PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

DETERMINANT EXPRESSION OF SELBERG ZETA FUNCTIONS (III)

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(Communicated by William Adams)

ABSTRACT. We will prove that for $\mathrm{PSL}(2, \mathbf{R})$ and its cofinite subgroup, the Selberg zeta function is expressed by the determinant of the Laplacian. We will also give an explicit calculation in case of congruence subgroups, and deduce that the part of the determinant of the Laplacian composed of the continuous spectrum is expressed by Dirichlet L-functions.

1. INTRODUCTION

The first discovery of the relation between the Selberg zeta function and the determinant of the Laplacian was by physicists [3, 4, 7]. Sarnak [14] and Voros [15] obtained the determinant expression of Selberg zeta functions for compact Riemann surfaces with torsionfree fundamental groups. In those cases, all the spectrum of the Laplacians are discrete. The determinant was defined via the holomorphy at the origin of the spectral zeta function of Minakshisundaram and Pleijel [13]. For noncompact but finite Riemann surfaces with torsion-free fundamental groups, these results are generalized by Efrat [5]. In this case there exist both discrete and continuous spectrum. He constructs the spectral zeta function composed not only of eigenvalues but some values concerning continuous spectrum, which are decided by all the poles of the scattering determinant in the Selberg trace formula. The determinant of the Laplacian is defined by the standard method with the holomorphy of the spectral zeta function at the origin. The aim of the present paper is to generalize his results to the case with any fundamental group Γ (§3) and to give some arithmetic examples of the determinant of the Laplacians (§4). In §4, we restrict ourselves to the case when Γ is a congruence subgroup of $\mathrm{PSL}(2, \mathbf{Z})$. In this case the partial spectral zeta function composed of only eigenvalues is also holomorphic at the origin [11, Theorem 3.3]. Hence we have a decomposition of the determinant into parts corresponding to the discrete and continuous spectrum. The scattering determinant is expressed very explicitly by Huxley [8] in terms of Dirichlet L-functions $L(s, \chi)$. Almost all the poles of the scattering determinant are described by nontrivial zeros of $L(s, \chi)$. The continuous part of the determinant

Received by the editors July 25, 1989 and, in revised form, June 27, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F72.

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can also be expressed by $L(s, \chi)$, in which the arithmetic information of Γ appears, whereas the geometric information goes into the discrete part.

Another example of this type of the decomposition is in [10, §6], where Γ is some arithmetic subgroup of $\mathrm{PSL}(2, \mathbf{C})$ acting on the real three-dimensional hyperbolic space.

In the general case, the decomposition of the determinant is unknown. We need more information about the properties of the scattering determinant. The background and the principle of the theory are described in [9].

2. The theorem of Efrat

We start by reviewing the result of Efrat [5], for torsion-free Γ. The Laplacian

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

for the upper half plane $\mathbf{H}$ on $L^2(\Gamma\backslash\mathbf{H})$ has both discrete and continuous spectrum. The latter is described by the scattering matrix $\Phi(s)$, whose entries come from the constant terms of the Eisenstein series. Let $\phi(s)$ be the determinant of $\Phi(s)$. Then we list three types of sequences:

(1) The set S_1 of $s_n \in \mathbf{C}$ such that $s_n(1-s_n) = \lambda_n$, where λ_n is the discrete spectrum of Δ.

(2) The set S_2 of poles $\rho_m = \beta_m + i\gamma_m$ of $\phi(s)$ with $\beta_m < \frac{1}{2}$.

(3) The set $S_3 = \{\eta_1, \dots, \eta_N\}$ of exceptional poles of $\phi(s)$ in $(\frac{1}{2}, 1]$.

