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L^2-DETERMINANT CLASS AND APPROXIMATION OF L^2-BETTI NUMBERS

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ABSTRACT. A standing conjecture in L^2-cohomology says that every finite CW-complex X is of L^2-determinant class. In this paper, we prove this whenever the fundamental group belongs to a large class $\mathcal{G}$ of groups containing, e.g., all extensions of residually finite groups with amenable quotients, all residually amenable groups, and free products of these. If, in addition, X is L^2-acyclic, we also show that the L^2-determinant is a homotopy invariant — giving a short and easy proof independent of and encompassing all known cases. Under suitable conditions we give new approximation formulas for L^2-Betti numbers.

1. INTRODUCTION

For a finite CW-complex X with fundamental group π, L^2-invariants of the universal covering $\tilde{X}$ are defined in terms of the combinatorial Laplacians Δ_* on $C^*_{(2)}(\tilde{X}) = C^*_{cell}(\tilde{X}) \otimes_{\mathbb{Z}\pi} l^2(\pi)$, which, after the choice of a cellular base, is a finite direct sum of copies of $l^2(\pi)$. In this way

$$\Delta_p = (c_p \otimes \mathrm{id})^*(c_p \otimes \mathrm{id}) + (c_{p-1} \otimes \mathrm{id})(c_{p-1} \otimes \mathrm{id})^*$$

becomes a matrix over $\mathbb{Z}\pi \subset \mathcal{N}\pi$, which acts on $l^2(\pi)^d$ via multiplication from the left. Here $\mathcal{N}\pi$ is the group von Neumann algebra with its natural trace tr_π, defined as follows:

1.1. Definition. For $\Delta = (a_{ij}) \in M(d \times d, \mathcal{N}\pi)$ set

$$\mathrm{tr}_\pi(\Delta) := \sum_i \mathrm{tr}_\pi(a_{ii}),$$

where $\mathrm{tr}_\pi(a) := a_1 = (a, 1)_{l^2(\pi)}$ is the coefficient of the trivial group element if $a = \sum_{g\in\pi} \lambda_g g \in \mathcal{N}\pi \subset l^2(\pi)$.

Particularly important are the spectral density functions

$$F_p(\lambda) := F_{\Delta_p}(\lambda) := \mathrm{tr}_\pi\, \chi_{[0,\lambda]}(\Delta_p). \tag{1.2}$$

The L^2-Betti numbers are defined as

$$b_p^{(2)}(X) := b_p^{(2)}(\Delta_p) := F_{\Delta_p}(0) = \dim_\pi(\ker(\Delta_p)).$$

These are invariants of the homotopy type of X.

Another important invariant is the *regularized determinant*:

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1.3. Definition. For a positive and self-adjoint operator $\Delta \in M(d \times d, \mathcal{N})$ over a finite von Neumann algebra $\mathcal{N}$ with spectral density function F_Δ, define

$$\ln \det_{\mathcal{N}}(\Delta) := \begin{cases} \int_{0^+}^{\infty} \ln(\lambda) dF_\Delta(\lambda) & \text{if the integral converges,} \\ -\infty & \text{otherwise.} \end{cases}$$

Sometimes, this regularized determinant is called the *Fuglede-Kadison determinant.*

This gives rise to the next definition:

1.4. Definition. A self-adjoint operator Δ as above is said to be of π*-determinant class* if and only if

$$\int_{0^+}^{1} \ln \lambda \, dF_\Delta(\lambda) > -\infty.$$

The space X is said to be of *determinant class* if the Laplacian Δ_p is of π-determinant class for every p.

1.5. Conjecture. *Every finite CW-complex is of determinant class.*

If X is of π-determinant class and all L^2-Betti numbers are zero, then we can define its (additive) L^2-Reidemeister torsion

$$T^{(2)}(X) := \sum_p (-1)^p p \ln \det_\pi \Delta_p.$$

Burghelea et al. [2] show that L^2-Reidemeister torsion is equal to L^2-analytical torsion (for closed manifolds) and therefore is a generalization of the volume of a hyperbolic manifold in the following sense: for odd-dimensional compact manifolds whose interior admits a complete hyperbolic metric of finite volume, the analytic torsion is proportional to the volume of the interior [15] with non-zero constant of proportionality [9].

Lück and Rothenberg [14, 3.12] show that this torsion is an invariant of the simple homotopy type of X. They conjecture

1.6. Conjecture. *L^2-Reidemeister torsion is a homotopy invariant.*

They prove [14, 3.11]:

1.7. Theorem. *L^2-Reidemeister torsion is a homotopy invariant of L^2-acyclic finite CW-complexes with fundamental group π if and only if for every $A \in Gl(d \times d, \mathbb{Z}\pi)$ the regularized determinant is zero:* $\ln \det_\pi(A^* A) = 0$.

1.8. *Remark.* In fact, $\ln \det_\pi(\cdot^* \cdot)$ factors through the Whitehead group of π. The corresponding homomorphism is denoted Φ by Lück and Rothenberg, but we will write $\ln \det_\pi$ for the map on $Wh(\pi)$ as well.

1.9. *Remark.* Mathai and Rothenberg [16, 2.5] extend the study of L^2-determinants from the L^2-acyclic to the general case, dealing with determinant lines instead of complex numbers. Without any difficulty, this could be done in our more general situation as well. Because this would only complicate the notation and the generalization is transparent, we will not carry this out.

Suppose that π is residual, i.e. it contains a nested sequence of normal subgroups $\pi = \pi_1 \supset \pi_2 \supset \dots$ such that $\bigcap_i \pi_i = \{1\}$. Then we construct the corresponding coverings X_i of X with fundamental group π_i. L^2-invariants can be defined for arbitrary normal coverings (the relevant von Neumann algebra is that of the group of deck transformations). We conjecture:

1.10. Conjecture. *In the situation just described, for every p, the L^2-Betti numbers $b_p^{(2)}(X_i)$ converge as $i \to \infty$, and*

$$\lim_{i\to\infty} b_p^{(2)}(X_i) = b_p^{(2)}(X).$$

The projections $\pi \to \pi/\pi_i$ induce maps $p_i : M(d \times d, \mathbb{Z}\pi) \to M(d \times d, \mathbb{Z}\pi/\pi_i)$, and the Laplacian on X_i (considered as such a matrix) is just the image of the Laplacian on $\tilde{X}$. Therefore, Conjecture 1.10 follows from the following conjecture about such matrices (and, in fact, is equivalent, as follows from the proof of [13, 2.2]).

1.11. Conjecture. *For $A \in M(d \times d, \mathbb{Z}\pi)$ set $A_i := p_i(A)$. Then*

$$\lim_{i\to\infty} \dim_{\pi/\pi_i}(\ker A_i) = \dim_\pi(\ker A).$$

Conjectures 1.5, 1.6 and 1.10 were proven for residually finite groups by Lück [11]. Using the ideas of Lück in a different context, Dodziuk and Mathai proved Conjecture 1.5 for amenable π [6, 0.2]. They also established an approximation theorem for L^2-Betti numbers of a slightly different type in this case.

Clair [3] proved Conjectures 1.5, 1.6 and 1.10 for residually amenable fundamental groups. To prove Conjecture 1.6 he used [16, Proposition 2.5], where Mathai and Rothenberg gave a proof of 1.6 for amenable π. Unfortunately, this proof is very complicated and not complete (some steps even seem to be wrong). In this paper we will obtain in particular an independent (and much easier) proof of this result (and of the results of Clair).

One can interpret the cohomology of the covering spaces as cohomology with coefficients in $l^2(\pi/\pi_i)$. Michael Farber [7] generalized Lück's results to certain other sequences of finite-dimensional coefficients which converge to $l^2(\pi)$.

The aim of this paper is to extend the above results (except Farber's) to the following larger class of groups.

1.12. Definition. Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following processes:

- If $U < \pi$ is any subgroup such that $U \in \mathcal{G}$ and the discrete homogeneous space π/U admits a π-invariant metric which makes it into an amenable discrete homogeneous space, then $\pi \in \mathcal{G}$.
- If $\pi = \operatorname{dirlim}_{i\in I} \pi_i$ is the direct limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$, too.
- If $\pi = \operatorname{invlim}_{i\in I} \pi_i$ is the inverse limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$, too.
- The class $\mathcal{G}$ is closed under taking subgroups.

