected, $f_2' | \{s_{i-1}, s_i\}$ extend to a map $f_2'\colon [s_{i-1}, s_i] \to U_i - A$. Assembly of these segmentally described maps produces an ε-approximation f_2' to f_2, with $f_2'(B^1) \subset X - f_1'(B^1)$.

COROLLARY 2.3. *If X is a locally compact* LC^0 *metric space such that for each $x \in X$ and for $i \in \{0, 1, 2\}$, $H_i(X, X - \{x\}) \cong 0$, then X has the* DAP.

In effect, the proof for Proposition 2.2 establishes the more interesting half of the following result.

PROPOSITION 2.4. *A locally compact metric space S has the* DAP *if and only if each map $f\colon B^1 \to S$ can be approximated by $F\colon B^1 \to S$ such that $F(B^1)$ is nowhere dense and* 0-LCC.

A characterization of the DADP can be derived by similar methods.

PROPOSITION 2.5. *For a locally compact metric space S, the following statements are equivalent:*

(1) *S has the* DADP.

(2) *Each map $f\colon B^1 \to S$ can be approximated by $F\colon B^1 \to S$ such that $F(B^1)$ is nowhere dense and both* 0-LCC *and* 1-LCC.

(3) *Each map $g\colon B^2 \to S$ can be approximated by $G\colon B^2 \to S$ such that $G(B^2)$ is nowhere dense and* 0-LCC.

LEMMA 2.6. *If X has the* DAP, *then each map of a finite* 1-*complex P into X can be approximated by an embedding e such that $e(P)$ is* 0-LCC.

Proof. Cannon's proof for Theorem 2.1 in [12], concerning approximations of singular 2-cells in a space satisfying the DDP, readily translates to a proof for the matter at hand.

COROLLARY 2.7. *If X has the* DAP, *then every map of a compact 1-dimensional metric space into X can be approximated, arbitrarily closely, by embeddings.*

This follows by mimicking an argument of Hurewicz-Wallman [22, Theorem V2].

PROPOSITION 2.8. *If X has the* DAP, *then $X \times E^1$ has the* DADP.

Proof. Let $p_X\colon X \times E^1 \to X$ and $p_E\colon X \times E^1 \to E^1$ denote the projection mappings. Given a map $f\colon B^1 \to X \times E^1$, one can apply Lemma 2.6 to approximate $p_X f$ by a 0-LCC embedding $e_X\colon B^1 \to X$. We shall prove that the embedding $e\colon e_X \times p_E f\colon B^1 \to X \times E^1$ gives a 1-LCC embedded arc.

First, however, we shall explain why the arc $A = e_X(B^1) \times \{0\}$ is 1-LCC embedded in $X \times E^1$. Let $V \times (-\delta, \delta)$ be a neighborhood of $a \in A$. Since X is LC^1, we can produce a path-connected open set U in X such that every loop in U is null homotopic in V and that $U \times (-\delta, \delta)$ is another neighborhood of a. Because $e_X(B^1)$ is 0-LCC, an argument like the final part of Proposition 2.2 shows that $U - e_X(B^1)$ is also path-connected. Hence, every loop in $U \times (-\delta, \delta) - A$ can be expressed as the composition of loops from $U \times (-\delta, \delta) - (e_X(B^1) \times [0, \delta))$ and from $U \times (-\delta, \delta) - (e_X(B^1) \times (-\delta, 0])$. In either case, such loops are contractible in $V \times (-\delta, \delta) - A$. As a result, the image of $\pi_1(U \times (-\delta, \delta) - A)$ in $\pi_1(V \times (-\delta, \delta) - A)$ is trivial, and A is 1-LCC embedded at a.

With comparative ease one might observe that A is also 0-LCC.

The Klee trick [23] or Brown's graph-pushing [8] reveals $e(B^1)$ and A to be equivalently embedded in $X \times E^1$, by means of a homeomorphism $\theta\colon X \times E^1 \to X \times E^1$ such that $p_X \theta = p_X$. Certainly then $e(B^1)$ is 0-LCC and 1-LCC. According to Proposition 2.5, $X \times E^1$ has the DADP.

LEMMA 2.9. *Suppose X has the* DADP, *f and g are maps of B^2 to X, P is a finite 1-complex in B^2, and $\varepsilon > 0$. Then there exist maps $F, G\colon B^2 \to X$ such that $\rho(F, f) < \varepsilon$, $\rho(G, g) < \varepsilon$ and $F(P) \cap G(B^2) = \varnothing$.*

Proof. It is an elementary consequence of the DADP that there exist maps $F\colon P \to X$ and $G\colon B^2 \to X$, approximating $f|P$ and g, respectively, such that $F(P) \cap G(B^2) = \varnothing$. When X is an ANR, it is well-known that F can be obtained so close to $f|P$ that it extends to $F: B^2 \to X$ within ε of f: under the operant hypothesis here that X be LC^1, one can extend F over the skeleta of a triangulation of B^2 containing P as a subcomplex to reach the same conclusion.

PROPOSITION 2.10. *If X has the* DADP, *then $X \times E^1$ has the* DDP.

Proof. Again let $p_X: X \times E^1 \to X$ and $\varepsilon > 0$. Name a triangulation T of B^2 having mesh so small that diam $f(\sigma) < \varepsilon$ and diam $g(\sigma) < \varepsilon$ for each $\sigma \in T$, and let P denote its 1-skeleton.

Since X has the DADP, there exist maps $f_1, g_1: B^2 \to X \times E^1$ within ε of f and g and satisfying

$$(*) \qquad p_X f_1(P) \cap p_X g_1(B^2) = \varnothing .$$

Repeating, while limiting the motion so as not to lose $(*)$, one produces maps $f_2, g_2: B^2 \to X \times E^1$ satisfying $\rho(f_2, f_1) < \varepsilon$, $\rho(g_2, g_1) < \varepsilon$, and

$$(**) \qquad p_X f_2(P) \cap p_X g_2(B^2) = \varnothing = p_X f_2(B^2) \cap p_X g_2(P) .$$

Note that, for each $\sigma \in T$, diam $f_2(\sigma) < 5\varepsilon$ and diam $g_2(\sigma) < 5\varepsilon$.

