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Determinant Bundles for Abelian Schemes

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Abstract. To a symmetric, relatively ample line bundle on an Abelian scheme one can associate a linear combination of the determinant bundle and the relative canonical bundle, which is a torsion element in the Picard group of the base. We improve the bound on the order of this element found by Faltings and Chai. In particular, we obtain an optimal bound when the degree of the line bundle d is odd and the set of residue characteristics of the base does not intersect the set of primes p dividing d, such that $p \equiv -1 \bmod(4)$ and $p \leqslant 2g-1$, where g is the relative dimension of the Abelian scheme. Also, we show that in some cases these torsion elements generate the entire torsion subgroup in the Picard group of the corresponding moduli stack.

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Let L be a relatively ample line bundle on an Abelian scheme $\pi\colon A \to S$, trivialized along the zero section. Assume that L is symmetric, i.e. $[-1]_A^*L \simeq L$. We denote by $\phi_L\colon A \to \hat{A}$ the corresponding self-dual homomorphism (where $\hat{A}$ is the dual Abelian scheme). Let $d = rk\pi_*L$, so that d^2 is the degree of ϕ_L. Then Faltings and Chai proved in [4], I, 5.1 the following equality in $\mathrm{Pic}(S)$:

$$8 \cdot d^3 \cdot \det(\pi_*L) = -4 \cdot d^4 \cdot \overline{\omega}_A$$

where we denote by $\overline{\omega}_A$ the restriction of the relative canonical bundle $\omega_{A/S}$ to the zero section. In other words, the element

$$\Delta(L) := 2 \cdot \det(\pi_*L) + d \cdot \overline{\omega}_A$$

of $\mathrm{Pic}(S)$ is annihilated by $4d^3$. It is known from the transformation theory of theta-functions (see [10]) that this result is sharp for principal polarizations ($d = 1$). In the case of analytic families of complex Abelian varieties A. Kouvidakis showed in [7] using theta functions that if the type of polarization is $(d_1, \dots, d_g)$ with $d_1 | \dots | d_g$ then $4 \cdot \Delta(L) = 0$ except when $3 | d_g$ and $d_{g-1} \not\equiv 0 \bmod(3)$. In the latter case he proved that $12 \cdot \Delta(L) = 0$. This suggests that one can try to eliminate the factor d^3 in general situation. We prove that one can do this outside certain set of prime divisors of d. In particular, we explain the appearance of factor 3 above algebraically (see Theorem 1.1). Here are the precise statements.

THEOREM 0.1 *Let L be a symmetric, relatively ample line bundle over an Abelian scheme A/S of relative dimension g, trivialized along the zero section. Then*

(1) $2^{n_2} \cdot d' \cdot \Delta(L) = 0$, where $d' = \prod p^{n_p}$ is the product of powers of primes p dividing d such that $p \equiv -1 \bmod(4)$ and $p \leqslant \max(2g-1, 3)$; $n_p = 1$ if $p \neq 3$ and $(2g+1/3) < p < 2g-1$, $n_p = 2$ if $p = 2g-1 \neq 3$, otherwise $n_p = v_p(d)$; $n_2 = 2 + 3v_2(d)$.

(2) *There exists an integer $N(g) > 0$ depending only on g, such that $N(g) \cdot \Delta(L) = 0$.*

We can get sharper bounds under some restrictions on the residue characteristics of S. For every Abelian group K and a prime number p we denote by $K^{(p)}$ the subgroup of elements of K annihilated by some power of p. Note that when p is not among the residue characteristics of S we can define the p-type of polarization as the type $(p^{n_1}, \ldots, p^{n_g})$ of the finite symplectic group $K(L)^{(p)}$, where $K(L) = \ker(\phi_L : A \to \hat{A})$. Here $n_1 \leqslant \cdots \leqslant n_g$ are locally constant functions on S.

THEOREM 0.2. *Let p be an odd prime number, $\Delta(L)^{(p)} \in \mathrm{Pic}(S)^{(p)}$ be the p-primary component of $\Delta(L)$. Let $S[1/p] \subset S$ be the open subscheme where p is invertible. Then $\Delta(L)^{(p)}|_{S[\frac{1}{p}]} = 0$ in $\mathrm{Pic}(S[\frac{1}{p}])$ unless $p = 3$ and the 3-type of the polarization over $S[\frac{1}{3}]$ is $(1, \ldots, 1, 3^k)$, $k > 0$. In the latter case one has $3 \cdot \Delta(L)^{(3)}|_{S[\frac{1}{3}]} = 0$ in $\mathrm{Pic}(S[\frac{1}{3}])$.*

Remarks. (1) In fact, the equalities of Theorems 0.1 and 0.2 can be realized by *canonical* (i.e. compatible with arbitrary base changes) isomorphisms of line bundles. More precisely, let $X_{g,d}$ be the moduli stack over $\mathrm{Spec}(\mathbb{Z})$ classifying data (A, L) as above. Since the construction of $\Delta(L)$ commutes with arbitrary base changes, we can consider $\Delta(L)$ as an element of the Picard group $\mathrm{Pic}(X_{g,d})$. Now we claim that the equality of Theorem 0.1 holds in $\mathrm{Pic}(X_{g,d})$ while the equality of Theorem 0.2 holds in $\mathrm{Pic}(X_{g,d}[(1/p)])$. This follows from the fact that there exists a PGL_N-torsor $\widetilde{X}_{g,d}$ over $X_{g,d}$ which is represented by a scheme. Indeed, $\widetilde{X}_{g,d}$ is obtained by adding a basis of $\pi_*(L^3)$ (considered up to constant) to the above data (A, L). Then the representing scheme can be constructed as in [17] using Hilbert schemes. Now it suffices to prove the triviality of the induced PGL_N-equivariant line bundle on $\widetilde{X}_{g,d}$. Applying Theorem 0.1 (resp. Theorem 0.2) to $\widetilde{X}_{g,d}$ we get some trivialization of this line bundle. However, as PGL_N has no non-trivial characters this trivialization is automatically compatible with PGL_N-action.

(2) These results can be extended to line bundles L which are not necessarily ample, but are *non-degenerate*, i.e. such that the corresponding homomorphism $\phi_L : A \to \hat{A}$ is an isogeny. In this case there is a locally constant function $i(L)$ on S such that

$R^i\pi_*(L) = 0$ for $i \neq i(L)$ and $R^{i(L)}\pi_*L$ is locally-free of rank d. One has by definition $\det \pi_*L := (-1)^{i(L)} \det(R^{i(L)}\pi_*L)$. Also we should replace d by $(-1)^{i(L)}d$ when defining $\Delta(L)$:

$$\Delta(L) := 2 \cdot \det \pi_*L + (-1)^{i(L)} \cdot d \cdot \overline{\omega}_A.$$

Then our argument goes through for such L. Note especially that in Section 2 we rely heavily on the fact that the function $n \mapsto \det \pi_*L^n$ has finite degree. However, this is true for any line bundle — see Lemma 2.1.

(3) When d is even one can consider an element $\Delta'(L) = \det \pi_*L + (d/2) \cdot \overline{\omega}_A$ in $\mathrm{Pic}(S)$ such that $\Delta(L) = 2 \cdot \Delta'(L)$. Kouvidakis proved in [7] that for a totally symmetric line bundle L one always has $3 \cdot \Delta'(L) = 0$ if $g \geqslant 3$ and the characteristic is zero (furthermore, $\Delta'(L) = 0$ if the 3-polarization type is not $(1, \ldots, 1, 3^k)$, $k > 0$). It would be nice to extend this result to the case of positive characteristics. So far, we only can control the behavior of these elements under isogeny of odd degree (see remark after Theorem 1.2). Also, in the principally polarized case (i.e. when $d = 1$) one still has $\Delta'(L^{2n}) = 0$ for $g \geqslant 3$ and any scheme S whose residue characteristics are prime to 2. This follows from the fact that the fibers of the relevant stacks over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ are smooth and irreducible (compare with the proof of Theorem 5.6).

The following corollary describes the cases when we get an optimal result.

COROLLARY 0.3. *Assume that d is odd and that every prime p dividing d, such that* $p \equiv -1 \bmod(4)$ *and* $p \leqslant \max(2g-1, 3)$, *is not among residue characteristics of S. Then one has* $4 \cdot \Delta(L) = 0$ *unless the* 3*-type of the polarization is* $(1, \ldots, 1, 3^k)$, $k > 0$. *In the latter case one has* $12 \cdot \Delta(L) = 0$.

Besides the idea of Faltings and Chai, the crucial step in the proof of these theorems is the relation between determinant bundles of L and α^*L where α is an isogeny of Abelian schemes, worked out in Section 1. Essentially this boils down to computation of norms of symmetric line bundles with cube structures over finite flat subgroups in Abelian schemes. Theorem 0.1 is proved in Sections 2 and 3, while Theorem 0.2 is proved in Section 4. In Section 5 we calculate elements $\Delta(L)$ in the case $g = 1$ and describe some linear relations between $\Delta(L^n)$ for different n in higher dimensions. In particular, we prove that for $d = 1$ and $g \geqslant 2$ one has the following relation $\Delta(L^n) = n^{g-1} \cdot \Delta(L)$ for any odd n. Also following Mumford's approach we determine the torsion subgroup in the Picard group of the moduli stack $\widetilde{\mathcal{A}}_g^+[\frac{1}{2}]$ of principally polarized Abelian varieties with even symmetric theta divisor (localized outside characterstic 2) for $g \geqslant 3$. Recall that the divisor (and the corresponding line bundle) is called even or odd depending on the parity of its multiplicity at zero. It turns out that the torsion subgroup in $\mathrm{Pic}(\widetilde{\mathcal{A}}_g^+[\frac{1}{2}])$ is cyclic of order 4, hence it is generated by our element $\Delta(L)$.

NOTATION. Throughout this paper S denotes the base scheme and for any scheme over S we denote by π its projection to S (hoping that this will not lead to confusion). For any morphism of schemes f we denote by f_*, f^* and $f^!$ the corresponding standard functors between derived categories of quasi-coherent sheaves (e.g. f_* denotes the right derived functor of the push-forward functor). For a vector bundle V of rank r we denote $\det V = \bigwedge^r(V)$. This definition can be naturally extended to complexes (see [2]). For all Abelian schemes over S we denote by e the zero section. For an Abelian scheme A (resp. morphism of Abelian schemes f) we denote by $\hat{A}$ (resp. $\hat{f}$) the dual Abelian scheme (resp. dual morphism). Also for every integer n we denote by $[n]_A : A \to A$ the multiplication by n on A, and by $A_n \subset A$ its kernel. For every line bundle L on A we denote by $\phi_L : A \to \hat{A}$ the corresponding morphism of Abelian schemes (see [12]). For a finite flat group scheme H/S we denote by $|H|$ its order which is a locally constant function on S, so for an integer k the condition $|H| > k$ means that the order of H is greater than k over every connected component of S. We mostly use additive notation for the group law in the Picard group.