Here we use the following definition (compare 4.1):

1.13. Definition. A discrete homogeneous space π/U is called *amenable* if on π/U there is a π-invariant metric $d : \pi/U \times \pi/U \to \mathbb{N}$ such that sets of finite diameter are finite and such that $\forall K > 0, \epsilon > 0$ there is a finite subset $\emptyset \neq X \subset \pi/U$ with

$$|\{x \in \pi/U;\ d(x, X) \le K \text{ and } d(x, \pi/U - X) \le K\}| \le \epsilon\,|X|\,.$$

This is in particular fulfilled if U is a normal subgroup and π/U is an amenable group.

It follows immediately from the definition that $\mathcal{G}$ contains all amenable groups, is closed under directed unions and is residually closed. Section 2 contains more results on the structure of $\mathcal{G}$.

The main theorem of the paper is the following:

1.14. Theorem. *Suppose π belongs to the class $\mathcal{G}$. Then for every CW-complex with fundamental group π which has finitely many cells in each dimension, Conjectures 1.5 and 1.6 are true. The approximation results 1.10 and 1.11 (and generalizations thereof, compare Section 6) are valid under the condition that all the groups which occur belong to $\mathcal{G}$.*

The more general form of the approximation result can be applied to get more cases and new proofs of the following conjecture, which is called the Atiyah conjecture [1]:

1.15. Conjecture. *Suppose π is torsion free. Then $\dim_\pi(\ker A) \in \mathbb{Z}$ whenever $A \in M(d \times d, \mathbb{Z}\pi)$. Equivalently, whenever X is a finite CW-complex with $\pi_1(X) = \pi$, then $b_p^{(2)}(X) \in \mathbb{Z}\ \forall p \in \mathbb{Z}$.*

We will address this in a forthcoming paper. Linnell [10] proved that the Atiyah conjecture is true for abelian groups, for free groups, and for extensions with elementary amenable quotient.

1.16. *Remark.* So far, no example of a countable group which does not belong to the class $\mathcal{G}$ has been constructed. Good candidates for such examples are finitely generated simple groups which are not amenable, e.g. groups containing a free group with two generators.

On the other hand, we cannot give an example of a non-residually-amenable group which belongs to $\mathcal{G}$, either. In any case, our description of $\mathcal{G}$ leads easily to many properties like being closed under direct sums and free products (Proposition 2.7), which (if true at all) are probably much harder to establish for the class of residually amenable groups.

In fact, we prove a little bit more than Theorem 1.14. Namely, we show that the relevant properties are stable under the operations characterizing the class $\mathcal{G}$. We use the following definitions:

1.17. Definition. Let C be any property of discrete groups. It is said to be

- *stable under direct/inverse limits* if C is true for π whenever π is a direct/inverse limit of a directed system of groups which have property C,
- *subgroup stable* if every subgroup $U < \pi$ of a group with property C shares this property, too, and
- *stable under amenable extensions* if π has property C whenever it contains a subgroup U with property C such that the homogeneous space π/U is amenable in the sense of Definition 1.13.

The properties we have in mind are listed in the following definition:

1.18. Definition. Let π be a discrete group. We say that

- π is *of determinant class* if Δ is of π-determinant class for every $\Delta \in M(d \times d, \mathbb{Z}\pi)$ which is positive and self-adjoint;
- π has *semi-integral determinant* if $\ln \det_\pi(\Delta) \geq 0$ for every $\Delta \in M(d \times d, \mathbb{Z}\pi)$ which is positive self-adjoint. In particular, every such Δ is of π-determinant class, i.e. π itself is of determinant class; and

- π has *Whitehead-trivial determinant* if $\ln \det_\pi(A^*A) = 0\ \forall A \in Wh(\pi)$.

1.19. Theorem. *The property "Whitehead-trivial" determinant is stable under direct and inverse limits and is subgroup stable.*

1.20. *Remark.* In the light of this theorem, we can use the fact that the Fuglede-Kadison determinant must be trivial on trivial Whitehead groups, e.g. for every torsion free discrete and cocompact subgroup of a Lie group with finitely many components [8, 2.1]. Waldhausen has shown that the Whitehead group is trivial for another class of groups, including torsion free one-relator groups and many fundamental groups of 3-manifolds [18, 17.5].

The validity of the isomorphism conjecture of Farrell and Jones would imply that the Whitehead group is trivial if π is torsion free.

We can extend $\mathcal{G}$ to the class $\mathcal{G}'$ which is the smallest class of groups containing $\mathcal{G}$ and in addition any class of groups whose Whitehead group is (known to be) trivial, and which is closed under taking subgroups and direct and inverse limits of directed systems. Then every group in $\mathcal{G}'$ has Whitehead trivial determinant; i.e., L^2-torsion is a homotopy invariant for L^2-acyclic finite CW-complexes with such a fundamental group.

1.21. Theorem. *The property "semi-integral determinant" is stable under direct and inverse limits, subgroup stable, and stable under amenable extensions.*

We now show that the complicated proofs of homotopy invariance of L^2-torsion in [11, 16, 3] can be replaced by a very short and easy argument.

1.22. Theorem. *If for a group π and $\forall A \in Wh(\pi)$ we have $\ln \det_\pi(A^*A) \geq 0$, then the Fuglede-Kadison determinant is trivial on $Wh(\pi)$.*

In particular, semi-integral determinant implies Whitehead-trivial determinant.

Proof. $A \in Wh(\pi)$ implies that A has an inverse $B \in Wh(\pi)$. Now by [12, 4.2]

$$0 = \ln \det_\pi(\mathrm{id}) = \ln \det_\pi((AB)^*AB) = \underbrace{\ln \det_\pi(A^*A)}_{\geq 0} + \underbrace{\ln \det_\pi(B^*B)}_{\geq 0},$$

and the statement follows. □

It follows from the induction principle 2.2 and the fact that the trivial group has semi-integral determinant (Lemma 6.8) that 1.21 and 1.22 imply the first part of our main theorem 1.14.

For the reader's convenience, the first section after this introduction could give a more detailed account of L^2-invariants, group von Neumann algebras and their dimension theory. Instead we refer to older sources, e.g. [11] and [6]. One of the reasons is that many results of this paper are obtained by an elaboration of the basic methods of Lück's paper and (to some extent) the paper of Dodziuk and Mathai.

The actual plan of the paper is as follows: In Section 2 we give a closer description of our class $\mathcal{G}$. Sections 3–6 contain the proofs of the theorems stated in this introduction. In Section 7 we discuss the question of how one can generalize the approximation result 1.11 to matrices over the complex group ring. Section 8 gives a slight generalization of the determinant class conjecture 1.5.

2. Properties of the class $\mathcal{G}$

We proceed with another definition of the class $\mathcal{G}$, similar to the description Linnell gave of his class $\mathcal{C}$ in [10, p. 570].

2.1. Definition. For each ordinal α, we define a class of groups $\mathcal{G}_\alpha$ inductively.

- $\mathcal{G}_0$ consists of the trivial groups.
- If α is a limit ordinal, then $\mathcal{G}_\alpha$ is the union of $\mathcal{G}_\beta$ with $\beta < \alpha$.
- If α has a predecessor $\alpha - 1$, then $\mathcal{G}_\alpha$ consists of all groups which
  - are subgroups of groups of $\mathcal{G}_{\alpha-1}$, or
  - contain a subgroup U which belongs to $\mathcal{G}_{\alpha-1}$ such that the quotient space is an amenable homogeneous space as defined in 1.13, or
  - are direct or inverse limits of directed systems of groups in $\mathcal{G}_{\alpha-1}$.

By definition, a group is in $\mathcal{G}$ if it belongs to $\mathcal{G}_\alpha$ for some ordinal α.