Enumerate the set Σ of 2-simplexes of T as $\sigma_1, \cdots, \sigma_m$. For each $\sigma_i \in \Sigma$ choose a point $s_i \in E^1$ so that $f_2(\sigma_i)$ is within 5ε of $X \times \{s_i\}$ and another point $t_i \in E^1$ so that $g_2(\sigma_i)$ is within 5ε of $X \times \{t_i\}$ and that no t_j belongs to $\{s_i | i = 1, \cdots, m\}$. Now one can easily exhibit maps f_3 and g_3 such that $f_3|P = f_2|P$, $g_3|P = g_2|P$ and, for $i = 1, \cdots, m$, $f_3(\sigma_i)$ is contained in

$$[p_X f_2(\partial\sigma_i) \times (s_i - 5\varepsilon, s_i + 5\varepsilon)] \cup [p_X f_2(\sigma_i) \times \{s_i\}] ,$$

a set of diameter $< 15\varepsilon$, while $g_3(\sigma_i)$ is contained in

$$[p_X g_2(\partial\sigma_i) \times (t_i - 5\varepsilon, t_i + 5\varepsilon)] \cup [p_X g_2(\sigma_i) \times \{t_i\}] .$$

It follows from $(**)$ and from the choice of points s_i and t_j that $f_3(B^2) \cap g_3(B^2) = \varnothing$. Moreover, $\rho(f_3, f) < 17\varepsilon$ and $\rho(g_3, g) < 17\varepsilon$.

The author is indebted to Luis Montejano for a simplification of an earlier argument for Proposition 2.10.

THEOREM 2.11. *If X has the* DAP, *then $X \times E^2$ has the* DDP.

See Propositions 2.8 and 2.10.

COROLLARY 2.12. *If X has the* DAP, *then $X \times I^2$ has the* DDP.

The primary justification for this section involves its bearing upon a generalized n-manifold Y, but a secondary justification occurs when Y is the cell-like image of a Q-manifold in which points have infinite codimension, that is, when $H_*(Y, Y - \{y\}) \cong 0$ for every $y \in Y$.

COROLLARY 2.13. *If Y is a generalized n-manifold, $n \geqq 3$, then $Y \times E^2$ has the* DDP.

COROLLARY 2.14. *If G is a cell-like decomposition of a Q-manifold M such that each point of M/G has infinite codimension, then $(M/G) \times I^2$ has the* DDP.

Proof. By [2, Lemma 2.1] the cell-like property implies that M/G is a LC^1 metric space.

3. **Applications.** To avoid unremitting repetition, we shall state here at the outset that all of the theorems in this section are founded on Edwards' Cell-like Approximation Theorem [17], which, in one form, asserts that if G is a cell-like decomposition of an n-manifold M ($n \geqq 5$) for which M/G has the DDP and is finite dimensional, then G is shrinkable, implying that the decomposition map $\pi: M \to M/G$ can be approximated by homeomorphisms. Frequently we shall find it convenient to exploit the equivalence: when G is a cell-like decomposition of a finite dimensional ANR X, $\dim(X/G) < \infty$ iff X/G is an ANR (see [25, Corollary 3.2] and [25, p. 510] or [24]; that $\pi: X \to X/G$ cannot raise dimension when X/G is an ANR results directly from the statement that π is hereditary proper homotopy equivalence).

THEOREM 3.1. *If G is a cell-like decomposition of an n-manifold M ($n \geqq 3$) such that M/G is finite dimensional, then $G \times E^2$ is shrinkable and $(M/G) \times E^2$ is homeomorphic to $M \times E^2$.*

Proof. The decomposition spaces $(M/G) \times E^2$ and $(M \times E^2)/(G \times E^2)$ are naturally equivalent. By Corollary 2.12, each has the DDP.

Armed with the powerful result of F. Quinn, one can improve this as follows.

THEOREM 3.2. *If Y is a generalized n-manifold ($n \geqq 3$), then $Y \times E^2$ is an $(n + 2)$-manifold.*

Proof. By Quinn's work [28], $Y \times E^2$ is topologically equivalent to the space associated with a cell-like decomposition of an $(n + 2)$-

manifold, and by Corollary 2.13, $Y \times E^2$ has the DDP.

The beacon originally attracting interest concerning products of decomposition spaces remains unanswered: if G is a cell-like decomposition of an n-manifold such that $\dim(M/G)$ is finite, is $(M/G) \times E^1$ homeomorphic to $M \times E^1$? Important cases in which the answer is affirmative can be identified with the aid of further terminology. For an upper semicontinuous decomposition G of a separable metric space X, endowed with the natural $\pi: X \to X/G$, it is somewhat standard parlance to say that G is *k-dimensional* if $\dim \pi(N_G) \leqq k$ and that G is *closed-k-dimensional* if $\dim Cl\pi(N_G) \leqq k$. The Sum Theorem from elementary dimension theory [22, Theorem III 2] indicates that X/G is finite dimensional if both G and X are.

In the next two theorems, to understand their proximity to best-possible results, one should remember that if G is a cell-like decomposition of an n-manifold M such that $\dim(M/G) < \infty$, then $\dim(M/G) \leqq n$ [24], and G itself must be n-dimensional.

THEOREM 3.3. *If G is an $(n-3)$-dimensional cell-like decomposition of an n-manifold M ($n \geqq 4$), then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homeomorphic to $M \times E^1$.*

Proof. In order to show that $(M/G) \times E^1$ has the DDP, we shall apply Proposition 2.10, after we show that M/G has the DADP. In order to prove that M/G has the DADP, we shall approximate an arbitrary map $f: B^2 \to M/G$ by $F: B^2 \to M/G$ such that $\dim F(B^2) \leqq n-2$. For any open subset U of M/G, Lemma 2.1 will imply that $H_1(U, U - F(B^2)) = 0$. Then the proof of Proposition 2.2 will reveal $F(B^2)$ to be 0-LCC, and Proposition 2.5 will certify M/G to have the DADP.