The spectral zeta function $\zeta(w, s)$ is defined by

$$\zeta(w, s) := \sum_{\sigma \in S} (\sigma(1-\sigma) - s(1-s))^{-w},$$

where $S = S_1 \cup S_2 - S_3$. The determinant of the Laplacian is defined to be

$$\det(\Delta - s(1-s))^2 := \exp\left(-\frac{\partial}{\partial w}\Big|_{w=0} \zeta(w, s) \right),$$

after proving the regularity of $\zeta(w, s)$ at $w = 0$. The Selberg zeta function of a group Γ (or of a Riemann surface $\Gamma\backslash\mathbf{H}$) is defined as

$$Z(s) := \prod_P \prod_{n=0}^{\infty} (1 - N(P)^{-s-n}),$$

where P runs through all the representatives of primitive hyperbolic conjugacy classes of Γ, and $N(P) := \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$). The gamma-factor of $Z(s)$ is the function from the identity term of the trace formula:

$$Z_I(s) = \left(\frac{\Gamma_2(s)^2 (2\pi)^s}{\Gamma(s)} \right)^{\mathrm{vol}(\Gamma\backslash\mathbf{H})/2\pi},$$

where $\Gamma_2(s)$ is the double gamma function of Barnes [1, 2]. Efrat's theorem gives the relation between the determinant of the Laplacian and the Selberg zeta function.

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Theorem 2.1. (Efrat [5]). *When Γ is torsion free and cofinite, we have the identity*

$$\det^2(\Delta - s(1-s)) = \phi(s)Z(s)^2 Z_I(s)^2 \Gamma\left(s+\tfrac{1}{2}\right)^{-2K}(2s-1)^A \\ \times \exp(B(2s-1)^2 + C(2s-1) + D),$$

where K is the number of inequivalent cusps and

$$A = K - \operatorname{tr}(\Phi(1/2)), \qquad B = -\frac{\operatorname{vol}(\Gamma\backslash\mathbf{H})}{4\pi},$$
$$C = K\log 2, \qquad D = \frac{\operatorname{vol}(\Gamma\backslash\mathbf{H})}{\pi}\left(2\zeta'(-1) - \log\sqrt{2\pi}\right) + 2K\log\sqrt{2\pi} - A\log 2.$$

3. The contribution of elliptic elements

We treat the case when Γ may have elliptic elements. Also in this case $\zeta(w, s)$ is regular at $w = 0$. We have the same definition of $\det(\Delta - s(1-s))$ as that with Γ torsionfree. The corresponding terms of the Selberg trace formula have some effects on the gamma-factor of the Selberg zeta function. Their contribution is computed by Fischer [6, Corollary 2.3.5] and the author [11, (5.9)] by taking different test functions;

$$Z_E(s) = \prod_R \prod_{l=0}^{\nu_R - 1} \Gamma\left(\frac{s+l}{\nu_R}\right)^{\frac{2l+1-\nu_R}{\nu_R}},$$

where ν_R is the order of R and R runs through all the primitive elliptic conjugacy classes. We can describe Efrat's theorem in the case when Γ may contain elliptic conjugacy classes.

Theorem 3.1. *When Γ is cofinite, we have the identity*

$$\det^2(\Delta - s(1-s)) = \phi(s)Z(s)^2 Z_I(s)^2 Z_E(s)^2 \Gamma\left(s+\frac{1}{2}\right)^{-2K} \\ \times (2s-1)^A \exp(B(2s-1)^2 + C(2s-1) + D),$$

where K, A, B, C are the same constants as in Theorem 2.1, and

$$D = \frac{\operatorname{vol}(\Gamma\backslash\mathbf{H})}{\pi}(2\zeta'(-1) - \log\sqrt{2\pi}) + 2K\log\sqrt{2\pi} - A\log 2 \\ + 2\left(\frac{n_2}{4}\log 2 + \frac{4n_3}{9}\log 3\right),$$

where n_2 and n_3 denote the number of the elliptic conjugacy classes of order 2 and 3, respectively.