The class $\mathcal{G}$ is defined by (transfinite) induction. Therefore, properties of the groups in $\mathcal{G}$ can be proven by induction, too. More precisely, the following induction principle is valid:

2.2. Proposition. *Suppose a property C of groups is shared by the trivial group, and the following are true:*

- *whenever K has property C and $K < \pi$ with π/K an amenable homogeneous space, then π has property C as well;*
- *whenever π is a direct or inverse limit of a directed system of groups with the property C, then π has property C; and*
- *property C is inherited by subgroups.*

Then property C is shared by all groups in the class $\mathcal{G}$.

Proof. The proof of the induction principle is done by transfinite induction.

By assumption, C holds for $\mathcal{G}_0$. We have to establish C for every group in $\mathcal{G}_\alpha$, granted its validity for groups in $\mathcal{G}_\beta$ for all $\beta < \alpha$. If α is a limit ordinal, this is trivial. If α has a predecessor $\alpha - 1$, the assumptions just match the definition of $\mathcal{G}_\alpha$, so the statement follows. $\square$

Now, we study properties of the class $\mathcal{G}$.

2.3. Proposition. *The class $\mathcal{G}$ is closed under directed unions.*

Proof. A directed union is a special case of a directed direct limit. $\square$

2.4. Proposition. *The class $\mathcal{G}$ is residually closed. This means that if π contains a nested sequence of normal subgroups $\pi_1 \supset \pi_2 \supset \dots$ with trivial intersection and if $\pi/\pi_i \in \mathcal{G}$ $\forall i$, then also $\pi \in \mathcal{G}$.*

Proof. The inverse system of groups π/π_i has some inverse limit G. The system of maps $\pi \to \pi/\pi_i$ induces a homomorphism $\pi \to G$. If $g \in \pi$ is mapped to $1 \in G$, then g has to be mapped to $1 \in \pi/\pi_i$ $\forall i$, i.e. $g \in \bigcap_i \pi_i = \{1\}$. As a directed limit, $G \in \mathcal{G}$, and as a subgroup of G, also $\pi \in \mathcal{G}$. $\square$

2.5. Theorem. *If U belongs to $\mathcal{G}$ and $\phi : U \to U$ is any group homomorphism, then the "mapping torus" extension of U with respect to ϕ,*

$$\pi = \langle u \in U, t \mid t^{-1}ut = \phi(u), u \cdot v = (uv); \forall u, v \in U\rangle,$$

also belongs to $\mathcal{G}$ (if ϕ is injective, this is a special example of an HNN-extension).

Proof. There is a canonical projection $\pi \to \mathbb{Z}$ sending $u \in U$ to 0 and t to 1. Denote its kernel by K. We will show that K belongs to $\mathcal{G}$; then so does π because it is an extension of K with amenable quotient $\mathbb{Z}$.

Now K is the direct limit of the sequence

$$U \xrightarrow{\phi} U \xrightarrow{\phi} U \xrightarrow{\phi} U \dots$$

and belongs to $\mathcal{G}$, which is closed under taking direct limits. $\square$

2.6. *Remark.* Although such a mapping-torus extension of a finitely presented group is finitely presented again, the kernel K we used in the proof may very well not even admit a finite set of generators. This is one instance where it is useful to allow arbitrary groups, even if one is only interested in fundamental groups of finite CW-complexes.

For the next property, we use the induction principle.

2.7. Proposition. *$\mathcal{G}$ is closed under forming*

1. *direct sums and direct products,*
2. *arbitrary inverse limits, and*
3. *free products.*

Proof. We have to check the conditions for the induction principle 2.2. Fix $\pi \in \mathcal{G}$.

1. If $U < G$ then $U \times \pi < G \times \pi$. If $G \times \pi \in \mathcal{G}$, the same is true for $U \times \pi$. If G is the (direct or inverse) limit of the directed system of groups G_i, then $G \times \pi$ is the limit of the system $G_i \times \pi$ (compare Lemma 2.9 or 2.10). If (by assumption) $G_i \times \pi \in \mathcal{G}$, then $G \times \pi \in \mathcal{G}$. Finally, if $U < G$, and G/U is amenable, then $U \times \pi < G \times \pi$ with the same amenable quotient. Therefore, $U \times \pi \in \mathcal{G} \implies G \times \pi \in \mathcal{G}$.

   The induction principle (and induction on the number of factors) now implies that $\mathcal{G}$ is closed under direct sums. Since a direct product is the inverse limit of finite direct sums (indexed by the directed system of finite subsets of the indexing set), $\mathcal{G}$ is closed under direct products, too.
2. An arbitrary inverse limit is by construction a subgroup of a direct product. Therefore the first assertion implies the second.
3. The case of free products is a little bit more complicated. As our first step we prove that $*_{i\in I}\mathbb{Z}/4 \in \mathcal{G}$ for every index set I. This is the direct limit of finite free products of copies of $\mathbb{Z}/4$; therefore we have to prove the statement for finite I. Now $*_{i=1}^{n}\mathbb{Z}/4$ is a subgroup of $\mathbb{Z}/4 * \mathbb{Z}/4$ (contained in the kernel of the projection onto one factor by 2.8), and $\mathbb{Z}/4 * \mathbb{Z}/4$ is virtually free by [5, IV.1.9] and [17, I Theorem 7], i.e. an extension of a free and therefore by [4, page 57] residually finite group (the free group) with an amenable (finite) group. Therefore $\mathbb{Z}/4 * \mathbb{Z}/4$ belongs to $\mathcal{G}$, and this implies the first step.

   Next we show that $\pi * (*_{j\in J}\mathbb{Z}/4) \in \mathcal{G}$ for every set J. We prove this using the induction principle. For $\pi = 1$ this is the conclusion of the first step. If π is a limit of $(\pi_i)_{\in I}$, or a subgroup of G, then $\pi * (*_{j\in J}\mathbb{Z}/4)$ is a subgroup of the limit of $\pi_i * (*_{j\in J}\mathbb{Z}/4)$ (compare Lemma 2.9 and Lemma 2.10) or a subgroup of $G * (*_{j\in J}\mathbb{Z}/4)$, and we can use the fact that $\mathcal{G}$ is subgroup-closed. If $U < \pi$ and π/U is amenable, $\pi * (*_{j\in J}\mathbb{Z}/4)$ acts on π/U. We get a new point stabilizer, which is isomorphic to the free product of U with $*_{G/U}(*_{i\in I}\mathbb{Z}/4)$ by Lemma 2.8. Fortunately, the induction hypothesis applies with the free product of an arbitrary number of copies of $\mathbb{Z}/4$.

As the next step we show that $*_{i\in I}\pi \in \mathcal{G}$. This follows (as $*_{i\in I}\pi$ is a direct limit) from the corresponding statement for I finite, and these are subgroups of $\pi * \mathbb{Z}/4$, contained in the kernel of the projection onto $\mathbb{Z}/4$ by Lemma 2.8.

$\pi_1 * \pi_2$ is contained in $(\pi_1 \times \pi_2) * (\pi_1 \times \pi_2)$, and the general statement follows by induction and taking limits. □

In the proof of Proposition 2.7 we have used the following well known lemmas.

2.8. Lemma. *Let K be the kernel of the group homomorphism $q : G \to Q$, and H a group. Then the kernel of $q * 1 : G * H \to Q$ is the free product*

$$(*_{i\in Q} H^{g_i}) * K,$$

where $\{g_i\}_{i\in Q}$ is a system of representatives for $G/K \cong Q$.

Proof. The subgroup generated by the conjugates H^{g_i} and K is contained in the kernel of $q*1$. For the converse, take an element of the kernel and write it in normal form $a_1h_1 \dots a_nh_n$ with $a_k \in G$ and $h_k \in H$. Check that it is possible to rewrite it in a unique way as a product of factors in H^{g_i} and K. □

2.9. Lemma. *If π is the direct limit of a system of groups π_i and G is any group, then $\pi * G$ is the direct limit of $\pi_i * G$, and $\pi \times G$ is the direct limit of $\pi_i \times G$.*

Proof. There are obvious maps from $\pi_i * G$ to $\pi * G$ and from $\pi_i \times G$ to $\pi \times G$.