For support in producing the desired approximation F, consider the complete metric space C of maps from B^2 to M/G. Determine open subsets $W_1, W_2 \cdots$ of M/G, with $\pi(N_G) \subset W_{i+1} \subset W_i$ for all i, such that $\dim(\cap W_i) \leqq n-3$, and find straight line segments $L_1, L_2, \cdots$ in B^2 such that $\dim(B^2 - \cup L_i) = 0$. Define

$$0_k = \{h \in C: h \text{ is a } (1/k)\text{-map over } (M/G) - W_k\}$$

$$P_k = \left\{h \in C: h \mid \bigcup_{i=1}^{k} L_i \text{ is a } (1/k)\text{-map}\right\}.$$

Routine arguments show both 0_k and P_k to be open subsets of C. Moreover, each is dense: that 0_k is dense follows when $n \geqq 5$ because each h can be approximated by the projection of an embedding into M and when $n = 4$, thinking of the worst possible situation where N_G is dense in M, because each h can be approximated by the projection of an immersion in M whose self-intersections lie in N_G; that P_k is dense follows from Corollary 2.3 and Lemma 2.6, combined with an ε-controlled version of the Borsuk Homotopy Extension Theorem, giving first an embedding of $\bigcup_{i=1}^{k} L_i$ so close to $h \mid \bigcup_{i=1}^{k} L_i$ that later one discovers it extends to a map $h'\colon B^2 \to M/G$ close to h. By the Baire Category Theorem, each $f \in C$ can be approximated by $F \in (\cap 0_k) \cap (\cap P_k)$. Then $F\colon B^2 \to M/G$ will be a map for which each point inverse $F^{-1}(x)$ is 0-dimensional (being the union of at most one point from $\cup L_i$ together with a subset of $B^2 - \cup L_i$) and for which the image $S_2(F)$ of its singular set (i.e., $S_2(F) = \{x \in F(B^2) \mid F^{-1}(x) \neq \text{point}\}$) is contained in $\cap W_i$ and, therefore, has dimension at most $n-3$. According to a result of Freudenthal [21] (cf. [27, Theorem 24-2]),

$$\dim F(B^2) \leqq \max \{\dim B^2,\ \dim S_2(F) + 1\} \leqq n - 2 .$$

In case G is a closed-$(n-2)$-dimensional cell-like decomposition, it is comparatively easy to obtain an approximation F to a given map $f\colon B^2 \to M/G$ such that $\dim F(B^2) \leqq n-2$. Then the first part of the preceding argument gives an alternate proof for a result of Cannon [12, Theorem 10.1].

THEOREM 3.4. (Cannon-Edwards.) *If G is a closed-$(n-2)$-dimensional cell-like decomposition of an n-manifold $M (n \geqq 4)$, then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homeomorphic to $M \times E^1$.*

With the techniques developed in § 2, we can set forth a concise explication of a fact known to many practitioners of this subject. Similar results are proved by Cernavskii-Seebeck-Ferry [20, Theorem 5] or by Quinn [28] when M/G is known to be a manifold or, at least, when $(M/G) \times E^1$ is. Cannon [12, § 9] also states a result which nearly implies this one, and the argument supplied here should suggest how to fill in details for the DDP analogue of the theorem he gives.

THEOREM 3.5. *Suppose M is an n-manifold $(n \geqq 4)$, G is a cell-like decomposition of $M \times \{0\} \subset M \times E^1$ such that $(M \times \{0\})/G$ is an* ANR, *and G^* is the trivial extension of G over $M \times E^1$. Then $(M \times E^1)/G^*$ has the* DDP.

Proof. Let π denote the decomposition map $M \times E^1 \to (M \times E^1)/G^*$. Simplifying the notation, we let $Y = \pi(M \times E^1)$, $Y_+ = \pi(M \times [0, \infty))$, $Y_- = \pi(M \times (-\infty, 0])$ and $X = \pi(M \times \{0\})$. By [7, Chapter 5] $\pi(M \times [0, 1])$, which is the mapping cylinder of the obvious map

$M \to X$, is an ANR. Consequently, Y_+ and Y_-, which are homeomorphic to open subsets of $\pi(M \times [0, 1])$, are ANR's as well. Similarly, Y is also an ANR.

As a result, to prove this theorem it suffices to prove that Y has the DDP.

Consider maps f_1 and f_2 of B^2 to Y. Since X is nowhere dense in Y, one can modify f_i ($i = 1, 2$) slightly so that $f_i^{-1}(X)$ is nowhere dense in B^2. Define $Z_i = f_i^{-1}(X)$, $T_i = f_i^{-1}(Y_+)$ and $L_i = f_i^{-1}(Y_-)$.

According to Corollary 2.3 and Lemma 2.7, the map $f_1 \cup f_2$ on the disjoint union of Z_1 and Z_2 can be approximated by an embedding $F_1 \cup F_2: Z_1 \cup Z_2 \to X$. This can be done so that $f_1 \cup f_2$ and $F_1 \cup F_2$ are homotopic via a short homotopy. It follows from the controlled version of the Borsuk Homotopy Extension Theorem suggested earlier that $F_1 \cup F_2$ can be determined so close to $f_1 \cup f_2 | Z_1 \cup Z_2$ that for $i = 1, 2$ $F_i | Z_i$ extends to maps $F_i: T_i \to Y_+$ and $F_i: L_i \to Y^-$ approximating the restriction of F_i to the appropriate domains. From the easily-verified observation that X is 0-LCC and 1-LCC in both Y_+ and Y_-, it follows that the extensions F_i above can be further adjusted, first, so as to satisfy $Z_i = F_i^{-1}(X)$, by requiring $F_i(T_i - Z_i) \subset Y_+ - X$ and $F_i(L_i - Z_i) \subset Y_- - X$, and second, because each of the latter ranges is a manifold of dimension $\geqq 5$, so as to satisfy $F_1(B^2 - Z_1) \cap F_2(B^2 - Z_2) = \emptyset$. Now, because the initial maps were obtained so that $F_1(Z_1) \cap F_2(Z_2) = \emptyset$, the images $F_1(B^2)$ and $F_2(B^2)$ must be disjoint.

The same argument gives a Disjoint Disks Property worth stating about Q-manifold decompositions.

PROPOSITION 3.6. *Suppose M is a Q-manifold, G is a cell-like decomposition of $M \times \{0\}$ such that $M \times \{0\}/G$ is an* ANR *having the* DAP, *and G^* is the trivial extension of G over $M \times [-1, 1]$. Then $(M \times [-1, 1])/G^*$ has the* DDP.

4. **Nicely spherical decompositions.** Let X denote a decomposition space associated with an upper semicontinuous decomposition of an n-manifold. In accordance with one aspect of the tradition, we say that X is *spherical* (some authors prefer *spheroidal*, but to be precise, probably one should call it *locally peripherally* whichever) if each $x \in X$ has arbitrarily small neighborhoods U_x whose frontiers $Fr U_x$ are $(n-1)$-spheres; furthermore, we say that X is *nicely spherical* if, in addition, U_x is LC^1 at each point of $Fr U_x$ (equivalently, $Fr U_x$ is 1-LCC in $Cl U_x$). What is accomplished in this section provides technical improvement to a result of J. W. Cannon [9, Theorem 62]; part of that improvement, one should note, depends

upon a change in the term "nicely spherical" from the definition given in [9], where each component of $X - FrU_x$ was required to be LC^1 at FrU_x. Because of that change in terminology, there is no change in the statement of the main result: if G is a cell-like decomposition of an n-manifold $M(n \geqq 5)$ such that M/G is nicely spherical, then M/G is an n-manifold.