1. The Behavior Under Isogenies

In this section we study the relation between $\Delta(L)$ and $\Delta(\alpha^*L)$ where $\alpha : A \to B$ is an isogeny of Abelian schemes, L is a relatively ample, symmetric, line bundle on B trivialized along the zero section. Let $d = \mathrm{rk}\, \pi_*L$. The main result of this section is the following theorem.

THEOREM 1.1. (1) *One has* $gcd(12, \deg(\alpha)) \cdot (\Delta(\alpha^*L) - \deg(\alpha) \cdot \Delta(L)) = 0$. (2) *If* $\deg(\alpha)$ *is odd, then* $\det \pi_*(\alpha^*L) = \deg(\alpha) \cdot \det \pi_*L + d \cdot \det \pi_*\mathcal{O}_{\ker \alpha} + \zeta$ *where* $gcd(3, \deg(\alpha)) \cdot \zeta = 0$.

More precisely, these equalities in $\mathrm{Pic}(S)$ *are realized by canonical isomorphisms of line bundles, compatible with arbitrary base changes.*

Remark. In the case $d = 1$ this follows from the result of Moret-Bailly in [9], VIII, 1.1.3. When d is even the second equality of the theorem can be rewritten as

$$gcd(3, \deg(\alpha)) \cdot (\Delta'(\alpha^*L) - \deg(\alpha) \cdot \Delta'(L)) = 0,$$

where $\Delta'(L) := \det \pi_*L + d2 \cdot \overline{\omega}_A$.

Outside of characteristic 3 we can improve Theorem 1.1 for some isogenies of degree divisible by 3.

THEOREM 1.2. *Assume that* 3 *is not among residue characteristics of* S *and that* d *is relatively prime to* 3. *Assume that* $3^k \cdot \ker(\alpha) = 0$, *that* $K(\alpha^*L)^{(3)}$ *is annihilated by* $\frac{1}{3} \cdot |K(\alpha^*L)^{(3)}|^{\frac{1}{2}}$, *and that* $|K(\alpha^*L)^{(3)}/(\ker(\alpha) + 3K(\alpha^*L)^{(3)})| > 3$. *Then one has a canoni-*

cal isomorphism of line bundles on S realizing the equality $\Delta(\alpha^* L) = \deg(\alpha) \cdot \Delta(L)$ *in* $\mathrm{Pic}(S)$.

Remark. When d is even the similar relation holds for elements $\Delta'(\cdot)$ instead of $\Delta(\cdot)$.

COROLLARY 1.3. *Let L be an ample symmetric line bundle on an Abelian scheme* A/S*, trivialized along the zero section. Then for any odd integer* $n > 0$ *which is not divisible by* 3 *one has* $\Delta(L^{n^2}) = n^{2g} \cdot \Delta(L)$. *Also one has* $\Delta(L^4)^{(3)} = 4^g \cdot \Delta(L)^{(3)}$ *and* $\Delta(L^9)^{(2)} = 9^g \cdot \Delta(L)^{(2)}$.

Furthermore, if $g > 1$ *and* 3 *is prime to* $d = \mathrm{rk}\, \pi_* L$ *and to the residue characteristics of S then* $\Delta(L^9) = 9^g \cdot \Delta(L)$.

We are going to use the relative Fourier–Mukai transform, so let us briefly recall some of its properties. For details the reader should consult [11] and [8]. The Fourier–Mukai transform is the functor

$$\mathcal{F}_A = \mathcal{F}_{A/S} : \mathcal{D}^b(A) \to \mathcal{D}^b(\hat{A}) : X \mapsto p_2^*(p_1^* X \otimes \mathcal{P}),$$

where $\mathcal{P}$ is the (normalized) relative Poincaré bundle on $A \times_S \hat{A}$. It is compatible with arbitrary base changes and satisfies the following fundamental property:

$$\mathcal{F}_{\hat{A}} \circ \mathcal{F}_A \simeq (-\mathrm{id}_A)^*(\cdot) \otimes \pi^* \overline{\omega}_A^{-1}[-g]$$

where g is the relative dimension of π. Because of this the relative canonical bundles of A and $\hat{A}$ often appear when working with the Fourier–Mukai transform so it is useful to know that there is a canonical isomorphism $\overline{\omega}_{\hat{A}} \simeq \overline{\omega}_A$ (see [8], 1.1.3).

Also for any homomorphism $f : A \to B$ of Abelian schemes over S one has the following canonical isomorphisms

$$\mathcal{F}_B \circ f_* \simeq \hat{f}^* \circ \mathcal{F}_A, \tag{1.1}$$

$$\mathcal{F}_A \circ f^! \simeq \hat{f}_* \circ \mathcal{F}_B. \tag{1.2}$$

The particular case of (1.1) is the isomorphism $e^* \circ \mathcal{F}_A \simeq \pi_*$, where $e : S \to A$ is the zero section, $\pi : A \to S$ is the projection.

The following lemma will be used in the proof of Theorem 1.2.

LEMMA 1.4. *For any relatively ample line bundle L on B trivialized along the zero section one has a canonical isomorphism* $\phi_L^* \mathcal{F}_B(L) \simeq \pi^* \pi_* L \otimes L^{-1}$.

Proof. Making the base change $\phi_L : B \to \hat{B}$ of the projection $p_2 : B \times \hat{B} \to \hat{B}$ we can write

$$\phi_L^* \mathcal{F}_B(L) \simeq \phi_L^* p_{2*}(p_1^* L \otimes \mathcal{P}) \simeq p_{2*}(p_1^* L \otimes (\mathrm{id}_B, \phi_L)^* \mathcal{P})$$

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where in the latter expression p_2 denotes the projection of the product $B \times_S B$ on the second factor. But we have an isomorphism

$$\mu^* L \simeq p_1^* L \otimes p_2^* L \otimes (\mathrm{id}_B, \phi_L)^* \mathcal{P}$$

since a trivialization of L along the zero section is equivalent to a cube structure on it (see [1]). Hence,

$$\phi_L^* \mathcal{F}_B(L) \simeq p_{2*}(p_2^* L^{-1} \otimes \mu^* L) \simeq L^{-1} \otimes p_{2*}\mu^*(L) \simeq L^{-1} \otimes \pi^* \pi_* L$$

as required. □

Proof of Theorem 1.1. Applying (1.2) to the isogeny $\alpha : A \to B$ we obtain $\mathcal{F}_A(\alpha^! L) \simeq \hat{\alpha}_* \mathcal{F}_B(L)$. Restricting this isomorphism to the zero section we obtain

$$\pi_*(\alpha^! L) \simeq e^* \mathcal{F}_A(\alpha^! L) \simeq \pi_*(\mathcal{F}_B(L)|_H),$$

where $H = \ker(\hat{\alpha}) \subset \hat{B}$. Taking the determinant of this isomorphism we get

$$\det \pi_*(\alpha^! L) = \det \pi_*(\mathcal{F}_B(L)|_H).$$

Using the fact that $\alpha^! L \simeq \alpha^* L \otimes \omega_{A/S} \otimes \omega_{B/S}^{-1}$ we can rewrite the left-hand side as follows:

$$\det \pi_*(\alpha^! L) = \det \pi_*(\alpha^* L) + n \cdot d \cdot \omega_\alpha\,,$$

where $\omega_a = \overline{\omega}_A - \overline{\omega}_B$, $n = \deg \alpha$.

Recall (see e.g. [2]) that for any vector bundle E of rank r on H one has

$$\det \pi_*(E) = r \cdot \det \pi_* \mathcal{O}_H + \mathrm{N}_{H/S}(\det E) \tag{1.3}$$

where $\mathrm{N}_{H/S} : \mathrm{Pic}(H) \to \mathrm{Pic}(S)$ is the corresponding norm homomorphism.

Consider the line bundle $M = \det \mathcal{F}_B(L) \otimes \pi^*(\det \pi_* L)^{-1}$ on $\hat{B}$. Then M has a canonical trivialization along the zero section. Furthermore, since the Fourier transform commutes with $[-1]^*$, we have an isomorphism $[-1]_{\hat{B}}^* M \simeq M$ compatible with the symmetry structure on L. Applying Equation (1.3) to $E = \mathcal{F}_B(L)|_H$, we get

$$\begin{aligned}\det \pi_*(\mathcal{F}_B(L)|_H) &= d \cdot \det \pi_* \mathcal{O}_H + N_H/S(\pi^* \det \pi_* L + M|_H)\\ &= d \cdot \det \pi^* \mathcal{O}_H + n \cdot \det \pi_* L + N_H/S(M|_H).\end{aligned}$$

Hence,

$$\det \pi_*(\alpha^* L) + n \cdot d \cdot \omega_\alpha = n \cdot \det \pi_* L + \mathrm{N}_{H/S}(M|_H) + d \cdot \det \pi_* \mathcal{O}_H. \tag{1.4}$$

This implies that

$$\Delta(\alpha^* L) - n \cdot \Delta(L) = 2 \cdot \mathrm{N}_{H/S}(M|_H) + 2 \cdot d \cdot \det \pi_* \mathcal{O}_H - n \cdot d \cdot \omega_\alpha.$$

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Recall that one has the canonical isomorphism $(\pi_*\mathcal{O}_H)^\vee \simeq \omega_H \otimes \pi_*\mathcal{O}_H$, where $\omega_H = (\pi_*\mathcal{O}_{\hat{H}})^H$ is the line bundle of (relative) invariant measures on H. Passing to determinants we get $2\det\pi_*\mathcal{O}_H = -n\cdot\omega_H$. Hence,

$$\Delta(\alpha^*L) - n\cdot\Delta(L) = 2\cdot \mathrm{N}_{H/S}(M|_H) - n\cdot d\cdot\omega_H - n\cdot d\cdot\omega_\alpha = 2\cdot\mathrm{N}_{H/S}(M|_H),$$

since $\omega_H \simeq \omega_{\hat{\alpha}} \simeq \omega_\alpha^{-1}$. Also using that $\ker(\alpha)$ is Cartier dual to H we deduce from (14) the following equality

$$\det\pi_*(\alpha^*L) = n\cdot\det\pi_*L + d\cdot\det\pi_*\mathcal{O}_{\ker\alpha} + \zeta$$

where $\zeta = \mathrm{N}_{H/S}(M|_H)$. It remains to show that $gcd(24, 2n)\cdot\mathrm{N}_{H/S}(M|_H) = 0$ and that $gcd(3, n)\cdot\mathrm{N}_{H/S}(M|_H) = 0$ if $\deg(\alpha)$ is odd. To this end let us decompose H into a product of two group schemes $H \simeq H' \times_S H''$ such that the order of H' is odd, while the order of H'' is a power of 2. Then the cube structure on M induces the decomposition of $M|_H$ into the external tensor product of $M|_{H'}$ and $M|_{H''}$. Hence, we obtain

$$\mathrm{N}_{H/S}(M|_H) = |H''|\cdot\mathrm{N}_{H'/S}(M|_{H'}) + |H'|\cdot\mathrm{N}_{H''/S}(M|_{H''}).$$