Suppose one has consistent maps from $\pi_i * G$ (or $\pi_i \times G$) to some group X. Since π_i and G both are subgroups of $\pi_i * G$ (or of $\pi_i \times G$), this means that we have a consistent family of maps on π_i multiplied with a fixed map on G. Therefore (from the properties of products) there exists exactly one map from $\pi * G$ (or $\pi \times G$) to X making all the diagrams commutative (for the abelian product, note that the union of the images of the π_i in X commutes with the image of G). □

2.10. Lemma. *If π is an inverse limit of a system of groups π_i and G is any group, then $\pi * G$ is contained in the inverse limit X of $\pi_i * G$.*

The inverse limit of $\pi_i \times G$ is $\pi \times G$.

Proof. First, we look at the free products:

We have a consistent family of homomorphisms from $\pi * G$ to $\pi_i * G$, therefore a homomorphism from $\pi * G$ to X. An element $x = p_1g_1 \dots p_ng_n \in \pi * G$ with $p_i \in \pi$ and $g_i \in G$ is in the kernel of this homomorphism if and only if it is mapped to $1 \in \pi_i * G$ for every $i \in I$. This cannot happen if $1 \neq p_1 \in \pi$. It remains to check the case $1 \neq g_1 \in G$. We may assume that $g_2 \neq 1$ if and only if $p_2 \neq 1$. If $\phi_i : \pi \to \pi_i$ is the natural homomorphism, then x is mapped to $\phi_i(p_1)g_1\phi_i(p_2)g_2 \dots g_n \in \pi_i * G$. If this is trivial, but $g_1 \neq 1$, necessarily $\phi_i(p_2) = 1\ \forall i$. This implies that $p_2 = 1$, i.e. $x = p_1g_1$, since we wrote x in normal form. But then $(\phi_i * \mathrm{id})(x) \neq 1\ \forall i$, and the kernel of the map to X is trivial, as required.

For the abelian product, let X be a group together with a consistent family of morphisms to $\pi_i \times G$. These have the form $x \mapsto (\phi_i(x), f_i(x))$. Composition with the projections to π_i or to G shows that ϕ_i is a consistent family of morphisms to π_i, and $f = f_i : X \to G$ all coincide. Let $\phi : X \to \pi$ be the limit. Then $\phi \times f : X \to \pi \times G$ is a unique homomorphism which makes all relevant diagrams commutative. By the universal property of inverse limits, the statement follows. □

3. Passage to subgroups

Suppose $U \subset \pi$ is a subgroup of a discrete group. A positive self-adjoint matrix $A \in M(d\times d, \mathbb{Z}U)$ can also be considered as a matrix over $\mathbb{Z}\pi$. Denote the operators by A_U and A_π, respectively. Recall the following well known fact:

3.1. Proposition. *The spectral density functions of A_U and A_π coincide.*

Proof. Choose a set of representatives $\{g_i\}_{i\in I}$ for $U\backslash\pi$ with $0 \in I$ and $g_0 = 1$, to write $\pi = \coprod_{i\in I} Ug_i$. Then

$$l^2(\pi)^d = \bigoplus_{i\in I} l^2(U)^d g_i.$$

With respect to this splitting, the action of A_π on $l^2(\pi)$ is diagonal and, restricted to each of the summands $l^2(U)^d g_i$, is multiplication by A_U from the left. It follows that every spectral projection $\chi_{[0,\lambda]}(A_\pi)$ is diagonal with $\chi_{[0,\lambda]}(A_U)$ on the diagonal. Then (where $e_k = (0,\ldots,0,1,0,\ldots,0)$)

$$\begin{aligned} F_{A_\pi}(\lambda) &= \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_\pi)e_k^\pi, e_k^\pi\rangle = \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_\pi)e_k^U\cdot 1, e_k^U\cdot 1\rangle \\ &= \sum_{k=1}^d \langle \chi_{[0,\lambda]}(A_U)e_k^U, e_k^U\rangle = F_{A_U}(\lambda). \qquad \square \end{aligned}$$

3.2. Corollary. *The properties of Definition 1.18 are inherited by subgroups.*

In particular, we have proven the subgroup part of Theorems 1.19 and 1.21.

4. Amenable extensions

4.1. Definition. A discrete homogeneous space π/U is called *amenable* if on π/U there is a π-invariant integer-valued metric $d : \pi/U \times \pi/U \to \mathbb{N}$ such that

- sets of finite diameter are finite, and
- for every $K > 0$, $\epsilon > 0$ there is a non-empty finite subset $\emptyset \neq X \subset \pi/U$ with
$$|N_K(X)| \leq \epsilon\,|X|\,,$$
where $N_K(X) := U_K(X)\cap U_K(\pi/U - X)$ is the K-neighborhood of the boundary of X (with $U_K(X) := \{x \in \pi/U;\ d(x,X) \leq K\}$).

A nested sequence of finite subsets $K_1 \subset K_2 \subset \ldots$ is called an *amenable exhaustion* of π/U if $\bigcup K_n = \pi/U$ and if for all $K > 0$ and $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $|N_K(K_i)| \leq \epsilon\,|K_i|\ \forall i \geq N$.

4.2. Lemma. *Every amenable homogeneous space π/U admits an amenable exhaustion.*

Proof. For $n, K \in \mathbb{N}$ we find $X_{n,K}$ with $|N_K(X_{n,K})| \leq \frac{1}{n}\,|X_{n,K}|$. Fix some base point $o \in \pi/U$. Since π acts transitively on π/U and the metric is π-invariant, we may assume after translation that the base point is contained in each of the $X_{n,K}$. Now we construct the exhaustion E_i inductively. Set $E_1 := X_{1,1}$. For the induction, suppose $E_1,\ldots,E_n$ are constructed with $|N_k(E_k)| \leq \frac{1}{k}\,|E_k|$ for $k = 1,\ldots,n$. Suppose $E_n \subset U_d(o)$ with $d \in \mathbb{N}$ and $d \geq n+1$. Set $E_{n+1} := E_n \cup U_n(o) \cup X_{n+1,2d}$. Then $U_d(E_{n+1}) \subset U_{2d}(X_{n+1,2d})$, by the triangle inequality and because

$o \in X_{n+1,2d}$. Moreover, $U_d(\pi/U - E_{n+1}) \subset U_{2d}(\pi/U - X_{n+1,2d})$. Consequently $N_{n+1}(E_{n+1}) \subset N_d(E_{n+1}) \subset N_{2d}(X_{n+1,2d})$. It follows that

$$|N_{n+1}(E_{n+1})| \le |N_{2d}(X_{n+1,2d})| \le \frac{1}{n+1}\,|X_{n+1,2d}| \le \frac{1}{n+1}\,|E_{n+1}|\,.$$

Since $U_n(o) \subset E_{n+1}$ we also have $\bigcup_{n\in\mathbb{N}} E_n = \pi/U$. The claim follows. $\square$

4.3. Example. If U is a normal subgroup and π/U is an amenable group, it is an amenable homogeneous space.

4.4. Definition. Suppose π is a group with subgroup U and amenable quotient π/U. Choose an amenable exhaustion $X_1 \subset X_2 \subset \cdots \subset \pi/U$. For $B \in M(d \times d, \mathcal{N}U)$ set

$$\operatorname{tr}_m(B) := \frac{1}{|X_m|}\operatorname{tr}_U(B).$$

For $\Delta \in M(d\times d, \mathcal{N}\pi)$ positive and self-adjoint, set $\Delta_m := P_m \Delta P_m$, where $P_m = \operatorname{diag}(p_m)$ with $p_m \in \mathcal{B}(l^2(\pi))$ given by projection onto the closed subspace generated by the inverse image of X_m. Then Δ_m no longer belongs to $\mathcal{N}\pi$ but still belongs to $\mathcal{N}U$, and we define (by slight abuse of notation)

$$F_{\Delta_m}(\lambda) := \operatorname{tr}_m(\chi_{[0,\lambda]}(\Delta_m)),$$
$$\ln\det{}_U(\Delta_m) := \int_{0^+}^{\infty} \ln(\lambda)\, dF_{\Delta_m}(\lambda) \qquad \text{using the new } F_{\Delta_m}.$$

Here Δ_m is considered as an operator on the image of P_m. This subspace is $\mathcal{N}U$-isomorphic to $l^2(U)^{d|X_m|}$.