We begin by recording the obvious.

LEMMA 4.1. *If G is a decomposition of an n-manifold M such that M/G is spherical, then M/G has dimension $\leqq n$.*

Consequently, in order to prove that M/G is a manifold when G is a cell-like decomposition, it suffices to verify that M/G satisfies the DDP.

LEMMA 4.2. *Let G be a decomposition of an n-manifold $M(n \geqq 5)$ such that M/G is nicely spherical, f_1 and f_2 maps of B^2 to M/G, C a closed subset of M/G such that $C_0 = f_1(B^2) \cap f_2(B^2) \cap C$ is 0-dimensional, and W an open subset of M/G containing C_0. Then there exist maps $F_1, F_2: B^2 \to M/G$ satisfying*

(1) $F_1(B^2) \cap F_2(B^2) \cap C = \varnothing$,

(2) $F_i f_i^{-1}(W) \subset W$, *and*

(3) $F_i | B^2 - f_i^{-1}(W) = f_i | B^2 - f_i^{-1}(W)$ $(i = 1, 2)$.

Proof. Step 1: *Finiteness Considerations.* Each point $c \in C_0$ has an open neighborhood U_c in W whose frontier FrU_c is an $(n-1)$-sphere 0-LCC and 1-LCC embedded in ClU_c. From the open cover $\{U_c | c \in C_0\}$ extract a finite subcover $\{U_j | j = 1, \cdots, r\}$ and trim the latter to another cover $\{V_j | j = 1, \cdots, r\}$ of C_0 by open sets in M/G, with $V_j \subset Cl V_j \subset U_j$.

Step 2: *A Reduction.* Given maps $f_1, f_2: B^2 \to M/G$ for which $f_1(B^2) \cap f_2(B^2) \cap C \subset \cup\{V_j | j = k, k+1, \cdots, r\}$, we shall describe maps $F_1, F_2: B^2 \to M/G$ such that

(1′) $F_1(B^2) \cap F_2(B^2) \cap C \subset \cup\{V_j | j = k+1, \cdots, r\}$,

(2′) $F_i f_i^{-1}(U_k) \subset U_k$, and

(3′) $F_i | B^2 - f_i^{-1}(U_k) = f_i | B^2 - f_i^{-1}(U_k)$ $(i = 1, 2)$.

Repeated application (r times) of this reduced version will establish Lemma 4.2.

Step 3: *Eradication of $f_i(B^2)$ from U_k.* Define $Z_i = f_i^{-1}(ClU_k)$ and $Y_i = f_i^{-1}(FrU_k)$. The map $f_i | Y_i$ of Y_i into the simply connected ANR FrU_k extends to a map $m_i: Z_i \to FrU_k$ $(i = 1, 2)$.

Step 4: *General Position Improvements.* To circumvent atypical complications when $n = 5$, we specialize to the case $n \geqq 6$, leaving the extra case to the reader for later meditation. Because $\dim Fr U_k \geqq 5$, m_1 and m_2 admit general position modifications, affecting no points of Y_1 or Y_2, so that $m_1(Z_1 - Y_1) \cap m_2(Z_2 - Y_2) = \emptyset$. Define maps $f_i'\colon B^2 \to M/G$ as m_i on Z_i and as f_i elsewhere $(i = 1, 2)$. Then any $c \in f_1'(B^2) \cap f_2'(B^2) \cap C$ belongs either to

$$[C \cap f_1(B^2) \cap f_2(B^2)] - V_k \subset C_0 - \bigcup_{j=1}^{k} V_j$$

or to

$$C \cap ([f_1(Y_1) \cap m_2(Z_2 - Y_2)] \cup [m_1(Z_1 - Y_1) \cap f_2(Y_2)]) .$$

Only the points of the second kind cause any further concern.

Step 5: *Final Improvements to m_i.* Define sets

$$X_1 = Z_1 \cap m_1^{-1}(C \cap m_1(Z_1) \cap f_2(Y_2))$$

and

$$X_1^* = X_1 - m_1^{-1}\Big(\bigcup_{j=k+1}^{r} V_j\Big) ,$$

and define sets X_2, X_2^* symmetrically. Then X_i^* is a compact subset of X_i and $X_i^* \subset Z_i - Y_i$ $(i = 1, 2)$, for if $x \in X_1^* \cap Y_1$, then

$$\begin{aligned} m_1(x) &\in [C \cap m_1(Y_1) \cap f_2(Y_2)] - \bigcup_{j=k}^{r} V_j \\ &\subset [C \cap f_1(Y_1) \cap f_2(Y_2)] - \bigcup_{j=k}^{r} V_j , \end{aligned}$$

which is void by hypothesis. As a result, in B^2 there exist neighborhoods T_i of X_i^* having closures in $Z_i - Y_i$ $(i = 1, 2)$. Then $m_1(Cl T_1) \cap m_2(Cl T_2) = \emptyset$. Since U_k is LC^1 at points of $m(T_i)$, these maps $m_i | T_i$ can be approximated by maps m_i' of T_i to U_k such that $m_1'(T_1)$ and $m_2'(T_2)$ remain disjoint, with the amount of adjustment damped to zero at $Fr T_i$ in B^2 so that the extension m_i' of this map elsewhere over Z_i via m_i is continuous $(i = 1, 2)$. It follows that

$$\begin{aligned} C \cap m_1'(Z_1) \cap m_2'(Z_2) &= C \cap m_1'(Z_1 - T_1) \cap m_2'(Z_2 - T_2) \\ &\subset C \cap m_1(X_1 - X_1^*) \cap m_2(X_2 - X_2^*) \\ &\subset C \cap \Big(\bigcup_{j=k+1}^{r} V_j\Big) . \end{aligned}$$

Consequently, the maps F_i defined as m_i' on Z_i and as f_i elsewhere fulfill the required conclusions of the reduction given in Step 2.