Proof. According to the method of Sarnak [14, §2.2], it suffices to examine the asymptotic behavior of $\log Z_E(s)^2$ as $s \to \infty$. Now all the elliptic elements in $\Gamma \subset \mathrm{PSL}(2, \mathbf{Z})$ are of order 2 or 3. From Stirling's formula, the behavior of

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$\log Z_E(s)^2$ is shown to be

$$\log Z_E(s)^2 = 2\left(\frac{n_2}{4} + \frac{4n_3}{9}\right)\log s - 2\left(\frac{n_2}{4}\log 2 + \frac{4n_3}{9}\log 3\right) + o(1).$$

As the constant D is the sum of the constant term in this expansion, the contribution of the elliptic factors is

$$2\left(\frac{n_2}{4}\log 2 + \frac{4n_3}{9}\log 3\right). \quad \text{Q.E.D.}$$

4. The contribution of the continuous spectrum

We treat the case when Γ is the image of one of the typical congruence subgroups below in $\mathrm{PSL}(2, \mathbf{R})$:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z});\ c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N);\ a \equiv d \equiv 1 \pmod N \right\},$$

$$\Gamma_2(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N);\ b \equiv 0 \pmod N \right\}.$$

These groups have elliptic elements in general. In these cases the scattering determinant is obtained by Huxley.

Theorem 4.1. (Huxley [8]). *The scattering determinant for $\Gamma_i(N)$ $(i = 0, 1, 2)$ is expressed by a product over even Dirichlet characters modulo N;*

$$\phi(s) = (-1)^{\frac{K-K_0}{2}} \left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^K \left(\frac{F}{\pi^K}\right)^{1-2s} \prod_\chi \frac{L(2-2s, \overline{\chi})}{L(2s, \chi)}, \tag{4.1}$$

where $K_0 := -\operatorname{tr}\Phi(\frac{1}{2})$, which is equal to the number of χ such that $L(s, \chi)$ has a pole as $s = 1$.

In this theorem the conditions on χ are described in [8], and the number of χ is equal to K. If χ is in the product, so is $\overline{\chi}$. The number F is a positive integer composed of prime factors of N. Put

$$L(s) := (F\pi^{-K})^{s/2} \prod_\chi \Gamma(s/2) L(s, \chi).$$

Then our scattering determinant is expressed as

$$\phi(s) = (-1)^{\frac{K-K_0}{2}} \frac{L(2-2s)}{L(2s)},$$

which leads us to the explicit form of the continuous part of the determinant of the Laplacian. The set S_2 is the sequence of all the nontrivial zeros of $L(2s, \chi)$ and $S_3 = \{1\}$ with multiplicity K_0. We define the spectral zeta function for

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each sequence by

$$\zeta_i(w, s) := \sum_{\sigma \in S_i} (\sigma(1-\sigma) - s(1-s))^{-w}, \qquad (i = 1, 2, 3).$$

We have the regularity of $\zeta_1(w, s)$ at $w = 0$ by the following theorem.

Theorem 4.2. *The discrete part,*

$$\zeta_1(w, s) := \sum_{\sigma \in S_1} (\sigma(1-\sigma) - s(1-s))^{-w},$$

of the spectral zeta function has the analytic continuation to the whole w-plane except at the pole at $w = 1$ of order 1 and the poles at $w = \frac{1}{2} - n$ ($n = 0, 1, 2, \dots$) of order 2.

Proof. As σ and $1-\sigma$ give the same eigenvalue,

$$\zeta_1(w, s) = 2\sum_{n=0}^{\infty} (\lambda_n - s(1-s))^{-w},$$

whose analytic continuation is proved in [11, Theorem 3.3] based on the method of Kurokawa [12, Theorem 3(1)]. Here we recall the proof briefly. We take the test function $h(r^2 + \frac{1}{4})$ in the Selberg trace formula as

$$h\left(r^2 + \tfrac{1}{4}\right) := \exp\left(-t\left(r^2 + \tfrac{1}{4} + s(s-1)\right)\right),$$

with $s > 1$, $t > 0$. It suffices to have the analytic continuation of the Mellin transformation of the term concerning discrete spectrum in the trace formula. All we have to do is to examine all the other terms in the trace formula. The contribution of the continuous spectrum and parabolic conjugacy classes is the sum of the following three terms:

$$CP_1(t) = \left(\frac{1}{\sqrt{4\pi t}} \log \frac{\pi^K}{2^K F} + \frac{K - K_0}{4}\right) \exp\left(-t\left(s - \frac{1}{2}\right)^2\right),$$