Note that there are two meanings of $F_{\Delta_m}(\lambda)$ and $\ln\det_U(\Delta_m)$ (using either tr_U or tr_m), but in the amenable case we will always use the variant where we divide by the volume of the sets X_m.

The following is one of the key lemmas which make our method work:

4.5. Lemma. *In the situation above, there is a $K \in \mathbb{R}$, independent of m, so that*

$$\|\Delta\| \le K \qquad \textit{and} \qquad \|\Delta_m\| \le K \qquad \forall m \in \mathbb{N}.$$

Proof. This is an immediate consequence of the fact that $\|P\| \le 1$ for every projection P and $\Delta_m = P_m \Delta P_m$ with projections P_m. $\square$

We now establish the second key lemma. It generalizes a corresponding result of Dodziuk and Mathai [6, 2.3], where U is trivial. We need the result only for matrices over $\mathbb{C}\pi$, but for possible other applications we prove a more general statement here.

4.6. Lemma. *Let $p(x) \in \mathbb{C}[x]$ be a polynomial. Suppose $\Delta \in M(d\times d, \mathcal{N}\pi)$. Then*

$$\operatorname{tr}_\pi p(\Delta) = \lim_{m\to\infty} \operatorname{tr}_m p(\Delta_m).$$

Proof. By linearity it suffices to prove the statement for the monomials x^N, $N \in \mathbb{N}$.

Pull the metric on π/U back to π to get some semi-metric on π. Denote the inverse image of X_k in π by X'_k.

We have to compare $(\Delta^N g e_k, g e_k)$ and $(\Delta_m^N g e_k, g e_k)$ for $g = g_i \in X'_m$, in particular for those g_i with $B_a(g_i) \subset X'_m$. Of course, we don't find $a \in \mathbb{R}$ such that the difference is zero. However, we will show that (for fixed N) we can find a such that the difference is sufficiently small.

First observe that $P_m g e_k = g e_k$ if $g \in X'_m$, and (since P_m is self-adjoint)

$$((P_m \Delta P_m)^N g e_k, g e_k) = (\Delta P_m \Delta \dots P_m \Delta g e_k, g e_k).$$

Now the following sum is a telescope, and therefore

$$\begin{aligned}\Delta P_m \Delta \dots P_m \Delta = \Delta^N - \Delta(1-P_m)\Delta^{N-1} \\ - \Delta P_m \Delta (1-P_m)\Delta^{N-2} - \dots - \Delta P_m \dots \Delta(1-P_m)\Delta.\end{aligned} \tag{4.7}$$

It follows that, for $g \in X'_m$,

$$\begin{aligned}\left|(\Delta^N g e_k, g e_k) - (\Delta_m^N g e_k, g e_k)\right| &\le \sum_{i=1}^{N-1} \left|((1-P_m)\Delta^i g e_k, (\Delta^* P_m)^{N-i} g e_k)\right| \\ &\le \sum_{i=1}^{N-1} \left|(1-P_m)\Delta^i g e_k\right| \cdot \|\Delta^*\|^{N-i}.\end{aligned} \tag{4.8}$$

Here we used the fact that the norm of a nontrivial projector is 1 and $|g e_k| = 1$.

Fix $\epsilon > 0$. For $i = 1, \dots, N-1$ and $k = 1, \dots, d$ we have $\Delta^i g e_k \in l^2(\pi)^d$. It follows that we can find an $R > 0$ so that

$$\left|(1 - P_{B_R(g)})\Delta^i g e_k\right| \le \epsilon, \tag{4.9}$$

where $P_{B_R(g)}$ is the projector onto the closed subspace spanned by the elements in $\bigcup_{k=1}^d B_R(g) e_k$. Since Δ and the semi-metric are π-invariant, this holds for every $g \in \pi$ with R independent of g. If the range of $P_{B_R(g)}$ is contained in the range of P_m, i.e. if $B_R(g) \in X'_m$, then (4.9) implies

$$\left|(1 - P_m)\Delta^i g e_k\right| \le \epsilon \tag{4.10}$$

(since we replace by zero even more Fourier coefficients (in the standard orthonormal base of $l^2(\pi)^d$ coming from π)). Using (4.8) and (4.10), we get

$$\begin{aligned}
&\left|\operatorname{tr}_\pi \Delta^N - \operatorname{tr}_m \Delta_m^N\right| \le \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m} \left|(\Delta^N g_i e_k, g_i e_k) - ((\Delta_m)^N g_i e_k, g_i e_k)\right| \\
&\le \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m} \sum_{j=1}^{N-1} \left|(1-P_m)\Delta^j g_i e_k\right| \cdot \|\Delta^*\|^{N-j} \\
&\le \frac{1}{|X_m|} \sum_{k=1}^d \sum_{i \in X_m - N_R(X_m)} \sum_{j=1}^{N-1} \underbrace{\left|(1-P_m)\Delta^j g_i e_k\right|}_{<\epsilon \text{ since } B_R(g_i) \subset X'_m \text{ by def. of } N_R(X_m)} \|\Delta^*\|^{N-j} \\
&\quad + \frac{1}{|X_m|} \sum_{i \in N_R(X_m)} \sum_{k=1}^d \sum_{j=1}^{N-1} \left|(1-P_m)\Delta^j g_i e_k\right| \cdot \|\Delta^*\|^{N-j} \\
&\le \epsilon \underbrace{dN \max_{j=1,\dots,N-1} \{\|\Delta^*\|^j\}}_{=:C_N} + \frac{|N_R(X_m)|}{|X_m|} \underbrace{dN \underbrace{\|1-P_m\|}_{\le 2} \max_{j=1\dots N} \{\|\Delta\|^j \cdot \|\Delta^*\|^{N-j}\}}_{=:C'_N}.
\end{aligned}$$

Note that C_N and C'_N are independent of m and ϵ. Since X_m is an amenable exhaustion, for every R there is an m_R so that $\frac{|N_R(X_m)|}{|X_m|}$ is smaller than ϵ for every $m \ge m_R$. Since ϵ was arbitrary, the lemma follows. $\square$

5. Direct and inverse limits

5.1. *Remark.* In this section, we study the properties of Definition 1.18 for direct and inverse limits. However, we will only deal with the apparently weaker statements that each of the conditions holds for every Δ of the form $\Delta = A^*A$. The general case is a consequence of this since we can easily compare the self-adjoint Δ with $\Delta^2 = \Delta^*\Delta$, because $F_\Delta(\lambda) = F_{\Delta^2}(\lambda^2)$.

Now we describe the situation we are dealing with in this section:

5.2. Definition. Suppose the group π is the direct or inverse limit of a directed system of groups π_i, $i \in I$. The latter means that we have a partial ordering $<$ on I, and $\forall i, j \in I$ we can find $k \in I$ with $i < k$ and $j < k$. In the case of a direct limit, let $p_i : \pi_i \to \pi$ be the natural maps; in the case of an inverse limit, $p_i : \pi \to \pi_i$.

Suppose $A \in M(d \times d, \mathbb{C}\pi)$ is given.

If π is an inverse limit, let $A_i = p_i(A)$ be the image of A under the projection $M(d,\pi) \to M(d,\pi_i)$. Set $\Delta := A^*A$. Then $\Delta_i = (A_i)^*A_i$ (this follows from the algebraic description of the adjoint [11, p. 465]). In particular, all of the operators Δ_i are positive. Define

$$\operatorname{tr}_i(\Delta_i) := \operatorname{tr}_{\pi_i}(\Delta_i).$$

$F_{\Delta_i}(\lambda)$ is defined using the trace on the von Neumann algebra of π_i $(i \in I)$.

If we want to give a similar definition in the case where π is a direct limit, we have to make additional choices. Namely, let $A = (a_{kl})$ with $a_{kl} = \sum_{g\in\pi} \lambda_{kl}^g g$. Then only finitely many of the λ_{kl}^g are nonzero. Let V be the corresponding finite collection of $g \in \pi$. Since π is the direct limit of π_i, we find $j_0 \in I$ such that $V \subset p_{j_0}(\pi_{j_0})$. Choose an inverse image for each g in π_{j_0}. This gives a matrix $A_{j_0} \in M(d \times d, \pi_{j_0})$, which is mapped to $A_i \in M(d \times d, \pi_i)$ for $i > j_0$. Now we apply the above constructions to this net $(A_i)_{i>j_0}$. Note that this definitely depends on the choices.