Recall that since M has a cube structure and $[-1]^*M \simeq M$, it follows that $[n]^*M \simeq M^{n^2}$ for any n. Now the multiplication by 3 is an automorphism of H'', hence

$$\mathrm{N}_{H''/S}(M|_{H''}) = \mathrm{N}_{H''/S}([3]^*_{\hat{B}}M|_{H''}) = \mathrm{N}_{H''/S}(M^9|_{H''}) = 9\cdot\mathrm{N}_{H''/S}(M|_{H''}).$$

Thus, $8\cdot\mathrm{N}_{H''/S}(M|_{H''}) = 0$. Similarly, using the multiplication by 2 we obtain

$$\mathrm{N}_{H'/S}(M|_{H'}) = \mathrm{N}_{H'/S}([2]^*_{\hat{B}}M|_{H'}) = \mathrm{N}_{H'/S}(M^4|_{H'}) = 4\cdot\mathrm{N}_{H'/S}(M|_{H'}),$$

hence $3\cdot\mathrm{N}_{H'/S}(M|_{H'}) = 0$. It remains to note that

$$\begin{aligned} 2\cdot N_H/S(M|_H) &= N_H/S(M|_H) + N_H/S([-1]^*_{\hat{B}}M|_H) \\ &= N_H/S((M\otimes[-1]^*_{\hat{B}}M)|_H) = N_H/S((id_{\hat{B}}, \phi_M)^*\mathcal{P}|_H) \end{aligned}$$

due to an isomorphism $M\otimes[-1]^*_{\hat{B}}M \simeq (\mathrm{id}_{\hat{B}}, \phi_M)^*\mathcal{P}$. In particular, since H is annihilated by n it follows that

$$2n\cdot\mathrm{N}_{H/S}(M|_H) = 0. \qquad \square$$

LEMMA 1.5. *Let B be an Abelian scheme over S, where 3 is prime to the residue characteristics of S, L be an ample line bundle on B trivialized along the zero section, $H \subset B_3$ be a finite flat subgroup. Assume that H is isotropic with respect to the symplectic form e^{L^3} on $K(L^3)$ and that $|H| > 3$. Then there is a canonical isomorphism of $\mathrm{N}_{H/S}(L|_H)$ with the trivial line bundle on S.*

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Proof. Without loss of generality we can assume that L is symmetric. Indeed, using the isomorphism $\mathrm{N}_{H/S}([-1]_B^* L|_H) \simeq \mathrm{N}_{H/S}(L|_H)$ and the cube structure on L one can see that $\mathrm{N}_{H/S}(L|_H)$ is annihilated by some power of 3. Hence, it suffices to prove the assertion for $L \otimes [-1]_B^* L$ instead of L.

Next we notice that the $\mathbb{G}_m$-torsor $L^3|_H$ is symmetric and has a structure of commutative group extension. Since the order of H is odd this implies that $L^3|_H$ has a unique trivialization compatible with the symmetry and with the group structure.

Let us consider the finite étale covering $c: S' \to S$ corresponding to a choice of a non-trivial point $\sigma \in H$. The degree of this covering is prime to 3, so it suffices to prove the triviality of $\mathrm{N}_{H/S}(L|_H)$ after making the corresponding base change. Thus, we can assume that we have a non-trivial S-point $\sigma: S \to H$. To compute $\mathrm{N}_{H/S}(L)$ we decompose the projection $H \to S$ into the composition $H \to \overline{H} \to S$ where $\overline{H} = H/\langle\sigma\rangle$, $\langle\sigma\rangle \subset H$ is the cyclic subgroup in H generated by σ. Now we claim that

$$\mathrm{N}_{H/\overline{H}}(L) \simeq \pi^*(\sigma^* L^2) \tag{1.5}$$

where π is the projection to S. Indeed, to give an isomorphism of line bundles on $\overline{H}$ is the same as to give an isomorphism of their pull-backs to H compatible with the action of $\langle\sigma\rangle \subset H$. Note that we have an isomorphism of symmetric line bundles trivialized at zero

$$L \otimes t_\sigma^* L \otimes t_{-\sigma}^* L \otimes L_\sigma^{-2} \simeq L^3. \tag{1.6}$$

We have a unique symmetric lifting of H to the Mumford group of a line bundle on each side. These liftings induce the symmetric action of H on both sides. By uniqueness the isomorphism (1.6) is compatible with the action of H on each side. Therefore, combining (1.6) with the trivialization of $L^3|_H$ we get an H-equivariant trivialization of the left-hand side restricted to H. In particular, this trivialization is compatible with the symmetric action of the subgroup $\langle\sigma\rangle \simeq \mathbb{Z}^3$, hence, it descends to an isomorphism (1.5) on $\overline{H}$. But $\sigma^* L$ is annihilated by 3 in $\mathrm{Pic}(S)$ and the degree of the projection $\pi: \overline{H} \to S$ is divisible by 3 (here we use the assumption that $|H| > 3$). Thus, we obtain

$$\mathrm{N}_{H/S}(L) = \mathrm{N}_{\overline{H}/S}\mathrm{N}_{H/\overline{H}}(L) = \mathrm{N}_{\overline{H}/S}(\pi^*\sigma^* L^2) = 0$$

as required. □

Proof of Theorem 1.2. Let $H \subset B_{3^k}$ be the preimage of $\ker(\hat{\alpha}) \subset \hat{B}_{3^k}$ under the isomorphism $\phi_L|_{B_{3^k}}: B_{3^k} \to \hat{B}_{3^k}$. As the proof of Theorem 1.1 shows we only have to check the triviality of the norm of M restricted to $\ker(\hat{\alpha})$. Since $\phi_L^* M \simeq L^{-d}$ by Lemma 1.4, this is equivalent to proving the triviality of $\mathrm{N}_{H/S}(L|_H)$. Consider the subgroup $K = \alpha^{-1}(H) \subset A$. Then $K \subset A_{3^{2k}}$ and the definition of H implies that

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$K = K(\alpha^* L)^{(3)}$. Also since d is prime to 3, it follows that $\ker(\alpha)$ is a maximal isotropic subgroup in K. Now we claim that after making an étale base change of degree prime to 3 we can find a finite flat subgroup $K_1 \subset K$ containing $\ker(\alpha)$, with the following two properties:

(1) K is annihilated by $3 \cdot |K_1/\ker(\alpha)|$,
(2) the quotient K/K_1 is annihilated by 3 and has order > 3.

Indeed, since $|K/\ker(\alpha)| = |K|^{\frac{1}{2}}$, all we need is to find K_1 such that $\ker(\alpha) + 3K \subset K_1 \subset K$ and $|K/K_1| = 9$. To get such K_1 we just make an étale covering of S (of degree prime to 3) corresponding to a choice of a subgroup of index 9 in $K/(\ker(\alpha) + 3K)$.

Let us denote $K_1' = K_1/\ker(\alpha) \subset H \subset B$ and $H' = K/K_1 = H/K_1'$. Consider the isogeny $f: B \to B' = B/K_1'$ and let L' be a line bundle on B' defined by

$$L' = \mathrm{N}_{B/B'}(L) \otimes \pi^* \mathrm{N}_{K_1'/S}(L|_{K_1'})^{-1}.$$

Then L' is trivialized along the zero section and we have

$$\mathrm{N}_{H/S}(L|_H) = \mathrm{N}_{H'/S}(\mathrm{N}_{B/B'} L|_{H'}) = \mathrm{N}_{H'/S}(L'|_{H'}) + |H'| \cdot \mathrm{N}_{K_1'/S}(L|_{K_1'}).$$

The latter term is trivial, since $\mathrm{N}_{K_1'/S}(L|_{K_1'})$ is annihilated by 3 (see the proof of Theorem 1.1). Thus, it remains to prove the triviality of $\mathrm{N}_{H'/S}(L'|_{H'})$. Since $|H'| > 3$ we can apply Lemma 1.5 to L' and H' provided that H' is isotropic with respect to the standard symplectic form on $K(L'^3)$. This is equivalent to asking that H is isotropic in $K((f^*L')^3)$. But the symplectic structure on the latter group is determined by the the polarization associated with $(f^*L')^3$ (see [12]). Now since $f^*\mathrm{N}_{B/B'}(L)$ is algebraically equivalent to $L^{\deg(f)}$ we obtain that $K((f^*L')^3) = K(L^{3\deg(f)})$ as symplectic groups. Now $H \subset K(L^{3\deg(f)})$ is isotropic iff $K = \alpha^{-1}(H) \subset K(\alpha^* L^{3\deg(f)})$ is isotropic. But $K \subset K(\alpha^* L)$, so this follows from the fact that K is annihilated by $3\deg(f) = 3 \cdot |K_1'|$ by assumption. □

2. The Method of Faltings and Chai

In this section we start proving Theorem 0.1.