LEMMA 4.3. *Suppose G is a decomposition of an n-manifold*

$M(n \geqq 5)$ such that M/G is nicely spherical, C is a closed q-dimensional subset of M/G, f_1 and f_2 are maps of B^2 to M/G, W is an open subset of M/G containing $C \cap f_1(B_2) \cap f_2(B^2)$, and $\varepsilon > 0$. Then there exist maps $F_1, F_2: B^2 \to M/G$ satisfying

(1) $C \cap F_1(B^2) \cap F_2(B^2) = \varnothing$,

(2) $\rho(F_i, f_i) < \varepsilon$, *and*

(3) $F_i | B^2 - f_i^{-1}(W) = f_i | B^2 - f_i^{-1}(W)$ $(i = 1, 2)$.

Proof. The argument proceeds by induction on q, with Lemma 4.2 essentially resolving the initial case $q = 0$.

Assume this Lemma holds for all $(q-1)$-dimensional closed subsets of M/G. For the q-dimensional set C, determine a countable collection $\{A_j\}$ of $(q-1)$-dimensional closed subsets such that $C - \cup A_j$ is 0-dimensional. Apply the inductive hypothesis repeatedly, exercising controls pertaining to the limit, to obtain maps $f_i^*: B^2 \to M/G$ $(i = 1, 2)$ such that $f_i^*(B^2) \cap (\cup A_j) = \varnothing$, $\rho(f_i^*, f_i) < \varepsilon/2$, and $f_i^* | B^2 - f_i^{-1}(W) = f_i | B^2 - f_i^{-1}(W)$. Then $C \cap f_1^*(B^2) \cap f_2^*(B^2)$, being a subset of $C - \cup A_j$, is 0-dimensional. Now apply Lemma 4.2, but with open set W^* in W containing $C \cap f_1^*(B^2) \cap f_2^*(B^2)$ and having no component of diameter larger than $\varepsilon/2$, to obtain F_1 and F_2.

This readily yields Cannon's result [9, Theorem 62].

THEOREM 4.4. *If G is a cell-like decomposition of an n-manifold M $(n \geqq 5)$ such that M/G is nicely spherical, then G is shrinkable and M/G is homeomorphic to M.*

Proof. Thinking of M/G as C, one can interpret Lemma 4.3 as asserting M/G to have the DDP. Theorem 4.4 follows from [17].

Scrutiny of the preceding arguments will reveal the validity of the improvements to Theorem 4.4 stated below.

THEOREM 4.5. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 5)$ such that each point $x \in M/G$ has arbitrarily small neighborhoods U_x whose frontiers $Fr\,U_x$ are simply connected* ANR's *having the* DDP *and for which U_x is LC^1 at each point of $Fr\,U_x$. Then G is shrinkable and M/G is homeomorphic to M.*

THEOREM 4.6. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 5)$ such that each point $x \in \pi(N_G)$ has arbitrarily small neighborhoods U_x as in Theorem 4.5. Then G is shrinkable and M/G is homeomorphic to M.*

REMARK. In proving Theorem 4.6, one should consider only maps $f_1, f_2: B^2 \to M/G$ arising as πF_1 and πF_2, where F_1 and F_2 are disjoint embeddings of B^2 of M. Then each point of $f_1(B^2) \cap f_2(B^2)$ will belong to $\pi(N_G)$ and will possess the appropriate local peripheral structure.

With additional effort comparable to that required earlier for the case $n = 5$, one also can establish the following result.

THEOREM 4.7. *Suppose G is a cell-like decomposition of an n-manifold such that each point $x \in \pi(N_G)$ has arbitrarily small neighborhoods U_x such that $Fr U_x$ is a simply connected $(n-1)$-manifold and U_x is LC^1 at points of $Fr U_x$. Then G is shrinkable and M/G is homeomorphic to M.*

5. Decomposition spaces locally encompassed by manifolds. Essential to the development of this section is a result due to Cannon, Bryant and Lacher [13, Theorem 7.2], stated below in a form slightly modified from theirs.

THEOREM 5.1 [13]. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 4)$, N is an $(n-1)$-manifold contained in M/G as a closed subset, and $G(N)$ is the decomposition induced over N. Then $G(N) \times E^1$ is a shrinkable decomposition of $M \times E^1$.*

Recall that $G(N)$ is defined to be the decomposition of M consisting of the sets $\pi^{-1}(x)$, $x \in N$, and the singletons of $M - \pi^{-1}(N)$.

COROLLARY 5.2. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 4)$ and P is an $(n-1)$-complex embedded in M/G as a closed subset that contains $\pi(N_G)$. Then $G \times E^1$ is a shrinkable decomposition of $M \times E^1$.*

Proof. Underlying the $(n-2)$-skeleton of P is a (closed) subset A such that $P - A$ is an $(n-1)$-manifold. Let $G(A)$ denote the decomposition induced over A by π. According to Theorem 3.4, $G(A) \times E^1$ is shrinkable. Hence, there exists a map θ of $M \times E^1$ to itself that is a limit of homeomorphisms and for which $\{\theta^{-1}(x) | x \in M \times E^1\} = G(A) \times E^1$. Name the induced map $\psi = \pi\theta^{-1}: M \times E^1 \to (M/G) \times E^1$ and note that ψ is $1-1$ over $A \times E^1$.

Since the set $N = P - A$ is an $(n-1)$-manifold embedded in $(M/G) - A$ as a closed subset, it follows from Theorem 5.1 that the decomposition of $(M \times E^1) - \psi^{-1}(A \times E^1)$ induced over $N \times E^1$ by ψ, which is equivalent to the product with E^1 of the decomposition of $M - \pi^{-1}(A)$ induced over N by π, is also shrinkable. In particular, there exists a homeomorphism π' of $M \times E^1 - \psi^{-1}(A \times E^1)$ onto $((M/G) - A) \times E^1$ approximating ψ so closely that the extension π^* of π' to the rest of $M \times E^1$ via ψ is continuous. By insisting at the outset that ψ be an approximation to π, one can conclude at the end that π^* is an approximation to π. Such a map π^*, which is a homeomorphism, shows that $G \times E^1$ is shrinkable.

COROLLARY 5.3. *Suppose G is a cell-like decomposition of an n-manifold M ($n \geqq 4$), P is an $(n-1)$-complex embedded in M/G as a closed subset, and $G(P)$ is the decomposition induced over P. Then $G(P) \times E^1$ is a shrinkable decomposition of $M \times E^1$.*

Let G be a decomposition of an n-manifold M. Then M/G is said to be *locally encompassed by manifolds* if each point $x \in M/G$ has arbitrarily small neighborhoods U_x whose frontiers are $(n-1)$-manifolds. For example, if M/G is spherical (nicely or not), it is certainly encompassed by manifolds.