For notational convenience, we choose some $j_0 \in I$ also when we deal with an inverse limit.

Now, we will establish in this situation the two key lemmas corresponding to Lemma 4.5 and Lemma 4.6.

For the first lemma, instead of working with the norm of operators, we will use another invariant which gives an upper bound for the norm but is much easier to read off:

5.3. Definition. Let π be a discrete group, and let $\Delta \in M(d \times d, \mathbb{Z}\pi)$. Set

$$K(\Delta) := d^2 \max_{i,j}\{|a_{i,j}|_1\}, \quad \text{where } |\cdot|_1 \text{ is the } L^1\text{-norm on } \mathbb{C}\pi \subset l^1(\pi).$$

5.4. Lemma. *Adopt the situation of Definition 5.2. One can find $K \in \mathbb{R}$, independent of i, such that*

$$\|A_i\| \le K \quad \forall i > j_0 \qquad \text{and} \qquad \|A\| \le K.$$

Proof. Lück [11, 2.5] showed that $\|A_i\| \le K(A_i)$. It follows from the construction of A_i that $K(A_i) \le K(A)$ in the case of an inverse limit, and $K(A_i) \le K(A_{j_0})$ in the case of a direct limit, with j_0 as above. In both cases, we obtain a uniform bound for $\|A_i\|$. $\square$

5.5. Lemma. *Adopt the situation of Definition 5.2. Let $p(x) \in \mathbb{C}[x]$ be a polynomial. There exists $i_0 \in I$, depending on the matrix A and on p, such that*

$$\operatorname{tr}_\pi(p(A)) = \operatorname{tr}_i(p(A_i)) \qquad \forall i > i_0.$$

Proof. Suppose π is the inverse limit of the π_i. We follow [11, 2.6]. Let $p(A) = \left(\sum_{g\in\pi} \lambda_g^{kl} g\right)_{k,l=1,\dots,d}$. Then

$$\operatorname{tr}_\pi(p(A)) = \sum_k \lambda_1^{kk} \qquad \text{and} \qquad \operatorname{tr}_{\pi_i}(p(A_i)) = \sum_k \sum_{g\in\ker p_i} \lambda_g^{kk}.$$

Since only finitely many $\lambda_g^{ij} \neq 0$ and π is the inverse limit of the π_i, we find $i_0 \in I$ such that $\lambda_g^{kk} \neq 0$ and $g \in \ker p_{i_0}$ implies $g = 1$. For $i > i_0$ the assertion is true.

If π is the direct limit, we have chosen A_{j_0} with $p_{j_0}(A_{j_0}) = A$. Then $p_{j_0}(p(A_{j_0})) = p(A)$. However, there may be a $g \in \pi_{j_0}$ with nontrivial coefficient in $p(A_{j_0})$ with $p_{j_0}(g) = 1$, and this means that the relevant traces may differ. But still there are only finitely many g with nontrivial coefficient in $p(A_{j_0})$, and since π is the direct limit of $(\pi_i)_{i>j_0}$ we find an i_0 such that $p_{i_0}(g) = 1$ for $g \in \pi_{i_0}$ with nontrivial coefficient in $p(A_{i_0})$ implies $g = 1$. Then the above reasoning shows that $\operatorname{tr}_{\pi_i}(p(A_i)) = \operatorname{tr}_\pi(p(A))\ \forall i > i_0$. □

6. Approximation properties and proof of stability statements

In this section, we use the information gathered so far to prove the statements of the introduction, in particular Theorem 1.21. This is done by studying limits of operators; therefore the same treatment yields approximation results for L^2-Betti numbers. The precise statements and conditions are given below.

We have already dealt with the passage to subgroups.

For the rest of this section, assume the following situation:

6.1. Situation. The group π is the direct or inverse limit of a directed system of groups π_i, or an amenable extension $U \to \pi \to \pi/U$ (write $\pi_i = U$ also in this case).

As described in 4.4 or 5.2, any matrix Δ over $\mathbb{C}\pi$ then gives rise to matrices Δ_i over π_i (after the choice of an inverse image in the case of a direct limit, and after the choice of an amenable exhaustion for amenable extensions). Without loss of generality we assume that $\Delta = A^*A$ for another matrix A over $\mathbb{C}\pi$ (this is explained in Remark 5.1). We also get spectral density functions $F_{A_i}(\lambda)$ defined using the group π_i (remember that in the amenable case there is an additional normalization).

The problem now is to obtain information about $F_\Delta(\lambda)$ from the family $F_{\Delta_i}(\lambda)$.

In particular, we want to show that $F_{A_i}(0)$ converges to $F_A(0)$. (Translated to geometry, this means that certain L^2-Betti numbers converge.)

In short, we have

- a group π and a matrix $\Delta \in M(d\times d, \mathbb{C}\pi)$,
- a family $(\Delta_i)_{i\in I}$ of matrices over $\mathbb{C}\pi_i$ which approximate Δ (I is a directed system),
- positive and normal trace functionals tr_i (on a von Neumann algebra containing Δ_i) which are normalized in the sense that if $\Delta = \mathrm{id} \in M(d\times d, \mathbb{Z}\pi)$, then $\operatorname{tr}_i(\Delta_i) = d\ \forall i$
- If Δ lives over $\mathbb{Z}\pi$, then Δ_i is a matrix over $\mathbb{Z}\pi_i$.

6.2. Definition. Define

$$\overline{F_\Delta}(\lambda) := \limsup_{i\in I} F_{\Delta_i}(\lambda),$$
$$\underline{F_\Delta}(\lambda) := \liminf_{i\in I} F_{\Delta_i}(\lambda).$$

Remember that

$$\limsup_{i\in I}\{x_i\} = \inf_{i\in I}\{\sup_{j>i}\{x_j\}\}, \qquad \liminf_{i\in I}\{x_i\} = \sup_{i\in I}\{\inf_{j>i}\{x_j\}\}.$$

6.3. Definition. Suppose $F : [0,\infty) \to \mathbb{R}$ is monotone increasing (e.g. a spectral density function). Then set

$$F^+(\lambda) := \lim_{\epsilon\to 0^+} F(\lambda+\epsilon),$$

i.e., F^+ is the right continuous approximation of F. In particular, we have defined $\overline{F_\Delta}^+$ and $\underline{F_\Delta}^+$.

6.4. *Remark.* Note that by our definition a spectral density function is right continuous, i.e. unchanged if we perform this construction.

To establish the first step in our program we have to establish the following functional analytical lemma (compare [11] or [3]):

6.5. Lemma. *Let $\mathcal{N}$ be a finite von Neumann algebra with positive normal and normalized trace $\operatorname{tr}_{\mathcal{N}}$. Choose $\Delta \in M(d\times d,\mathcal{N})$ positive and self-adjoint.*

If for a function $p_n : \mathbb{R} \to \mathbb{R}$

$$\chi_{[0,\lambda]}(x) \le p_n(x) \le \frac{1}{n}\chi_{[0,K]}(x) + \chi_{[0,\lambda+1/n]}(x) \qquad \forall 0 \le x \le K \tag{6.6}$$

and if $\|\Delta\| \le K$, then

$$F_\Delta(\lambda) \le \operatorname{tr}_{\mathcal{N}} p_n(\Delta) \le \frac{1}{n}d + F_\Delta(\lambda+1/n).$$

Here $\chi_S(x)$ is the characteristic function of the subset $S \subset \mathbb{R}$.

Proof. This is a direct consequence of positivity of the trace, of the definition of spectral density functions, and of the fact that $\operatorname{tr}_{\mathcal{N}}(1 \in M(d\times d,\mathcal{N})) = d$ by the definition of a normalized trace. □

Now we give one of the key technical results. Corresponding special cases are [11, 2.3(1)], [6, 2.1(1)], [3, 2.1], and the proof is essentially always the same.