$$CP_2(t) = \frac{1}{\sqrt{\pi t}} \sum_{\chi} \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n} \exp\left(-t\left(s - \frac{1}{2}\right)^2 - \frac{(\log n)^2}{t}\right),$$

$$CP_3(t) = -\frac{K}{2\pi} \int_{-\infty}^{\infty} \left(\psi(\tfrac{1}{2} + ir) + \psi(1 + ir)\right) \exp\left(-t\left(r^2 + (s - \tfrac{1}{2})^2\right)\right) dr,$$

where $\Lambda(n)$ is the von Mangoldt function and $\psi = \Gamma'/\Gamma$. The key point is the treatment of CP_3. Its Mellin transformation is

$$\int_0^{\infty} \left(r^2 + (s - \tfrac{1}{2})^2\right)^{-w} (\mathrm{Re}(\psi(1 + 2ir)) - \log 2)\, dr,$$

up to gamma-factor. If we divide this integral into $\int_0^{s-1/2}$ and $\int_{s-1/2}^{\infty}$, the former is finite for all $w \in \mathbf{C}$. The latter is equal to

$$\int_0^1 y^{w-3/2}(y+1)^{-w} \left(\mathrm{Re}\left(\psi\left(1 + \frac{2s-1}{\sqrt{y}} i\right)\right) - \log 2\right) dy,$$

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up to constant, by $r = \left(s - \frac{1}{2}\right) y^{-1/2}$. The Stirling-Binet formula [16, 12.3, p. 252] shows that

$$\mathrm{Re}(\psi(1 + ir)) = \log r + \sum_{n=1}^{N} \frac{\alpha_n}{r^{2n}} + R_N(r) \qquad (\alpha_n \in \mathbf{R}),$$

with $|R_N(r)| \leq M_n/r^{2N+1}$ $\left(r \geq \frac{1}{2}\right)$, where α_n are constants expressd via Bernoulli numbers. Moreover, we can apply the binomial expansion formula to get

$$(y+1)^{-w} = \sum_{k=0}^{\infty} \binom{-w}{k} y^k,$$

where

$$\binom{-w}{k} := \begin{cases} \frac{-w(-w-1)\cdots(-w-k+1)}{k!} & (k \geq 1), \\ 1 & (k = 0). \end{cases}$$

Then the above integral $\int_{s-1/2}^{\infty}$ can be written as

$$\sum_{k=0}^{\infty} \binom{-w}{k} \int_0^1 y^{w+k-3/2} \left(-\frac{\log y}{2} + \sum_{n=0}^{N} \frac{\alpha_n}{(2s-1)^{2n}} y^n + R_N\left(\frac{2s-1}{\sqrt{y}}\right)\right) dy,$$

where $\alpha_0 := \log(2s-1)$. The remainder term is holomorphic in $\mathrm{Re}(w) > -N$. An elementary calculation shows that the integral is entire in the whole w-plane except at double poles at $w = \frac{1}{2} - k$ $(k = 0, 1, 2, \ldots)$. Next we treat all the other terms in the trace formula. Possible poles of the Mellin transormation come from the behavior of each term as $t \to 0$, because $h(r^2 + \frac{1}{4})$ is exponentially small as $t \to \infty$. The hyperbolic term and CP_2 do not make poles, for they are exponentially small as $t \to 0$. We can expand other terms using the expansion of the exponential function around the origin. Consequently, their behavior as $t \to 0$ is expressed as

$$\sum_{n=-1}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^{n-1/2} \qquad (a_n, b_n \in \mathbf{R}).$$

The Mellin transformation tells that it has simple poles at $w = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$, among which the poles at nonpositive integers are those of the gamma-factor. Q.E.D.

The result of Efrat and Theorem 4.2 show that we can define the discrete and the continuous part of the determinant of the Laplacian separately. The continuous part of the determinant of the Laplacian is defined by

$$\det_C(\Delta - s(1-s)) := \frac{\det_2(\Delta - s(1-s))}{\det_3(\Delta - s(1-s))},$$

where

$$\det_i(\Delta - s(1-s)) := \exp\left(-\frac{\partial}{\partial w}\Big|_{w=0} \zeta_i(w, s)\right).$$

We deduce the following theorem.