The following definition serves as a basic fixture in the primary technical results to be used. Given an open subset U of a space X, a subset A of $Fr\,U$, and $F \subset A$, one says that U *is* 1-LC *in* $U \cup F$ *at* A if, for each $a \in A$ and each neighborhood V of a, there exists a neighborhood V' of a such that every loop in $V' \cap U$ is contractible in $V \cap (U \cup F)$.

The next lemma, which is elementary, is stated without proof. It follows from the hypothesis that $Cl\,U$ is LC^1. Hence, one can model an argument on Proposition 1 in [15].

LEMMA 5.4. *Suppose U is an open subset of an LC^1 metric X whose frontier Σ is a manifold (more generally, an* ANR) *and suppose D is a dense subset of E^1. Then $U \times E^1$ is* 1-LC *in $(U \times E^1) \cup (\Sigma \times D)$ at $\Sigma \times E^1$.*

LEMMA 5.5. *Suppose X is an LC^1 metric space, U_1 is an open subset of X with frontier Σ, $U_2 = X - Cl\,U_1$, and U_e is* 0-LC *at Σ ($e = 1, 2$). Suppose P^* is an open subset of B^2 that can be expresed as the union of (relatively) closed sets A_1 and A_2 and $f\colon B^2 \to X \times E^1$ is a map such that $f(A_e) \subset Cl\,U_e \times E^1$. Finally, suppose D_e is a subset of $\Sigma \times E^1$ ($e = 1, 2$) such that $U_e \times E^1$ is* 1-LC *in $(U_e \times E^1) \cup D_e$ at $\Sigma \times E^1$.*

Then for each $\varepsilon > 0$ there exists a map $f_\varepsilon\colon B^2 \to X \times E^1$ satisfying

(i) $\rho(f_\varepsilon, f) < \varepsilon$,

(ii) $f_\varepsilon|(B^2 - P^*) \cup (A_1 \cap A_2) = f|(B^2 - P^*) \cup (A_1 \cap A_2)$,

(iii) $f_e(A_e) \subset Cl\,U_e \times E^1$, *and*

(iv) $f_e(A_e - (A_1 \cap A_2)) \cap (\Sigma \times E^1) \subset D_e$ $(e = 1, 2)$.

Proof. Carefully extend the map $f|(B^2 - P^*) \cup (A_1 \cap A_2)$ over successive skeleta of a fine triangulation $T_e (e = 1, 2)$ of $A_e - (A_1 \cap A_2)$ whose mesh approaches 0 near $(B^2 - P^*) \cup (A_1 \cap A_2)$.

THEOREM 5.6. *If G is a cell-like decomposition of an n-manifold $(n \geqq 4)$ such that M/G is locally encompassed by manifolds, then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homeomorphic to $M \times E^1$.*

Proof. That M/G is locally encompassed by manifolds certainly implies that $\dim (M/G) \leqq n$. Hence, to establish this theorem, it suffices to verify that $(M/G) \times E^1$ satisfies the DDP. Towards that end, consider maps $f_1, f_2 \colon B^2 \to (M/G) \times E^1$ and a rational positive number ε.

Step 1: *An alteration of the decomposition map.* Because M/G is locally encompassed by manifolds, one can find a collection $\{N_i | i = 1, 2, \cdots\}$ of compact $(n-1)$-manifolds in M/G such that $\dim ((M/G) - \cup N_i) \leqq 0$. By Theorem 5.1, the decompositions $G(N_i) \times E^1$ of $M \times E^1$ are shrinkable $(i = 1, 2, \cdots)$; by Theorem 3.5 the decompositions $(G \times \{t\})^*$ induced over $(M/G) \times \{t\}$ are shrinkable for all $t \in E^1$. It follows from a rather fundamental principle exploited repeatedly by Edwards [17] (to the effect that if the decompositions induced by a proper cell-like map $\pi \colon M \times E^1 \to X$ (M an n-manifold, $n \geqq 4$) over each member of a countable collection $\{A_i\}$ of closed sets from X is shrinkable, then π can be approximated by a cell-like map that is $1-1$ over $\cup A_i$) that we can obtain a cell-like map $p \colon M \times E^1 \to (M/G) \times E^1$ that is $1-1$ over the sets $N_i \times E^1$ and $(M/G) \times \{q\}$, for every rational number q. Now $p(N_p)$ is contained in the product of the irrationals with $((M/G) - \cup N_i)$. Consequently, there exist 0-dimensional F_σ sets X and X' in M/G and E^1, respectively, such that $p(N_p) \subset X \times X'$.

REMARK. The fundamental principle mentioned above was certainly noticed by Cannon and his students. It appears, for example, in the thesis of D. Everett [18, Theorem 3]; in the published version of his thesis, Everett relegates this principle to the middle of a proof [19, pp. 365–366].

Step 2: *Modifications of the maps f_e.* Determine a triangulation T of B^2 such that $\operatorname{diam} f_e(\sigma) < \varepsilon$ for all $\sigma \in T$ and $e = 1, 2$. Since $(M/G) \times E^1$ satisfies the DADP, we may assume that

$$f_1(B^2) \cap f_2(T^{(1)}) = \varnothing = f_1(T^{(1)}) \cap f_2(B^2) ;$$

approximating each of these maps by the image under p of disjoint embeddings in $M \times E^1$, we may assume that $f_1(B^2) \cap f_2(B^2) \subset X \times X'$; noting that $X \times E^1$ is 1-dimensional and σ-compact, implying that it is 0-LCC in $(M/G) \times E^1$, we can assume further that $f_e(T^{(1)}) \cap (X \times E^1) = \varnothing$ $(e = 1, 2)$, without affecting the set $S = f_1(B^2) \cap f_2(B^2)$; finally, since $X \times \{t\}$ is 1-LCC in $(M/G) \times E^1$ for each $t \in E^1$ (this follows from a localized Van-Kampen argument because $(M/G \times [t, \infty)) - (X \times \{t\})$ is locally contractible and because any small loop in $(M/G \times E^1) - (X \times \{t\})$ can be expressed as a composition of (small) loops from $(M/G \times [t, \infty)) - (X \times \{t\})$ and $(M/G \times (-\infty, t]) - (X \times \{t\})$, we may assume, in addition, that for $e = 1, 2$ and k an integer