6.7. Proposition. *For every $\lambda \in \mathbb{R}$ we have*

$$\overline{F_\Delta}(\lambda) \le F_\Delta(\lambda) = \underline{F_\Delta}^+(\lambda) \le \underline{F_\Delta}^+(\lambda),$$
$$F_\Delta(\lambda) = \underline{F_\Delta}^+(\lambda) = \overline{F_\Delta}^+(\lambda).$$

Proof. The proof only depends on our key lemmas 4.5, 4.6, 5.4, 5.5. These say

- $\exists K \in \mathbb{R}$ such that $\|\Delta_i\| \le K\ \forall i \in I$, and
- for every polynomial $p \in \mathbb{C}[x]$ we have $\operatorname{tr}_\pi(p(\Delta)) = \lim_i \operatorname{tr}_i(p(\Delta_i))$.

For each $\lambda \in \mathbb{R}$ choose polynomials $p_n \in \mathbb{R}[x]$ such that inequality (6.6) is fulfilled. Note that by the first key lemma we have a uniform upper bound K for the spectrum of all of the Δ_i. Then by Lemma 6.6

$$F_{\Delta_i}(\lambda) \le \operatorname{tr}_i(p_n(\Delta_i)) \le F_{\Delta_i}(\lambda+\frac{1}{n}) + \frac{d}{n}$$

We can take the limits inferior and superior and use the second key lemma to get

$$\overline{F_\Delta}(\lambda) \leq \mathrm{tr}_\pi(p_n(\Delta)) \leq \underline{F_\Delta}(\lambda + \frac{1}{n}) + \frac{d}{n}.$$

Now we take the limits for $n \to \infty$. We use the fact that tr_π is normal and $p_n(\Delta)$ converges strongly inside a norm bounded set to $\chi_{[0,\lambda]}(\Delta)$. Therefore the convergence holds even in the ultra strong topology.

This implies that

$$\overline{F_\Delta}(\lambda) \leq F_\Delta(\lambda) \leq \underline{F_\Delta}^+(\lambda).$$

For $\epsilon > 0$ we can now conclude, since $\underline{F_\Delta}$ and $\overline{F_\Delta}$ are monotone, that

$$F_\Delta(\lambda) \leq \underline{F_\Delta}(\lambda + \epsilon) \leq \overline{F_\Delta}(\lambda + \epsilon) \leq F_\Delta(\lambda + \epsilon).$$

Taking the limit as $\epsilon \to 0^+$ gives (since F_Δ is right continuous)

$$F_\Delta(\lambda) = \overline{F_\Delta}^+(\lambda) = \underline{F_\Delta}^+(\lambda).$$

Therefore both of the inequalities are established. □

The next step is to prove convergence results without taking right continuous approximations (at least for $\lambda = 0$). We are able to do this only under additional assumptions:

- From now on, Δ, and therefore also Δ_i $\forall i \in I$, are matrices over the integral group ring.

The following well known statement is used as start for the induction (observed e.g. in [11]).

6.8. Lemma. *The trivial group has semi-integral determinant.*

Proof. Take $\Delta \in M(d \times d, \mathbb{Z})$ positive and self-adjoint. Then $\det_1(\Delta)$ is the product of all nonzero eigenvalues, and therefore the lowest nonzero coefficient in the characteristic polynomial. In particular, it is an integer $\neq 0$, and $\ln \det_1(\Delta) \geq 0$. □

Now we give a proof of Theorem 1.21 and prove the corresponding approximation result.

6.9. Theorem. *Suppose π_i has semi-integral determinant $\forall i \in I$. Then the same is true for π, and* $\dim_\pi(\ker \Delta) = F_\Delta(0) = \lim_i F_{\Delta_i}(0)$.

Proof. Choose $K \in \mathbb{R}$ such that $K > \|\Delta\|$ and $K > \|\Delta_i\|$ $\forall i$. This is possible because of Lemma 4.5 or 5.4. Then

$$\ln \det_{\pi_i}(\Delta_i) = \ln(K)(F_{\Delta_i}(K) - F_{\Delta_i}(0)) - \int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda.$$

If this is (by assumption) ≥ 0, then, since $F_{\Delta_i}(K) = \mathrm{tr}_i(\mathbf{1}_d) = d$ by our normalization,

$$\int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda \leq \ln(K)(d - F_{\Delta_i}(0)) \leq \ln(K)d.$$

We want to establish the same estimate for Δ. If $\epsilon > 0$, then

$$\int_\epsilon^K \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda}\, d\lambda = \int_\epsilon^K \frac{\underline{F_\Delta}^+(\lambda) - F_\Delta(0)}{\lambda}\, d\lambda = \int_\epsilon^K \frac{\underline{F_\Delta}(\lambda) - F_\Delta(0)}{\lambda}$$

(since the integrand is bounded, the integral over the left continuous approximation is equal to the integral over the original function)

$$\begin{aligned}
&\le \int_\epsilon^K \frac{\underline{\underline{F_\Delta}}(\lambda) - \overline{F_\Delta}(0)}{\lambda}\\
&= \int_\epsilon^K \frac{\liminf_i F_{\Delta_i}(\lambda) - \limsup_i F_{\Delta_i}(0)}{\lambda}\\
&\le \int_\epsilon^K \frac{\liminf_i (F_{\Delta_i}(\lambda) - F_{\Delta_i}(0))}{\lambda}\\
&\le \liminf_i \int_\epsilon^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \le d\ln(K).
\end{aligned}$$

Since this holds for every $\epsilon > 0$, we even have

$$\begin{aligned}
\int_{0^+}^K \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda} \le& \int_{0^+}^K \frac{\underline{\underline{F_\Delta}}(\lambda) - \overline{F_\Delta}(0)}{\lambda}\, d\lambda\\
&\le \sup_{\epsilon>0} \liminf_i \int_\epsilon^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda\\
&\le d\ln(K).
\end{aligned}$$

The second integral would be infinite if $\lim_{\delta\to 0} \underline{F_\Delta}(\delta) \neq \overline{F_\Delta(0)}$. It follows from 6.7 that $\limsup_i F_{\Delta_i}(0) = F_\Delta(0)$. Since we can play the same game for every subnet of I, also $\liminf_i F_{\Delta_i}(0) = F_\Delta(0)$, i.e. the approximation property is true.

To estimate the determinant, note that in the above inequality

$$\begin{aligned}
&\sup_{\epsilon>0} \liminf_i \int_\epsilon^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda\\
&\qquad \le \liminf_i \sup_{\epsilon>0} \int_\epsilon^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda = \liminf_i \int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda\\
&\qquad \le \ln(K)(d - F_{\Delta_i}(0)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\ln \det_\pi(\Delta) &= \ln(K)(d - F_\Delta(0)) - \int_{0^+}^K \frac{F_\Delta(\lambda) - F_\Delta(0)}{\lambda}\, d\lambda\\
&\ge \ln(K)(d - \lim_i F_{\Delta_i}(0)) - \liminf_i \int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda\\
&= \limsup_i \left(\ln(K)(d - F_{\Delta_i}(0)) - \int_{0^+}^K \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, d\lambda \right)\\
&= \limsup_i \ln \det_{\pi_i}(\Delta_i) \ge 0. \qquad \square
\end{aligned}$$

6.10. *Remark.* We can establish the approximation results only under the assumption that the groups π_i have good properties, e.g. belong to the class $\mathcal{G}$. It is interesting to note that for amenable groups every quotient group is amenable and belongs to $\mathcal{G}$. It follows that Conjecture 1.10 holds without additional assumptions if $\pi_1(X)$ is amenable.

6.11. *Remark.* It would be possible to give geometrical interpretations of the more general approximation results similar to 1.10. However, this seems to be rather artificial and therefore is omitted here.

Proof of Theorem 1.19. It remains to show that the property "Whitehead-trivial determinant" is stable under direct and inverse limits.

We are still in the situation described at the beginning of this section, and, in addition, we assume that Δ is invertible in $M(d\times d,\mathbb{Z}\pi)$ with inverse $B\in M(d\times d,\mathbb{Z}\pi)$. If π is an inverse limit then Δ_i and B_i are images of projections of Δ and B, and therefore remain inverse to each other.