Fix an odd prime number p. Following the proof of Faltings and Chai we consider the homomorphism $f_p: \mathbb{Z} \to \mathrm{Pic}(S)^{(p)}$, such that $f_p(n)$ is the p-primary component of $\Delta(L^n)$. This is a 'polynomial' function in n, which means that $\delta^i f_p = 0$ for some i where δ is the difference operator: $\delta\phi(n) = \phi(n+1) - \phi(n)$. As was noticed in [4] this can be seen by embedding A into the product of projective bundles $\mathbb{P}(\pi_* L^a) \times_S \mathbb{P}(\pi_* L^b)$ for relatively prime a and b (see Lemma 2.1 below for a more precise result). This implies immediately that the image of f_p belongs to some finitely generated subgroup of $\mathrm{Pic}(S)^{(p)}$ and that $f_p(n+p^N)=f_p(n)$ for sufficiently large N. By Serre duality one has $f_p(1)+(-1)^g\cdot f_p(-1)=0$, where g is the relative dimension of A/S. Thus, if we find an integer $k\equiv -1 \bmod (p^N)$ such that $f_p(k)=k^g\cdot f_p(1)$ this would imply that $\Delta(L)^{(p)}=0$. This is always possible when $p\equiv 1 \bmod(4)$. Indeed, we claim that if $p>3$ then $f_p(m^2n)=m^{2g}f_p(n)$ for all n and $m\neq 0$. This follows immediately from Theorem 1.1 applied to the isogeny $[m]_A: A\to A$ and the line bundle L^n (we don't have to worry about the factor 12 since we only consider p-primary component of the equality of Lemma 2.1 and $p>3$). If $p\equiv 1 \bmod(4)$ then we can find $k=m^2\equiv -1 \bmod(4)$, so we are done. Hence, we can assume that $p\equiv -1 \bmod(4)$. In this case one can always find some integers n and m such that $n^2+m^2\equiv -1 \bmod(p^N)$. Now let us consider the isogeny $\alpha: A^2\to A^2$ given by the matrix $\begin{pmatrix} [n]_A & [m]_A \\ [-m]_A & [n]_A \end{pmatrix}$. Then it is easy to see that $\alpha^*(L\boxtimes L)\simeq L^k\boxtimes L^k$ where $k=n^2+m^2$. Note that possibly changing initial n and m we can achieve that k is prime to 3. Applying Theorem 1.1 we find that

$$4\cdot\Delta(L^k\boxtimes L^k)=4\cdot k^{2g}\Delta(L\boxtimes L).$$

Hence, $d\cdot f_p(k)=d\cdot k^g\cdot f_p(1)$. As we noticed above this implies that $d\cdot\Delta(L)^{(p)}=0$. This finishes the first step in the proof of Theorem 0.1.

Now let us prove that $\Delta(L)^{(p)}=0$ for $p\geqslant 2g+1$, $p\neq 3$. The only new ingredient we need is the following lemma. Let us say that a function $\phi:\mathbb{Z}\to G$, where G is an Abelian group, has degree $\leqslant l$ if $\delta^{l+1}\phi=0$ where $\delta\phi(n)=\phi(n+1)-\phi(n)$.

LEMMA 2.1. *Let $\pi: X\to S$ be a smooth projective morphism of pure dimension g, L be a line bundle on X. Then the function $f:\mathbb{Z}\to \mathrm{Pic}(S)$ defined by $f(n)=\det\pi_*(L^n)$ has degree $\leqslant g+1$.*

Proof. This follows from Elkik's construction (based on ideas of Deligne in [2]), see [3], IV.1.3. □

Applying this Lemma to our Abelian scheme A/S and the line bundle L on it we deduce that f_p has degree $\leqslant g+1$. Now the vanishing of $\Delta(L)^{(p)}$ for $p\geqslant 2g+1$, $p\neq 3$, is implied by the following Lemma.

LEMMA 2.2. *Let p be a prime number, such that $p\geqslant 2g+1$, $p\neq 3$, $\phi:\mathbb{Z}\to\mathbb{Z}/p^k\mathbb{Z}$ be a function of degree $\leqslant g+1$, such that $\phi(n+p^N)=\phi(n)$ for sufficiently large N. Assume that $\phi(m^2n)=m^{2g}\cdot\phi(n)$ for all n and m. Then $\phi(n)=n^g\cdot\phi(1)$ for all n.*

Proof. Replacing ϕ by $\phi(n)-n^g\cdot\phi(1)$ we can assume that $\phi(1)=0$. In this case the assertion of Lemma is that $\phi=0$. An easy induction in k shows that it suffices to prove this for $k=1$. Then we can find a polynomial $\phi'(x)\in\mathbb{Z}/p\mathbb{Z}[x]$ of degree $\leqslant p-1$ such that $\phi'(n)=\phi(n)$ for $n=0,1,\ldots,p-1$. Since ϕ is the function of degree $\leqslant p-1$, it is determined uniquely by the set of its p consequtive values.

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The same is true for ϕ' considered as a function $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. It follows that $\phi'(n) = \phi(n)$ for all n, in particular, $\phi(n)$ depends only on $n \bmod (p)$. Let us fix a non-quadratic residue a modulo p. We know that $\phi(n) = 0$ if n is a square modulo p, and that $\phi(n) = a^{-g} \cdot n^g \cdot \phi(a)$ if n is not a square modulo p. Hence, for some $\lambda \in \mathbb{Z}/p\mathbb{Z}$ we have $\phi(n) = \lambda \cdot n^g \cdot (1 - n^{\frac{p-1}{2}})$ for all n. Now if $\lambda \neq 0$ then the right-hand side is given by a polynomial of degree $g + (p-1/2) \leqslant p-1$. Therefore, we actually have an identity of polynomials in $\mathbb{Z}/p\mathbb{Z}[x]$ which implies that $\deg(\phi) = g + (p-1/2)$. But this contradicts to $\deg(\phi) \leqslant g+1$, hence, $\lambda = 0$ as required. □

Remark. The fact that the element $\Delta(L) \in \mathrm{Pic}(S)$ has finite order is proved in [4] along the same lines. One should consider the function $f_0 : \mathbb{Z} \to \mathrm{Pic}(S)/\mathrm{Pic}(S)^{tors}$ where $\mathrm{Pic}(S)^{tors}$ is the torsion subgroup of $\mathrm{Pic}(S)$, such that $f_0(n) = \Delta(L^n)$ mod $\mathrm{Pic}(S)^{tors}$. Then f_0 is a function of finite degree, hence, its image is a finitely generated free group. Then the identity $f_0(n^2) = n^{2g} \cdot f_0(1)$ for infinitely many n implies that $f_0(n) = n^g \cdot f_0(1)$ for all n. Applying this to $n = -1$ and using Serre duality as above we deduce that $f_0 = 0$. At last, the bound on the 2-primary torsion of $\Delta(L)$ is obtained by considering the isogeny $A^4 \to A^4$ given by a 4×4 matrix of multiplication by a quaternion $n + m \cdot i + p \cdot j + q \cdot k$ such that $n^2 + m^2 + p^2 + q^2 \equiv -1 (N)$ for sufficiently divisible N (see [4]).

3. Some Arithmetics

Let $\phi : \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ be a map, $g \geqslant 1$ be an integer. Let us say that ϕ is g-*special*, if ϕ has degree $\leqslant g+1$ and $\phi(m^2 n) = m^{2g} \cdot \phi(n)$ for all n and m. In particular, since ϕ has finite degree it factors through $\mathbb{Z}/p^N\mathbb{Z}$ for sufficiently large N. In this terminology Lemma 2.2 says that for $p > 3$, $g \leqslant (p-1/2)$ any g-special map has form $\phi(n) = n^g \cdot \phi(1)$. In this section we'll study g-special maps for other values of g. Our main result is the following theorem, which combined with results of the previous section implies the first part of Theorem 0.1, except for the fact that $\Delta(L)^{(p)}$ is annihilated by p^2 if $p = 2g-1$. The latter statement will be proved together with the second half of Theorem 0.1 in the end of this section.

THEOREM 3.1. *If $p > 3$ and $g < (3p-1/2)$, $g \neq (p+1/2)$, then for any g-special map $\phi : \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ one has $p \cdot \phi(n) = p \cdot n^g \cdot \phi(1)$.*

We will use the condition that the degree of ϕ is $\leqslant g+1$ in the following form. Let us consider the generating function $F(t) = \sum_{n \geqslant 0} \phi(n) t^n \in \mathbb{Z}/p^k\mathbb{Z}[[t]]$. Then the condition $\deg(\phi) \leqslant g+1$ implies that $F(t) \cdot (t-1)^{g+2} = P(t)$, where $P(t) \in \mathbb{Z}/p^k\mathbb{Z}[t]$ is a polynomial in t. In particular, if $\phi(n + p^N) = \phi(n)$ for all n then $F(t) = Q_\phi(t) \cdot (1 - t^{p^N})^{-1}$, where $Q_\phi(t) = \sum_{n=0}^{p^N - 1} \phi(n)$. Thus, $Q_\phi(t) \cdot (t-1)^{g+2}$ is divisible by $t^{p^N} - 1$ in $\mathbb{Z}/p^k\mathbb{Z}[t]$.

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We are particularly interested in the case $k = 1$. In this case we obtain that $Q_\phi(t)$ is divisible by $(t-1)^{p^N-g-2}$. Let us denote

$$S_r(t) = \sum_{n=0}^{p-1} n^r \cdot t^n \in \mathbb{Z}/p\mathbb{Z}[t]$$

for $r \geqslant 0$ (in case $r = 0$ our convention is that $0^0 = 1$). Note that for $r > 0$ one has $S_r(t) = S_{r+p-1}(t)$. For every polynomial $Q \in \mathbb{Z}/p\mathbb{Z}[t]$ we denote by $v_{(t-1)}(Q)$ the maximal power of $(t-1)$ dividing Q.

LEMMA 3.2. *One has* $v_{(t-1)}(S_r(t)) = p - 1 - r$ *for* $0 \leqslant r \leqslant p-1$.

Proof. For $r = 0$ we have an identity $S_0(t) = (t-1)^{p-1}$ which follows from the congruence $\binom{p-1}{i} \equiv (-1)^i \bmod(p)$. Now the identity $S_{r+1}(t) = t \cdot (d/dt)S_r(t)$ and an easy induction show that

$$S_r(t) \equiv (-1)^r \cdot r! \cdot t^r \cdot (t-1)^{p-1-r} \bmod((t-1)^{p-r})$$

for $0 \leqslant r \leqslant p-1$. □

The first step in the proof of Theorem 3.1 is the following Lemma.

LEMMA 3.3. *Let* $\phi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ *be a* g*-special map. If* $p > 3$ *and* $(p - 1/2) < g < 2p - 1$ *then*

$$\phi(n) = \lambda \cdot n^g + \mu \cdot n^{g-\frac{p-1}{2}} \bmod(p)$$

for some constants $\lambda, \mu \in \mathbb{Z}/p\mathbb{Z}$.