$$f_e(B^2) \cap (X \times \{k\varepsilon/2\}) = \varnothing .$$

Step 3: *Finiteness Considerations.* Choose a neighborhood W^* of X in M/G such that each component of W^* has diameter less than $\varepsilon/2$, and define W as $W^* \times (E^1 - \bigcup_{k \in Z} \{k\varepsilon/2\})$. For each point $s \in S = f_1(B^2) \cap f_2(B^2)$ there exists a neighborhood V_s of s in W such that $V_s = U_s \times J_s$, where $J_s = (k(s) \cdot \varepsilon/2, (k(s) + 1)\varepsilon/2)$ for some integer $k(s)$ and where U_s is a connected open subset with closure in W^*, having manifold frontier, and satisfying

$$(\mathrm{A}_0) \quad (U_s \times E^1) \cap (f_1(T^{(1)}) \cup f_2(T^{(1)})) = \varnothing ,$$
$$(\mathrm{B}_0) \quad (F_r U_s \times E^1) \cap f_1(B^2) \cap f_2(B^2) = \varnothing ,$$
$$(\mathrm{C}_0) \quad (U_s \times FrJ_s) \cap (f_1(B^2) \cup f_2(B^2)) = \varnothing .$$

Cover S by a finite number $\{V_1, V_2, \cdots, V_r\}$ of such sets $\{V_s\}$.

Step 4: *A Reduction.* For $j = 1, \cdots, r$ we plan to describe maps $f_{1,j}$ and $f_{2,j}$ of B^2 to $(M/G) \times E^1$ that agree with f_1 and f_2 over $((M/G) \times E^1) - W$ and that satisfy the analogues of (A_0), (B_0), (C_0) above, as well as

$$f_{1,j}(B^2) \cap f_{2,j}(B^2) \subset S - \left(\bigcup_{i=1}^{j} V_i \right) .$$

Of course, in the final case $j = r$ we will have the required disjoint approximations. As in the inductive proof of Lemma 4.2, this can be accomplished by proving the following: if h_1 and h_2 are maps of B^2 into $(M/G) \times E^1$ such that

$$(B_{i-1}) \quad h_1(B^2) \cap h_2(B^2) \cap \left[\bigcup_{m=1}^{r} (FrU_m) \times E^1 \right] = \varnothing$$

$$(C_{i-1}) \quad [h_1(B^2) \cup h_2(B^2)] \cap \left[\bigcup_{m=1}^{r} (U_m \times FrJ_m) \right] = \varnothing$$

where $V_m = U_m \times J_m$ $(m = 1, \cdots, r)$ and that

$$h_1(B^2) \cap h_2(B^2) \subset S - \bigcup_{j=1}^{i-1} V_j ,$$

then there exist maps $F_1, F_2\colon B^2 \to (M/G) \times E^1$ such that for $e=1, 2$

$$F_e^{-1}(W) = h_e^{-1}(W) ,$$
$$F_e | B^2 - F_e^{-1}(W) = h_e | B^2 - h_e^{-1}(W) ,$$
$$[F_1(B^2) \cup F_2(B^2)] \cap [\bigcup_{m=1}^{r} (U_m \times Fr J_m)] = \varnothing ,$$
$$F_1(B^2) \cap F_2(B^2) \subset (h_1(B^2) \cap h_2(B^2)) - V_i \subset S - \bigcup_{j=1}^{i} V_j .$$

Step 5: *Elimination of Intersections in* V_i. Choose a point $t_i \in J_i$ such that

$$[h_1(B^2) \cup h_2(B^2)] \cap (U_i \times \{t_i\}) = \varnothing .$$

For $e = 1, 2$, define $Z_e = h_e^{-1}(Cl V_i)$ and

$$Y_e = h_e^{-1}(Fr V_i) = h_e^{-1}((Fr U_i) \times J_i) .$$

According to [7, Chapter 4], $((Fr U_i) \times J_i) \cup (U_i \times \{t_i\})$ is an ANR. By Borsuk's Extension Theorem, the map $h_1 | Y_1$ of Y_1 into $(Fr U_i) \times J_i$ extends to

$$m_1\colon Z_1 \longrightarrow ((Fr U_i) \times J_i) \cup (U_i \times \{t_i\}) ,$$

since $h_1 | Y_1$ is homotopic to one that extends over Z_1 (h_1 followed by the "projection" of $Cl U_i \times J_i$ to $Cl U_i \times \{t_i\}$). Choose $s_i \in J_i$ such that

$$[m_1(Z_1) \cap h_2(B^2)] \cap (U_i \times \{s_i\}) = \varnothing .$$

As before, $h_2 | Y_2$ extends to

$$m_2\colon Z_2 \longrightarrow ((Fr U_i) \times J_i) \cup (U_i \times \{s_i\}) .$$

At this point, we have maps $m_e\colon Z_e \to (Cl U_i) \times J_i$ $(e = 1, 2)$ that extend over the rest of B^2 via h_e and the images of which intersect at no point of V_i, although they may intersect at points of $(Fr U_i) \times J_i$.

Step 6: *General Position Improvements.* As was done earlier, to circumvent complications peculiar to the case $n = 4$, we specialize to the case $n \geqq 5$. For $e = 1, 2$ let $R_e = m_e^{-1}((Fr U_i) \times J_i)$. Because $\dim (Fr U_i \times E^1) \geqq 5$ and because the sets $U_i \times \{s_i\}$ and $U_i \times \{t_i\}$ are disjoint, the maps $m_e | R_e$ into $U_i \times J_i$ admit general position modifications, affecting no points of $Fr R_e$, hence extending over $Z_e - R_e$

via m_e, so that $m_1(Z_1 - Y_1) \cap m_2(Z_2 - Y_2) = \emptyset$. Define maps $h'_e: B^2 \to (M/G) \times E^1$ as m_e on Z_e and as h_e elsewhere. Then any $s \in h'_1(B^2) \cap h'_2(B^2)$ belongs either to

$$[S \cap h_1(B^2) \cap h_2(B^2)] - V_i \subset S - \bigcup_{j=1}^{i} V_j$$

or to

$$[h_1(Y_1) \cap m_2(Z_2 - Y_2)] \cup [m_1(Z_1 - Y_1) \cap h_2(Y_2)] .$$

By expelling points of the second kind, without adding any others, we will achieve our goal.