In case π is a direct limit, we first lift Δ to some Δ_{j_0}. We may assume that we also can lift B to B_{j_0}. Then $\Delta_{j_0}B_{j_0}$ is mapped to the identity over π. Since it has only finitely many nonzero coefficients, there is j_1 such that the image of $\Delta_{j_0}B_{j_0}$ over π_{j_1} already is the identity, and similarly for $B_{j_0}\Delta_{j_0}$. Therefore, we may assume that the lifts Δ_{j_0} and B_{j_0} are inverse to each other. The same is then true for Δ_i and B_i for $i>j_0$, i.e. Δ_i represents an element in $Wh(\pi_i)$.

By assumption, $\ln\det_{\pi_i}(\Delta_i^*\Delta_i)=0$. Note that the proof of Theorem 6.9 applies to our situation, and we conclude that $\ln\det_\pi(\Delta^*\Delta)\geq 0$. Since $\Delta\in Wh(\pi)$ was arbitrary, Theorem 1.22 implies the result. □

6.12. *Remark.* It is not possible to proceed along similar lines in the case when U has an amenable quotient (even if π is amenable itself). The problem is that we approximate the matrix Δ (over $\mathbb{Z}\pi$) by matrices over $\mathbb{Z}U$ of larger and larger dimension. One can show that these matrices are invertible over $\mathcal{N}U$ if Δ itself was invertible. However, even if the inverse of Δ is a matrix over $\mathbb{Z}\pi$, in general this is not true for the approximating matrices over $\mathbb{Z}U$.

This finishes the proof of Theorems 1.19, 1.21 and 1.14. We conclude with some side remarks.

7. Complex approximation

In this section, we will address the question of whether the approximation results we obtained in section 6 are valid not only for matrices over the integral group ring, but also over the complex group ring. In particular, we adopt Situation 6.1: A group π is approximated by groups π_i, and a matrix $\Delta=A^*A$ by matrices Δ_i.

Essentially, we will give a positive answer to our question only for free abelian groups. We start with a general observation.

7.1. Lemma. *Suppose in the situation 6.1 that* $\ker\Delta=0$. *Then the approximation result holds without integrality assumptions:* $\lim_i F_{\Delta_i}(0)=0=\dim_\pi(\ker\Delta)$.

More generally, if λ is not an eigenvalue of Δ, then $\lim_i F_{\Delta_i}(\lambda)=F_\Delta(\lambda)$.

Proof. We know that $\underline{F}_\Delta^+(x)=F_\Delta(x)$ for every $x\in\mathbb{R}$. If $\underline{F}_\Delta(\lambda)\leq F_\Delta(\lambda)-\epsilon$, then $\underline{F}_\Delta^+(x)\leq F_\Delta(\lambda)-\epsilon$ for every $x<\lambda$, i.e. $F_\Delta(x)\leq F_\Delta(\lambda)-\epsilon\ \forall x<\lambda$. By assumption, the eigenspace of Δ to λ is trivial; therefore F_Δ is continuous at λ and ϵ can only be zero. □

7.2. Proposition. *If $\pi=\mathbb{Z}^n$ and is contained in the inverse limit of quotient groups $\{\pi_i\}_{i\in\mathbb{N}}$, then the approximation result holds for all $\Delta\in\mathbb{C}\pi$.*

Proof. Embed $\mathbb{C}[\mathbb{Z}^n]$ into its ring of fractions. Let $A\in M(d\times d,\mathbb{C}[\mathbb{Z}^n])$. Linear algebra gives $X,Y\in Gl(d\times d,\mathbb{C}(\mathbb{Z}^n))$ with $A=X\operatorname{diag}(1,\dots,1,0,\dots,0)Y$.

Collecting the denominators in X and Y, we find $0 \neq c \in \mathbb{C}[\mathbb{Z}^n] \cap Gl(1, \mathbb{C}(\mathbb{Z}^n))$ and $X', Y' \in M(d \times d, \mathbb{C}[\mathbb{Z}^n]) \cap Gl(d, \mathbb{C}(\mathbb{Z}^n))$ such that $cA = X'DY'$ with $D = \operatorname{diag}(1, \dots, 1, \underbrace{0, \dots, 0}_{r})$. The corresponding equation holds after passage to matrices c_i, A_i, X_i', Y_i' over π_i. Observe that, for $V = l^2(\pi_{(i)})^d$ and π- or π_i-morphisms $u, v : V \to V$, the exact sequence

$$0 \to \ker(v) \hookrightarrow \ker(uv) \xrightarrow{v} \ker(u),$$

the inclusions $\operatorname{im}(uv) \subset \operatorname{im}(u)$, and the exact sequences

$$0 \to \ker(u) \to V \xrightarrow{u} \operatorname{im}(u) \to 0, \qquad 0 \to \ker(uv) \to V \xrightarrow{uv} \operatorname{im}(uv) \to 0$$

together with additivity of the von Neumann dimension dim imply

$$\max\{\dim(\ker(u)), \dim(\ker(v))\} \leq \dim(\ker(uv)) \leq \dim(\ker(u)) + \dim(\ker(v)).$$

Because they are invertible over the field of fractions $\mathbb{C}(\mathbb{Z}^n)$, the operators c, X' and Y' have trivial kernel. The above inequalities applied to c_i, A_i, X_i', Y_i' then imply by Lemma 7.1 that

$$r = \lim_i \dim_{\pi_i}(\ker(X_i' D Y_i')) = \lim_i \dim_{\pi_i} \mathcal{G} \ker(c_i A_i) = \lim_i \dim_{\pi_i} \ker(A_i).$$

But we also have $r = \dim_{\mathbb{Z}^n}(\ker(A))$, and therefore the desired result. □

8. Quotients

To enlarge the class $\mathcal{G}$, it is important to find other operations under which our main properties —determinant class and semi-integrality— are inherited.

We indicate just one partial result:

8.1. Proposition. *Suppose $1 \to F \to \pi \xrightarrow{p} Q \to 1$ is an extension of groups and $|F| < \infty$, and π is of determinant class. Then also Q is of determinant class.*

Proof. We only indicate the proof, which was discussed with M. Farber during a conference in Oberwolfach, and which uses the theory of virtual characters of Farber [7]. $l^2(\pi)$ corresponds to the Dirac character δ_1. The representations V_k of the finite group F give rise to characters χ_k of π with support contained in F. Since $l^2(F) = \bigoplus \mu_k V_k$, it follows that $\delta_1 = \frac{1}{|F|} \sum \mu_k \chi_k$ with $\mu_k > 0$. Since the operator Δ we are interested in arises from $c^* \otimes \mathrm{id}$ on $C^* \otimes l^2(\pi)$,

$$F_\Delta(\lambda) = \sum \frac{\mu_k}{|F|} F_\Delta^{\chi_k}(\lambda)$$

(compare [7, 7.2]). Now the trivial representation V_1 of F corresponds to the quotient representation $l^2(Q)$, and $F_\Delta^{\chi_1}$ is just $F_{p(\Delta)}^Q$. By assumption

$$\int_{0+}^{\infty} \ln(\lambda)\, dF_\Delta(\lambda) > -\infty.$$

Since $\int \ln(\lambda)\, dF_\Delta^{\chi_k}(\lambda) < \infty\ \forall k$, it follows in particular that

$$\int_{0^+}^{\infty} \ln(\lambda)\, dF_{p(\Delta)}^Q(\lambda) > -\infty.$$

Since p is surjective, this is true for every matrix over $\mathbb{Z}\pi$ we have to consider. This concludes the proof. □

8.2. *Remark.* If we have an extension $1 \to \mathbb{Z}^n \to \pi \xrightarrow{p} Q \to 1$, we cannot write the character of $l^2(\pi)$ as a direct sum, but as a direct integral (over the dual space $\widehat{\mathbb{Z}^n} = T^n$). Then $F_\Delta(\lambda) = \int_{T^n} F_\Delta^{\chi_\eta}(\lambda)\ d\eta$, and $F_\Delta^{\chi_1}(\lambda) = F^Q_{p(\Delta)}(\lambda)$. If it would be possible to establish an appropriate continuity property, we could conclude as above that if π is of determinant class, then

$$\int_{0^+}^\infty \ln(\lambda)\ dF^Q_{p(\Delta)}(\lambda) > -\infty,$$

i.e. Q is also of determinant class.

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