Proof. It is easy to see that for any λ and μ the map

$$n \mapsto \lambda \cdot n^g + \mu \cdot n^{g-\frac{p-1}{2}} \bmod(p)$$

is g-special. Hence, if we write

$$\phi(n) = \lambda \cdot n^g + \mu \cdot n^{g-\frac{p-1}{2}} + \phi'(n)$$

for some λ and μ then ϕ' will also be a g-special map. Choosing λ and μ appropriately we can achieve that $\phi'(1) = \phi'(a) = 0$ for some a which is a not a square modulo p. Replacing ϕ by ϕ' we can assume that this condition holds for ϕ. Since $\phi(n+p) = \phi(n)$ for every $n \not\equiv 0 \bmod(p)$ (this follows from g-speciality and the fact that $(n+p)n^{-1}$ is a square in $\mathbb{Z}/p^N\mathbb{Z}$) we deduce that $\phi(n) = 0$ for all $n \not\equiv 0 \bmod(p)$. On the other hand, $\phi(p^2 n) = p^{2g}\phi(n) = 0$, hence the only non-trivial values of ϕ are $\phi(pn)$ for $n \not\equiv 0 \bmod(p)$. If $\deg \phi < p$, then this implies immediately that $\phi = 0$, so we can assume that $g \geqslant p-1$. For all $n \not\equiv 0 \bmod(p)$ we have $\phi(pn + p^2) = (1 + p \cdot n^{-1})^g \cdot \phi(pn) = \phi(pn)$. In particular, ϕ depends only on

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$p \bmod(p^2)$ and

$$Q_\phi(t) = \sum_{n=0}^{p^2-1} \phi(n)t^n = \sum_{n=1}^{p-1} \phi(pn)t^{pn}.$$

Now $n \mapsto \phi(pn)$ is a p-special map depending only on $n \bmod(p)$. Hence,

$$\phi(pn) = a \cdot n^g + b \cdot n^{g-\frac{p-1}{2}}$$

for some $a, b \in \mathbb{Z}/p\mathbb{Z}$. Therefore,

$$Q_\phi(t) = a \cdot S_g(t^p) + b \cdot S_{g-\frac{p-1}{2}}(t^p).$$

As we have seen above the fact that deg $\phi \leqslant g+1$ implies that $v_{(t-1)}(Q_\phi) \geqslant p^2 - g - 2$. Now we claim that for all g such that $p-1 \leqslant g \leqslant 2(p-1)$ the valuations of $S_g(t^p)$ and $S_{g-\frac{p-1}{2}}(t^p)$ at $(t-1)$ are less than p^2-g-2. This would imply that $a=b=0$, hence $\phi = 0$ as required. To prove our claim let us apply Lemma 3.2 to compute $v_{(t-1)}S_g$ and $v_{(t-1)}S_{g-\frac{p-1}{2}}$. For $g = p-1$ we get $v_{(t-1)}S_g = 0$, while for $p-1 < g \leqslant 2(p-1)$ we have $v_{(t-1)}S_g(t^p) = p \cdot (2(p-1)-g) < p^2-g-2$. Similarly, for $p-1 \leqslant g \leqslant (3(p-1)/2)$ we get $v_{(t-1)}S_{g-\frac{p-1}{2}}(t^p) = p \cdot (p-1-g+(p-1/2)) < p^2-g-2$, while for $(3(p-1)/2) < g \leqslant 2(p-1)$ we have $v_{(t-1)}S_{g-\frac{p-1}{2}}(t^p) = p \cdot (2(p-1)-g+(p-1/2))$. Thus, to finish the proof we need the inequality

$$p \cdot \left(\frac{5(p-1)}{2} - g\right) < p^2 - g - 2$$

in this case, but when $p > 3$ it follows from $g > (3(p-1)/2)$. □

Remark. In fact, for $p > 3$ one can prove that the conclusion of the previous lemma remains true when g belongs to one of the following intervals of integers: $[2p, \frac{5}{2}(p-1)]$, $[(5p+1/2), 3(p-1)]$, $[3p, \frac{7}{2}(p-1)]$, $[(7p+1/2), 4(p-1)]$, etc. (for given p the set of such g is finite).

Next step is to consider g-special maps depending only on $n \bmod(p^2)$.

LEMMA 3.4. *Let $\phi : \mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ be a g-special map such that $\phi(n+p^2) = \phi(n)$ for all n. Assume that $p > 3$ and $(p+1/2) < g < (3p-1/2)$. Then*

$$p \cdot \phi(n) = p \cdot n^g \cdot \phi(1)$$

for all n.

Proof. Replacing ϕ by $\phi - n^g \cdot \phi(1)$ we can assume that $\phi(1) = 0$. In this case we need to show that $\phi = 0 \bmod(p)$. Applying Lemma 3.3 to $\phi \bmod(p)$ we obtain

$$\phi(n) = c \cdot (n^g - n^{g-\frac{p-1}{2}}) + p \cdot \psi(n) \tag{3.1}$$

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for some constant $c \in \mathbb{Z}/p^2\mathbb{Z}$ and some map $\psi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Now for every $n \not\equiv 0 \bmod(p)$ we have

$$\phi(n+p) - \phi(n) = (1 + p \cdot n^{-}1)^g \cdot \phi(n) - \phi(n) = p \cdot g \cdot n^{-}1 \cdot \phi(n)$$
$$= p \cdot c \cdot g \cdot (n^g - 1 - n^{g - \frac{p+1}{2}}).$$

On the other hand, subtracting Equation (3.1) for n from that for $n+p$ we get

$$\phi(n+p) - \phi(n) = p \cdot c \cdot (g \cdot n^{g-1} - \left(g - \frac{p-1}{2}\right) \cdot n^{g-\frac{p+1}{2}}) + p \cdot (\psi(n+p) - \psi(n)).$$

It follows that

$$\psi(n+p) - \psi(n) = -c \cdot \frac{p-1}{2} \cdot n^{g-\frac{p+1}{2}} = \frac{c}{2} \cdot n^{g-\frac{p+1}{2}}.$$

Hence, for every m one has

$$\psi(n + p \cdot m) = \psi(n) + \frac{c}{2} \cdot m \cdot n^{g-\frac{p+1}{2}} \tag{3.2}$$

provided that $n \not\equiv 0 \bmod(p)$.

Also we claim that

$$\psi(p \cdot n) = \lambda \cdot n^g + \mu \cdot n^{g-\frac{p-1}{2}} \tag{3.3}$$

for some $\lambda, \mu \in \mathbb{Z}/p\mathbb{Z}$. Indeed, since $g > (p + 1/2)$ equation (3.1) shows that $\phi(p \cdot n) = p \cdot \psi(p \cdot n)$. Hence, the map $n \mapsto \psi(p \cdot n)$ is g-special and the assertion follows from Lemma 3.3.

Now (3.1) shows that ψ depends only on $n \bmod(p^2)$ and has degree $\leqslant g+1$. Hence, the corresponding polynomial $Q_\psi(t) = \sum_{n=0}^{p^2-1} \psi(n)t^n$ is divisible by $(t-1)^{p^2-g-2}$. Using (3.2) and (3.3) we can write

$$Q_\psi(t) = \sum_{n=1}^{p-1}\sum_{m=0}^{p-1} \psi(n + p \cdot m)t^{n+p\cdot m} + \sum_{n=1}^{p-1} \psi(p \cdot n)$$
$$= \left(\sum_{n=1}^{p-1} \psi(n)t^n\right) \cdot S_0(t^p) + \frac{c}{2} \cdot S_{g-\frac{p+1}{2}}(t) \cdot S_1(t^p) + \lambda \cdot S_g(t^p) + \mu \cdot S_{g-\frac{p-1}{2}}(t^p).$$

In the case $g \leqslant p-2$ we have $p^2 - g - 2 \geqslant p^2 - p$, hence, $v_{(t-1)}Q_\psi \geqslant p^2 - p$. To prove that $c = 0$ in this case it is sufficient to check that $v(g) := v_{(t-1)}S_{g-\frac{p+1}{2}}(t) + v_{(t-1)}S_1(t^p) < p^2 - p$ and that $v(g)$ differs from $v_{(t-1)}S_g(t^p)$ and $v_{(t-1)}S_{g-\frac{p-1}{2}}(t^p)$. One can check using Lemma 3.2 that this is indeed the case. When $g \geqslant p-1$ we can omit the first term in the above expression for $Q_\psi(t)$ when considering $Q_\psi(t) \bmod (t-1)^{p^2-g-2}$. Hence, to deduce that $c = 0$ one should check using Lemma

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3.2 that $v(g) < p^2 - g - 2$ and $v(g)$ differs from the valuations of two other terms. We omit the details of this simple computation. $\square$

Proof of Theorem 3.1. The case $g \leqslant (p - 1/2)$ follows from Lemma 2.2 so we only consider $g > (p + 1/2)$. Also as usual we can assume that $\phi(1) = 0$. An easy induction in k shows that it suffices to consider the case $k = 2$. In the latter case we have $\phi(p^2 \cdot n) = 0$ for all n and $\phi(n + p^2) = \phi(n)$ for $n \not\equiv 0 \bmod (p)$. According to Lemma 3.3 we can write

$$\phi(n) = c \cdot (n^g - n^{g - \frac{p-1}{2}}) + p \cdot \psi(n)$$

for some constant $c \in \mathbb{Z}/p^2\mathbb{Z}$ and some map $\psi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. In particular, $\phi(p \cdot n) = p \cdot \psi(p \cdot n)$ and $n \mapsto \psi(p \cdot n)$ is a g-special map. Now Lemma 3.3 implies that $\psi(p \cdot (n + p)) = \psi(p \cdot n)$, hence $\phi(p \cdot n + p^2) = \phi(p \cdot n)$. Therefore, $\phi(n)$ depends only on $n \bmod (p^2)$ and we can apply Lemma 3.4 to finish the proof. $\square$

Now we turn to the proof of the second half of Theorem 0.1. Note that for any prime p the function $f_p : n \mapsto \Delta(L^n)^{(p)}$ satisfies the following property:

$$f_p(m^2) = m^{2g} \cdot f_p(1)$$

for all m such that $m \not\equiv 0 \bmod (p)$. Indeed, changing m by $m + p^N$ if necessary we may assume that m is odd and is prime to 3, hence this follows from Theorem 1.1.

PROPOSITION 3.5. *Let* $f : \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ *be a function of degree* $\leqslant g + 1$, *such that* $f(m^2) = m^{2g} \cdot f(1)$ *for all* m *such that* $m \not\equiv 0 \bmod (p)$. *Then there exists an integer* $n(p, g)$ *depending only on* p *and* g *such that* $p^{n(p,g)} \cdot (f(n) - n^g \cdot f(1)) = 0$ *for all* n.