Step 7: *Final Modifications*. It follows from Condition (B_{i-1}) that $m_1(Y_1) \cap m_2(Y_2) = \emptyset$. Thus, for $e = 1, 2$ there exist disjoint open subsets P_e and Q_e in B^2 satisfying

$$\begin{aligned} &Y_e \subset P_e \subset (h'_e)^{-1}(W \cap [((M/G) - U_i) \times J_i]) , \\ &(m_e | Z_e - Y_e)^{-1}(m_1(Z_1) \cap m_2(Z_2)) \subset Q_e \subset Z_e - Y_e , \\ &h'_1(P_1) \cap h'_2(P_2) = \emptyset = h'_1(Q_1) \cap h'_2(Q_2) . \end{aligned}$$

Note that any point $b \in B^2$ for which

$$h'_e(b) \in [h_1(Y_1) \cap m_2(Z_2 - Y_2)] \cup [m_1(Z_1 - Y_1) \cap h_2(Y_2)]$$

belongs to $P_e \cup Q_e$.

Since $Fr U_i$ is an $(n-1)$-manifold separating M/G, properties of generalized manifolds guarantee that $(M/G) - Cl U_i$ is 0-LC at $Fr U_i$ [30] [1]. After choosing disjoint dense subsets D', D'' of $E^1 - \{s_i, t_i\}$, we can apply Lemma 5.5 to obtain maps $F_e: B^2 \to (M/G) \times E^1$ such that

$$\begin{aligned} &F_e | B^2 - (P_e \cup Q_e) = h'_e | B^2 - (P_e \cup Q_e) , \\ &F_e(P_e) \subset (W \cap [((M/G) - Cl U_i) \times J_i]) \cup [(Fr U_i) \times (D' \cap J_i)] , \\ &F_e(Q_e) \subset (U_i \times J_i) \cup [(Fr U_i) \times (D'' \cap J_i)] , \end{aligned}$$

all of which can be arranged with minor care in the approximation so that, in addition,

$$\begin{aligned} &F_1(P_1) \cap F_2(P_2) = \emptyset = F_1(Q_1) \cap F_2(Q_2) . \\ &F_1(P_1 \cup Q_1) \cap m_2(B^2 - (P_2 \cup Q_2)) = \emptyset = m_1(B^2 - (P_1 \cup Q_1)) \cap F_2(P_2 \cup Q_2). \end{aligned}$$

The disjointness of D', D'' then implies that

$$F_1(P_1 \cup Z_1) \cap F_2(P_2 \cup Z_2) = \emptyset .$$

Consequently, the maps F_1 and F_2 satisfy

$$F_1(B^2) \cap F_2(B^2) \subset h'_1(B^2) \cap h'_2(B^2)$$

and eradicate the intersections of the second kind identified in Step 6. Routine auditing of the entries will verify that these maps fulfill the requirements of the reduction outlined in Step 4.

This argument establishes the following improved version of Theorem 5.6.

THEOREM 5.7. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 4)$ and $\{A_i \mid i = 1, \cdots\}$ is a collection of closed subsets of M/G such that each point of $(M/G) - \cup A_i$ is locally encompassed by manifolds and for each decomposition $G(A_i)$ induced over A_i, $G(A_i) \times E^1$ is a shrinkable decomposition of $M \times E^1$. Then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homomorphic to $M \times E^1$.*

COROLLARY 5.8. *Suppose G is a cell-like decomposition of an n-manifold M $(n \geqq 4)$ and $\{P_i \mid i = 1, 2, \cdots\}$ is a collection of closed subsets of M/G such that each P_i is either an $(n-1)$-complex or an $(n-1)$-manifold and each point of $(M/G) - \cup P_i$ is locally encompassed by manifolds. Then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homeomorphic to $M \times E^1$.*

The cell-like, totally noncellular decompositions described by Cannon-Daverman in [14] satisfy the hypothesis of Corollary 5.8. As a result, this section gives another proof (at least for $n > 3$), quite unlike the one given in [14], that the product of the decomposition space and E^1 is a manifold.

COROLLARY 5.9. *If G is a cell-like decomposition of an n-manifold M $(n \geqq 4)$ that has a defining sequence in the sense defined in [14, § 2, 3], then $G \times E^1$ is shrinkable and $(M/G) \times E^1$ is homomorphic to $M \times E^1$.*

Appendix. Proof of Lemma 2.1. Clearly Lemma 2.1 is valid when $\dim A = -1$. Assume it to be true for all closed subsets of dimension $< k$. Given a k-dimensional closed subset A, consider $z \in H_j(X, X - A)$, where $0 \leqq j \leqq r - k$. We shall show that $z = 0$.

The key to the proof is the observation that, when A' and A'' are closed subsets of X for which $\dim(A' \cap A'') < k$, the Mayer-Vietoris sequence for the "excisive couple of pairs" $\{(X, X - A'), (X, X - A'')\}$ (see [29, p. 189]) yields an inclusion-induced isomorphism α

$$H_{j+1}(X, X - (A' \cap A'')) \longrightarrow H_j(X, X - (A' \cup A''))$$
$$\xrightarrow{\alpha} H_j(X, X - A') \oplus H_j(X, X - A'')$$

because of the inductive assumption that $H_e(X, X - (A' \cap A'')) = 0$ $(e = j, j + 1)$.

Fix a compact pair $(C, C') \subset (X, X - A)$ carrying a representative of z. Since $H_j(X, X - \{x\}) \cong 0$ for every $x \in X$, each $a \in A$ has a neighborhood N_a in A for which the image of z in $H_j(X, X - N_a)$ is trivial. Elementary dimension theory properties give a cover $\{C_i | i = 1, \cdots, m\}$ of $C \cap A$ by closed sets such that $\{C_i\}$ refines the cover $\{N_a | a \in C \cap A\}$, the interior (rel A) of $\cup C_i$ contains $C \cap A$, and the frontier of each C_i has dimension $\leqq k - 1$. Define A_i as $Cl(A - \bigcup_{i+1}^{m} C_j)$ $(i = 0, \cdots, m)$. Since A_0 does not intersect C, the image of z in $H_j(X, X - A_0)$ is trivial. Inductively, for $A' = A_{i-1}$ and $A'' = C_i$, we presume that the image of z in $H_j(X, X - A')$ is trivial, and we know it is trivial in $H_j(X, X - A'')$; by construction $\dim (A' \cap A'') \leqq \dim FrC_i \leqq k - 1$, and the Mayer-Vietoris argument above reveals that the image of z in $H_j(X, X - (A' \cup A'')) = H_j(X, X - A_i)$ is trivial. Of course, when $i = m$, this proves that z itself is trivial.

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