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Theorem 4.3. *For the present* Γ, *the continuous part of the determinant of the Laplacian is expressed by*

$$\det_C(\Delta - s(1-s))^2 = e^{d+d's(s-1)}L(2s)L(2-2s)(2s-1)^{2K_0},$$

with some constants d *and* d'.

Proof. The function $(2s(2s-1))^{K_0}L(2s)$ is entire and has the following expression as an infinite product;

$$(2s(2s-1))^{K_0}L(2s) = pe^{\alpha s}\prod_{\rho\in S_2}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

with some constants p and α. Replacing s by $1-s$ tells us that

$$((2-2s)(1-2s))^{K_0}L(2-2s) = pe^{\alpha(1-s)}\prod_{\rho\in S_2}\left(1-\frac{1-s}{\rho}\right)e^{(1-s)/\rho}.$$

Now we have the identity

$$\begin{aligned}&\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\det_2(\Delta - s(1-s))^2\\ &\quad= \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log L(2s)L(2-2s)(2s-1)^{2K_0}(s(s-1))^{K_0}.\end{aligned}$$

Indeed, a little calculation shows that both sides are equal to

$$-\sum_{\rho\in S_2}\frac{2s-1}{((\rho-s)(\rho-(1-s)))^2}.$$

On the other hand, it is easy to compute that

$$\det_3(\Delta - s(1-s))^2 = (s(s-1))^{K_0}.$$

Then the proof is accomplished. Q.E.D.

Proposition 4.4. *The constants* d *and* d' *in Theorem* 4.3 *are*

$$d \in -\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{4\pi} + 2K\log\frac{4\sqrt{2\pi}}{e} + 2\sqrt{\pi}\log\frac{\pi^K}{2^K F} + \frac{K-K_0}{2}\pi i + 2\pi i\mathbf{Z},$$

and

$$d' = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}.$$

Proof. The discrete part $\det_1$ is the same as the discrete part defined in [11], in which we have the identity

$$\begin{aligned}&\det_1(\Delta - s(1-s))L(2s)\left(s-\tfrac12\right)^{-\frac{K_0}{2}}e^{c+c's(s-1)}\\ &\quad= Z(s)Z_I(s)Z_E(s)2^{-Ks}\left(s-\tfrac12\right)^{\frac{K}{2}}\Gamma\left(s+\tfrac12\right)^{-K},\end{aligned}$$

where

$$\begin{aligned}c \in{}& \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{2\pi}\left(\frac{\log 2\pi}{2} - 2\zeta'(-1)\right) - \left(\frac{n_2}{4}\log 2 + \frac{4n_3}{9}\log 3\right)\\ &+ K\log\frac{2}{\sqrt{\pi}e} + \sqrt{\pi}\log\frac{\pi^K}{2^K F} + 2\pi i\mathbf{Z},\end{aligned}$$

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and

$$c' = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}.$$

By comparing the coefficent of $s(s-1)$ and the constant term in the logarithm of this and Theorem 2.1 into which we substitute the result of Theorem 4.3, we get

$$d' - 2c' = -\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}$$

and

$$\begin{aligned} d - 2c \in &-\frac{\mathrm{vol}(\Gamma\backslash\mathbf{H})}{\pi}\left(\frac{1}{4} + \log\sqrt{2\pi} - 2\zeta'(-1)\right) + 2K\log 2\sqrt{2}\pi \\ &+ 2\left(\frac{n_2}{4}\log 2 + \frac{4n_3}{9}\log 3\right) + \frac{K-K_0}{2}\pi i + 2\pi i\mathbf{Z}. \end{aligned}$$

Hence we get the conclusion. Q.E.D.

Acknowledgment

The author would like to express his hearty thanks to Professor N. Kurokawa. He introduced the author to the subject and recommended to publish this paper. Some extent of the author's debt to him is clear in the proof of Theorem 4.2.

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