Proof. First of all, replacing f by $(g + 1)! \cdot f$ we can assume that

$$f(n) - n^g \cdot f(1) = a_0 + n \cdot a_1 + \ldots + n^{g+1} \cdot a_{g+1}$$

for some $a_i \in \mathbb{Z}/p^k\mathbb{Z}$. Let $n_0, \ldots, n_{g+1}$ be the first $g + 2$ positive integers of the form m^2 with $m \not\equiv 0 \bmod (p)$. Then $f(n_i) - n_i^g \cdot f(1) = 0$ by assumption, hence every coefficient a_i is annihilated by the Vandermonde determinant $\Delta(n_0, \ldots, n_{g+1}) = \prod_{i<j}(n_j - n_i)$. It follows that we can take $n(p, g)$ to be $v_p((g + 1)! \cdot \Delta(n_0, \ldots, n_{g+1}))$. $\square$

The proof of the second part of Theorem 0.1 follows immediately from this proposition: we can take $N(g) = \prod_{p \leqslant 2g-1} p^{n(p,g)}$. To complete the proof of Theorem 0.1 it remains to prove the statement concerning the prime $p = 2g - 1$. This is the content of the following lemma.

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LEMMA 3.6. *Let* $p > 3$ *be a prime,* $f : \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$ *be a* g*-special map where* $g = (p + 1/2)$*. Then* $p^2 \cdot (f(n) - n^g \cdot f(1)) = 0$.

Proof. Since $g + 1 < p$, we have

$$f(n) - n^g \cdot f(1) = a_0 + n \cdot a_1 + \ldots + n^{g+1} \cdot a_{g+1}$$

for some $a_i \in \mathbb{Z}/p^k\mathbb{Z}$. Let $n_0 = 0, n_1, \ldots, n_{g+1}$ be the first $g + 1$ positive integers of the form m^2 with $m \not\equiv 0 \bmod(p)$. Then $f(n_i) - n_i^g \cdot f(1) = 0$ for all $i = 0, \ldots, g + 1$ by g-speciality, therefore every coefficient a_i is annihilated by the Vandermonde determinant $\Delta = \Delta(n_0, \ldots, n_{g+1}) = \prod_{i<j}(n_j - n_i)$. Since $g + 1 = (p - 1/2) + 2$, it follows that $v_p(\Delta) = 2$, hence the assertion. □

4. Eliminating of Primes Not Dividing Characteristic

In the case when an odd prime p is not among residue characteristics of the base we can evaluate $\Delta(L)^{(p)}$ over $S[\frac{1}{p}]$ using the following result.

THEOREM 4.1. *Let* L *be a symmetric, relatively ample line bundle over an Abelian scheme* A/S *trivialized along the zero section. Let* p *be an odd prime number which is not equal to any of residue characteristics of* S*. Then there exists a finite flat base change* $c : S' \to S$ *of degree prime to* p*, an isogeny of Abelian* S'*-schemes* $\alpha : A' \to B$*, where* A' *is obtained from* A *by this base change, such that* $\deg(\alpha)$ *is the power of* p*, and a symmetric line bundle* M *on* B *together with a symmetric isomorphism* $\alpha^* M \simeq L'$ *(where* L' *is obtained from* L *by the base change), such that* $\deg \phi_M$ *is prime to* p*.*

Let us deduce Theorem 0.2 from this. Choose a base change $c : S' \to S$ as in Theorem 4.1. Since $\mathrm{Pic}(S)^{(p)} \to \mathrm{Pic}(S')^{(p)}$ is injective and the construction of $\det \pi_*$ commutes with this base change we can work with A' instead of A. It remains to apply Theorem 1.1 to isogeny α and Theorem 0.1 to M to deduce that $\Delta(L')^{(p)} = 0$ if $p \neq 3$. In the case $p = 3$ by the same argument we always have $3 \cdot \Delta(L')^{(3)} = 0$. Now if the 3-type of the polarization is different from $(1, \ldots, 1, 3^k)$, then one can see easily from the construction of the isogeny α below that the conditions of Theorem 1.2 are satisfied for α and M. Hence, the triviality of $\Delta(L')^{(3)}$ in this case.

Proof of Theorem 4.1. We can assume that the base S is connected. Let $K = K(L)^{(p)}$ be the p-primary component of the finite flat group scheme $K(L)$ over S. Then K is étale over S (since p is not among residue characteristics of S) and ϕ_L induces a skew-symmetric isomorphism $K \simeq \hat{K}$. The fiber of K over a geometric point of S is a discrete group of the form $(\mathbb{Z}/p^{n_1}\mathbb{Z})^2 \times \ldots \times (\mathbb{Z}/p^{n_k}\mathbb{Z})^2$ where the factors $(\mathbb{Z}/p^{n_i}\mathbb{Z})^2$ are orthogonal to each other with respect to a symplectic form,

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$\mathbb{Z}/p^{n_i}\mathbb{Z} \times \{0\} \subset (\mathbb{Z}/p^{n_i}\mathbb{Z})^2$ is a Lagrangian subgroup for every i. Since S is connected the collection $n_1, \ldots, n_k$ doesn't depend on a point. Let n be the maximum of $n_1, \ldots, n_k$, so that n is the minimal number such that $p^n \cdot K = 0$. Now let us construct a canonical isotropic subgroup $I_0 \subset K$, étale over S, such that $K_0 = I_0^{\perp}/I_0$ is annihilated by p. If $n = 1$ we can take $I_0 = 0$ so let's assume that $n \geqslant 2$. Then $p^{n-1} \cdot K$ is an isotropic subgroup in K so we can consider the reduction $\overline{K} = (p^{n-1} \cdot K)^{\perp}/(p^{n-1} \cdot K)$ with its induced symplectic form. By induction we may assume that we already found an isotropic subgroup $\overline{I}_0 \subset K'$ such that $(\overline{I}_0)^{\perp}/\overline{I}_0$ is a p-group. Now take I_0 to be the preimage of $\overline{I}_0$ in K.

Our base change $c : S' \to S$ will be the finite flat covering corresponding to a choice of a Lagrangian subgroup in K_0. One can construct such a covering in the following way. Let p^{2r} be the order of K^0. Start with a subscheme $\widetilde{S}$ in $K_0 \times_S \ldots \times_S K_0$ ($2r$ times) corresponding to symplectic bases in K_0. Then $\widetilde{S} \to S$ is a $\mathrm{Sp}_{2r}(\mathbb{Z}/p\mathbb{Z})$-torsor. Now let $P \subset \mathrm{Sp}_{2r}(\mathbb{Z}/p\mathbb{Z})$ be the subgroup preserving the standard r-dimensional Lagrangian subgroup in $(\mathbb{Z}/p\mathbb{Z})^{2r}$, then we can take $S' = P\backslash\widetilde{S}$. The degree of the covering $S' \to S$ is equal to the number of Lagrangian subgroups in $(\mathbb{Z}/p\mathbb{Z})^{2r}$ which is easily seen to be equal to $\prod_{i=1}^{r}(p^i + 1)$ (first compute the number of isotropic flags and then divide by the number of flags in $(\mathbb{Z}/p\mathbb{Z})^r$), which is prime to p.

Let (A', L', K', I_0', K_0') be the data obtained from (A, L, K, I_0, K_0) by the base change $S' \to S$. Then by construction we have a Lagrangian subgroup $\overline{I} \subset K_0'$. Taking its preimage by the morphism $(I_0')^{\perp}/I_0' \to K_0'$ we obtain a Lagrangian subgroup $I \subset K'$. It remains to prove that L' descends to a symmetric bundle on $B = A'/I$. To this end note that the $\mathbb{G}_m$-torsor $L'|_I$ has a structure of the commutative group extension. Since the order of I is odd there exists a unique trivialization of $L'|_I$ as a group extension which is compatible with the symmetry $[-1]^*L' \simeq L'$. This trivialization gives a symmetric lifting of I to the Mumford group of L' as required. □

5. Complements

5.1. CASE OF ELLIPTIC CURVES

Let us evaluate determinant bundles in the case $g = 1$. It is known (see, e.g., [13]) that in characteristics $\neq 2, 3$ the Picard group of the moduli stack of elliptic curves (with one fixed point) is $\mathbb{Z}/12\mathbb{Z}$ and as generator one can take the line bundle $\overline{\omega}$ on this moduli stack that associates to every family of elliptic curves $\pi : E \to S$ the relative canonical bundle $\overline{\omega}_E \in \mathrm{Pic}(S)$. If S is connected then any symmetric line bundle L on E, trivialized along the zero section, is either isomorphic to $L_d(e) := \mathcal{O}(d \cdot e) \otimes \omega_{E/S}^d$ where $e : S \to E$ is the zero section, or to $L_d(\eta) := \mathcal{O}((d-1)\cdot e + \eta) \otimes \omega_{E/S}^{d-1}$ where $\eta : S \to E$ is an everywhere non-trivial point of order 2. Note that these line bundles are trivialized along the zero section, since $e^*\mathcal{O}(e) \simeq \overline{\omega}_E^{-1}$.

PROPOSITION 5.1. *One has*

$$\det \pi_*(L_d(e)) = \left(\frac{d(d-1)}{2} + 1\right) \cdot \overline{\omega}_E, \qquad \det \pi_*(L_d(\eta)) = \frac{d(d-1)}{2} \cdot \overline{\omega}_E.$$

In particular,

$$\Delta(L_d(e)) = (d^2 + 2) \cdot \overline{\omega}_E, \qquad \Delta(L_d(\eta)) = d^2 \cdot \overline{\omega}_E.$$

Furthermore, these equalities are represented by canonical isomorphisms of line bundles.

Proof. Considering the push-forward of the exact sequence

$$0 \to \mathcal{O}((d-1) \cdot e) \to \mathcal{O}(d \cdot e) \to e_* e^* \mathcal{O}(d \cdot e) \to 0,$$

we deduce that

$$\det \pi_* \mathcal{O}(d \cdot e) - \det \pi_* \mathcal{O}((d-1) \cdot e) = -d \cdot \overline{\omega}_E.$$

Since $\pi_* \mathcal{O}(e) \simeq \mathcal{O}_S$, it follows that

$$\det \pi_* \mathcal{O}(d \cdot e) = \left(1 - \frac{d(d+1)}{2}\right) \overline{\omega}_E,$$

hence $\det \pi_* L_d(e) = ((d(d-1))/2) + 1) \overline{\omega}_E$. The case of $\mathcal{O}_d(\eta)$ is considered similarly using the exact sequence

$$0 \to \mathcal{O}((d-1) \cdot e + \eta) \to \mathcal{O}(d \cdot e + \eta) \to e_* e^* \mathcal{O}(d \cdot e + \eta) \to 0$$

and the triviality of $\pi_* \mathcal{O}(\eta)$. □

Note that $\Delta(L_d(e))$ gives a line bundle on the moduli stack of elliptic curves $\mathcal{A}_1$, while $\Delta(L_d(\eta))$ lives on the stack $\widetilde{\mathcal{A}}_1$ classifying elliptic curves with a non-trivial point of order 2. Now Proposition 5.1 combined with Theorem 0.1 implies immediately that the order of $\overline{\omega}$ in $\mathrm{Pic}(\mathcal{A}_1)$ is 12, while the order of the pull-back of $\overline{\omega}$ to $\widetilde{\mathcal{A}}_1$ is 4. In particular, $\Delta(L_3(e)) = -\overline{\omega}$ in $\mathrm{Pic}(\mathcal{A}_1)$.

5.2. LINEAR RELATIONS BETWEEN DETERMINANT BUNDLES

Let us first consider the determinant bundles $\det \pi_*(L^n)$ on an Abelian scheme A/S of relative dimension $g = 2$ or $g = 3$, where L is a relatively ample, symmetric line bundle on A trivialized along the zero section.

PROPOSITION 5.2. *If $g = 2$ then one has*

$$\det \pi_* L^n = \frac{4n - n^3}{3} \cdot \det \pi_* L + \frac{n^3 - n}{6} \cdot \det \pi_* L^2 + \frac{n(n-1)(n-2)}{6} \cdot d \cdot \overline{\omega}_A,$$

where $d = \mathrm{rk}\, \pi_* L$. In particular,

$$\Delta(L^n) = \frac{4n - n^3}{3} \cdot \Delta(L) + \frac{n^3 - n}{6} \cdot \Delta(L^2).$$

If $g = 3$, then one has

$$\det \pi_* L^n = \frac{4n^2 - n^4}{3} \cdot \det \pi_* L + \frac{n^4 - n^2}{12} \cdot \det \pi_* L^2 + \frac{n^2(n-1)(n-2)}{6} \cdot d \cdot \overline{\omega}_A.$$

In particular, in this case

$$\Delta(L^n) = \frac{4n^2 - n^4}{3} \cdot \Delta(L) + \frac{n^4 - n^2}{12} \cdot \Delta(L^2).$$

Proof. This is easily deduced from the fact that the function $f : n \mapsto \det \pi_* L^n$ has degree $\leqslant g + 1$. Indeed, by Serre duality the values of this function at $n = -2, -1$ are expressed via those for $n = 1, 2$. Also $\det \pi_* \mathcal{O}_A = 0$, hence, we know values of f at $n \in [-2, 2]$ and we can interpolate the rest. □

COROLLARY 5.3. *If $g = 2$ or $g = 3$, then* $8 \cdot 9 \cdot \Delta(L) = 4 \cdot 9 \cdot \Delta(L^2) = 0$.

Proof. This is proved by considering separately 2-primary and 3-primary parts of $\Delta(L)$ and $\Delta(L^2)$, using the previous proposition and Theorem 1.1. □

If $g = 2$, $gcd(d, 3) = 1$ and n is odd then we also get from Proposition 5.2 that $\Delta(L^n) = n \cdot \Delta(L)$. If $g = 3$, $gcd(d, 3) = 1$, and the characteristic is zero then according to Kouvidakis one has $\Delta(L^2) = 0$, hence in this case for odd n we get $\Delta(L^n) = \Delta(L)$. It would be interesting to find similar dependences between $\Delta(L^n)$ in higher dimensions (see Section 5.3 for the case of even principal polarization). Recall (see Lemma 2.1) that $n \mapsto \det \pi_*(L^n)$ is a function of degree $\leqslant g + 1$. Hence, using Serre's duality, one can express all $\det \pi_*(L^n)$ as linear combinations of $\overline{\omega}$ and $\det \pi_*(L^i)$ where $0 \leqslant i \leqslant \frac{g}{2} + 1$. However, we expect much more relations between $\Delta(L^n)$. Here are some examples.

(1) For $g \geqslant 2$ one has $\det \pi_* \mathcal{O}_A = 0$.
(2) Let p be a prime, $p \equiv -1 \bmod(4)$. Then for any n such that $(n, p) = 1$ one has

$$\Delta(L^n)^{(p)} = \left(\frac{n}{p}\right) \cdot n^g \cdot \Delta(L)^{(p)},$$

where $\left(\frac{n}{p}\right) = \pm 1$ is the Legendre symbol.

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(3) Assume that $g \geqslant 2$. Let p be a prime such that $p \equiv -1 \bmod(4)$ and $p \geqslant (g+3)/2$. Then one has $\Delta(L^n)^{(p)} \in \mathbb{Z}\Delta(L)^{(p)}$ for all n.

(4) For odd n one has $\Delta(L^{n+8})^{(2)} = ((n+8)/n)^g \cdot \Delta(L^n)^{(2)}$. In particular, if n and d are odd then $\Delta(L^{n+8})^{(2)} = \Delta(L^n)^{(2)}$.

(1) follows from the fact that $R^i\pi_*\mathcal{O}_A = \bigwedge^i R^1\pi_*\mathcal{O}_A$. For the proof of (2) note that for $(n,p)=1$ Theorem 1.1 implies that $\Delta(L^n)^{(p)} = n^g \cdot \Delta(L)^{(p)}$ if n is a square modulo p, and that $\Delta(L^n)^{(p)} = (-n)^g \cdot \Delta(L^{-1})^{(p)}$ if $-n$ is a square modulo p. But Serre's duality implies that $\Delta(L^{-1}) = -(-1)^g \cdot \Delta(L)$, hence the assertion. To prove (3) note that for $p \geqslant (g+3)/2$ all the elements $\Delta(L^n)^{(p)}$ are linear combinations of $\Delta(L^i)^{(p)}$ with $|i| < p$. It remains to apply (1) and (2). At last, (4) follows from Theorem 1.1 since $(n+8/n)$ is a square modulo resp. 2^k.

PROPOSITION 5.4. $d \cdot (\Delta(L^3)^{(2)} + 3^g \cdot \Delta(L)^{(2)}) = 0$.

Proof. Let us denote $\Delta(L,n) = \det \pi_*(L^n) - n^g \cdot \det \pi_* L$. Then it is easy to see that $2\Delta(L,n) = \Delta(L^n) - n^g \cdot \Delta(L)$, in particular, $\Delta(L,n)$ is a torsion element in $\mathrm{Pic}(S)$. Note also that $n \mapsto \Delta(L,n)$ is a polynomial function. Let us choose a sufficiently divisible integer $N > 0$ such that both functions $\Delta(L^n)$ and $\Delta(L,n)$ of n are N-periodic and are annihilated by N, and N is divisible by 6. Now let l be a prime, such that N is not divisible by l and such that $l \equiv 3 \bmod(2^m)$ where $m >> 0$. Then there exists a solution (a,b) to the congruence $a^2 + lb^2 \equiv -1 \bmod(N)$. Consider the isogeny $\alpha : A^2 \to A^2$ given by the matrix $\begin{pmatrix} a & -lb \\ b & a \end{pmatrix}$. Then it is easy to see that $\alpha^*(L \boxtimes L^l) \simeq L^k \boxtimes L^{lk}$ where $k = a^2 + lb^2$. In particular, $\deg(\alpha) = k^{2g}$ and applying Theorem 1.1 we obtain the equality

$$\det \pi_*(L^k \boxtimes L^{lk}) = k^{2g} \cdot \det \pi_*(L \boxtimes L^l).$$

Now using the equalities $\Delta(L,k) = \Delta(L,-1)$, $\Delta(L,lk) = \Delta(L,-l)$, Serre's duality, the fact that $k \equiv -1 \bmod(N)$, and the conditions on N one can easily deduce that

$$d \cdot (\Delta(L^l) + l^g \cdot \Delta(L)) = 0.$$

It remains to take the 2-primary part of this equality and replace l by 3 in the obtained identity (this is justified by the congruence $l \equiv 3 \bmod(2^m)$ with $m >> 0$). □

COROLLARY 5.5. *Assume that n and d are odd. Then $\Delta(L^n)^{(2)} = n^{g-1} \cdot \Delta(L)^{(2)}$. In particular, if $d = 1$ and $g \geqslant 2$ then $\Delta(L^n) = n^{g-1} \cdot \Delta(L)$ for any odd n.*

Proof. Since $\Delta(L^n)^{(2)}$ for odd n depends only on n mod(8) the first equality follows from the case $n = 3$ considered above, Serre's duality, and the vanishing of $4 \cdot \Delta(L^n)^{(2)}$ for odd n. Now the second statement follows from the fact that for $d = 1$ and $g \geqslant 2$ one has $4 \cdot \Delta(L^n) = 0$ for any n. □

5.3. TORSION IN THE PICARD GROUP OF MODULI

Let $\widetilde{\mathcal{A}}_g$ be the moduli stack of the data $(A/S, \Theta)$ where A/S is an Abelian scheme of relative dimension g, $\Theta \subset A$ is an effective (relative) divisor which is symmetric and defines a principal polarization of A. One can normalize the line bundle $\mathcal{O}(\Theta)$ over the universal Abelian scheme over $\widetilde{\mathcal{A}}_g$ to obtain the line bundle L which is trivial along the zero section. In particular, we have an element $\Delta(L) \in \mathrm{Pic}(\widetilde{\mathcal{A}}_g)$. Let $\widetilde{\mathcal{A}}_g^+$ be the irreducible component of $\widetilde{\mathcal{A}}_g$ corresponding to even theta divisors, $\widetilde{\mathcal{A}}_g^+[\frac{1}{2}]$ be the localization of this stack over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$.

THEOREM 5.6. *Assume that $g \geqslant 3$. Then the torsion subgroup in* $\mathrm{Pic}(\widetilde{\mathcal{A}}_g^+[\frac{1}{2}])$ *is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and is generated by $\Delta(L)$.*

Proof. Since $\widetilde{\mathcal{A}}_g^+$ has smooth geometrically irreducible fibers over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ (cf. [4], IV 7.1) it is sufficient to prove this statement in characteristic zero. Indeed, it is known that the order of $\Delta(L)$ is precisely 4 (cf. [10]), hence it would follow that $\Delta(L)$ generates the entire torsion subgroup in the Picard group of the general fiber of $\widetilde{\mathcal{A}}_g^+$. Now it remains to prove that if some line bundle over $\widetilde{\mathcal{A}}_g^+$ is trivial over the general fiber then it is trivial everywhere. If $\widetilde{\mathcal{A}}_g^+$ were represented by a scheme then we could apply the argument from [15], p. 103, to prove this. Since it is not, we have to replace $\widetilde{\mathcal{A}}_g^+$ by a PGL_N-torsor over it which is representable (see remark 1 after Theorem 0.2), apply the cited argument, and use the fact that PGL_N has no non-trivial characters.

The corresponding analytic stack is the quotient (in the sense of stacks) of the Siegel's half-space $\mathfrak{H}_g$ by the subgroup $\Gamma_{1,2} \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ consisting of matrices whose reduction modulo 2 preserves the standard even quadratic form $\sum_{i=1}^{g} x_i y_i$. (cf. [9], VIII, 3.4). It follows that the torsion in the Picard group of this stack is an Abelian group dual to $\Gamma_{1,2}/[\Gamma_{1,2}, \Gamma_{1,2}]$ (cf. [13]). It remains to prove that the latter group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

As is shown in [18], Prop. 8.10, there is a normal subgroup $\Delta \subset \Gamma_{1,2}$ such that $\Gamma_{1,2}/\Delta \simeq \mathbb{Z}/4\mathbb{Z}$. Furthermore, it is shown there that Δ is generated by the matrices of the form $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$ where $A \in \mathrm{SL}_g(\mathbb{Z})$, $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$ where B is integral symmetric $g \times g$ matrix with even diagonal (here we use the standard symplectic basis $e_1, \dots, e_g, f_1, \dots f_g$ such that $(e_i, f_j) = \delta_{i,j}$). We claim that $\Delta \subset [\Gamma_{1,2}, \Gamma_{1,2}]$. For the proof let us introduce the relevant elementary matrices following the notation of [6] 5.3.1. Let $\mathbf{S}_{2g}$ be the set of pairs (i, j) where $1 \leqslant i, j \leqslant 2g$ which are not of the form $(2k-1, 2k)$ or $(2k, 2k-1)$. Then for for every $(i, j) \in \boldsymbol{S}_{2g}$ we

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define an elementary matrix E_{ij} as follows:

$$E_{2k,2l} = \begin{pmatrix} 1 & 0 \\ \gamma_{k,l} & 1 \end{pmatrix},$$

$$E_{2k-1,2l-1} = \begin{pmatrix} 1 & -\gamma_{k,l} \\ 0 & 1 \end{pmatrix},$$

$$E_{2k-1,2l} = \begin{pmatrix} e_{kl} & 0 \\ 0 & e_{lk}^{-1} \end{pmatrix}, \qquad E_{2l,2k-1} = E_{2k-1,2l}$$

where γ_{kl} has zero (α,β)-entry unless $(\alpha,\beta) = (k,l)$ or $(\alpha,\beta) = (l,k)$, in the latter case (α,β)-entry is 1; e_{kl} for $k \neq l$ is the usual elementary matrix with units on the diagonal and at (k,l)-entry and zeros elsewhere. Using these matrices one can say that Δ is generated by $E_{2k-1,l}$ with $k \neq l$, $E_{2k,2l}$ and $E_{2k-1,2l-1}$ with $k \neq l$, and $E_{i,i}^2$ for all $1 \leqslant i \leqslant 2g$. It remains to notice that all the matrices E_{ij} with $i \neq j$ belong to $\Gamma_{1,2}$ and use the following relations (cf. [6] 9.2.13):

(1) $[E_{ij}, E_{kl}] = E_{il}$, if $(j,k) \notin \boldsymbol{S}_{2g}$, j is even, and i, j, k, and l are distinct,
(2) $[E_{ij}, E_{ki}] = E_{ii}^2$, if $(j,k) \notin \boldsymbol{S}_{2g}$, j is even, and i, j, and k are distinct.

□

5.4. CASE OF PRINCIPALLY POLARIZED ABELIAN SURFACES

Let A/S be a relative Abelian surface, L be a symmetric line bundle trivialized along the zero section. Assume also that $d = 1$ that is L gives a principal polarization. Then $L \simeq \mathcal{O}(\Theta) \otimes \pi^*(\pi_* L)$ where $\Theta \subset A$ is theta-divisor.

PROPOSITION 5.7. *Assume that S is smooth. Then one has the following equalities in* Pic(S)*:* $\det \pi_* \mathcal{O}_\Theta = \overline{\omega}_A$, $5 \cdot \overline{\omega}_A = \delta + \Delta'(L^2)$, *where* $\Delta'(L^2) = \det \pi_*(L^2) + 2 \cdot \overline{\omega}_A$, *$\delta$ is the class of the divisor consisting of points $s \in S$ such that Θ_s is singular.*

Proof. First of all, we note that $\det \pi_* \mathcal{O}_\Theta = \det \pi_* \omega_\Theta$ by Serre's duality. Now by adjunction we have $\omega_\Theta = \mathcal{O}_\Theta(\Theta) \otimes \pi^* \overline{\omega}_A$, which implies the first equality due to triviality of $\det \pi_*(\mathcal{O}_A(\Theta))$ and $\det \pi_*(\mathcal{O}_A)$. We also deduce that

$$\det \pi_*(\omega_\Theta^2) = \det \pi_* \mathcal{O}_\Theta(2\Theta) + 6 \cdot \overline{\omega}_A.$$

Next, since $L^2 \simeq \mathcal{O}(2\Theta) \otimes \pi^*(\pi_* L)^2$ we obtain that

$$\det \pi_* \mathcal{O}_A(2\Theta) = \det \pi_* L^2 + 4 \cdot \overline{\omega}_A.$$

The exact sequence

$$0 \to \mathcal{O}_A(\Theta) \to \mathcal{O}_A(2\Theta) \to \mathcal{O}_\Theta(2\Theta) \to 0$$

shows that $\det \pi_* \mathcal{O}_\Theta(2\Theta) = \det \pi_* \mathcal{O}_A(2\Theta)$. Combining it with the above equalities we get

$$\det \pi_*(\omega_\Theta^2) = \det \pi_* L^2 + 10 \cdot \overline{\omega}_A = \Delta'(L^2) + 8 \cdot \overline{\omega}_A.$$

On the other hand, since Θ is a stable curve over S we have according to Mumford's Theorem 5.10 in [15]

$$\det \pi_*(\omega_\Theta^2) = 13 \cdot \det \pi_* \omega_\Theta - \delta = 13 \cdot \overline{\omega}_A - \delta.$$

Comparing this with the previous expression for $\det \pi_*(\omega_\Theta^2)$ we obtain the result. □

Let $\overline{\mathcal{M}}_2$ be the moduli stack of stable curves of genus 2, $\mathcal{M}_2$ be the open substack corresponding to smooth curves, $\mathcal{M}_2'$ be the substack of $\overline{\mathcal{M}}_2$ corresponding to curves which are either smooth or reducible. The Picard groups of these stacks can be described as follows (see [14–16]). $\mathrm{Pic}(\overline{\mathcal{M}}_2)$ is isomorphic to $\mathbb{Z}^2$ and is generated by the classes δ_0, δ_1 and λ, where δ_0 (resp. δ_1) is the class of the divisor of singular irreducible curves (resp. reducible curves), $\lambda = \det \pi_* \omega_{\mathcal{C}}$ where $\pi : \mathcal{C} \to \overline{\mathcal{M}}_2$ is the universal curve, with the only relation

$$10 \cdot \lambda = \delta_0 + 2 \cdot \delta_1. \tag{5.1}$$

It follows that $\mathrm{Pic}(\mathcal{M}_2')$ is generated by λ and δ_1 with the relation $10 \cdot \lambda = 2 \cdot \delta_1$, and $\mathrm{Pic}(\mathcal{M}_2)$ is generated by λ with the relation $10 \cdot \lambda = 0$. Note that the theta divisors Θ_s are either smooth or reducible, so in the above situation we get a morphism $f : S \to \mathcal{M}_2'$ and our computation shows that $\Delta'(L) = f^*(5 \cdot \lambda - \delta_1)$.

COROLLARY 5.8. *In the above situation* $\Delta(L^{2n}) = 0$ *for any n.*

Proof. For $n = 1$ this follows from the triviality of $10 \cdot \lambda - 2 \cdot \delta_1$ in $\mathrm{Pic}(\mathcal{M}_2')$. Now the triviality of $\Delta(L^{2n})$ in general follows from Proposition 5.2. □

Remark. Note that $L^2 \simeq L_\phi := (\mathrm{id}, \phi)^* \mathcal{P}$, where $\phi : A \to \hat{A}$ is the polarization corresponding to L. Hence, $\Delta'(L^2)$ is the pull-back of the line bundle $\Delta'(L_\phi)$ over the moduli stack $\mathcal{A}_2$. The explicit trivialization of $2 \cdot \Delta'(L_\phi) = 10 \cdot \overline{\omega} - 2 \cdot \delta$ in the analytic situation can be found by considering the following modular form of weight 5 on Siegel half-space $\mathfrak{H}_2$ (cf. [16])

$$f(Z) = \prod_{a,b \text{ even}} \theta \begin{bmatrix} a \\ b \end{bmatrix} (0, Z)$$

where the product is taken over all 10 even theta-characteristics. Then f defines a section of $\overline{\omega}^5$ vanishing precisely on the locus $\Delta \subset \mathfrak{H}_2$ corresponding to products of elliptic curves. It is known that f is a modular form for the group $\mathrm{Sp}_4(\mathbb{Z})$ with a non-trivial character $\chi_0 : \mathrm{Sp}_4(\mathbb{Z}) \to \{\pm 1\}$ (such a character is unique, and is obtained from the sign character on $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z}) \simeq S_6$). Thus, f^2 gives the

$\mathrm{Sp}_4(\mathbb{Z})$-equivariant trivialization of $\overline{\omega}^{10}(-2\Delta)$ which descends to a trivialization over $\mathcal{A}_2$. This argument also shows that the element $\Delta'(L_\phi) = 5 \cdot \overline{\omega} - \delta \in \mathrm{Pic}(\mathcal{A}_2)$ is non-trivial. In fact, it generates the torsion subgroup of $\mathrm{Pic}(\mathcal{A}_2)$ (cf. [5]). Furthermore, one can show that pull-backs of $\Delta'(L_\phi)$ to either of two irreducible components of $\widetilde{\mathcal{A}}_2$ are non-trivial. Indeed, it is sufficient to check that the subgroup in $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z}) \simeq S_6$ preserving a quadratic form q on $(\mathbb{Z}/2\mathbb{Z})^4$, such that $q(x+y) + q(x) + q(y)$ is the symplectic form, contains an odd permutation. Recall that the identification of $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z})$ with S_6 is obtained by considering the action on the set of 6 odd quadratic forms q as above. Using this it is easy to compute that the matrix E_{14} (see the proof of Theorem 5.6), preserving the standard even form $q_0 = x_1y_1 + x_2y_2$, corresponds to the product of three transpositions. Similarly, the matrix E_{11} preserves the odd form $x_1y_1 + x_2y_2 + x_2^2 + y_2^2$ and corresponds to a transposition.

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