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Determinant Representation for Correlation Functions of Spin-1/2 XXX and XXZ Heisenberg Magnets

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Abstract: We consider zero temperature correlation functions of the spin-$\frac{1}{2}$ XXZ Heisenberg chain in the critical regime $-1 < \Delta \leqq 1$ in a magnetic field. Starting from the algebraic Bethe Ansatz we derive representations for various correlation functions in terms of determinants of Fredholm integral operators.

1. Introduction

Despite the great advances made over the last sixty years in the study of integrable quantum models, evaluation of their correlation functions still poses a formidable problem. Quite recently there has been significant progress in this direction: the group at RIMS succeeded in deriving integral representations for some correlation functions of the Heisenberg XXZ model [2, 13, 46, 5, 16, 49–51, 56–58, 53] defined by the hamiltonian (1.1) for $\Delta > 1$ by taking advantage of the infinite quantum affine symmetry of the model on the infinite chain [10, 27]. (see e.g. [28, 12, 6, 7] for further developments). The isotropic (XXX) limit $\Delta \to 1$ was obtained in [45, 33]. These integral representations are most powerful for studying the *short distance* behaviour of correlators, whereas it is not obvious how to extract the large distance behaviour. Also it is not straightforward to extend this approach to the critical regime $-1 < \Delta < 1$ or to include an external magnetic field.

Precisely these issues can be very naturally addressed in the framework of a different approach to studying correlation functions in integrable models, which was carried out in [29, 30, 18–21, 23, 34, 35] for the example of the δ-function Bose gas [40, 41]. A detailed and complete exhibition of this work can be found in the book [32]. We call this method the *Dual Field Approach* (DFA). The DFA permits one to derive determinant representations for correlation functions of models of interacting fermions (the corresponding spectrum of the hamiltonian is not equivalent

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to free fermions). It can be considered as a generalization of the seminal work on correlation functions for the impenetrable δ-function Bose gas: using a determinant representation for zero temperature equal time quantum correlation functions [38, 39] M. Jimbo, T. Miwa, Y. Mori and M. Sato proved that the correlators are described by an ordinary differential equation reducible to the Painlevé transcendent [26].

The DFA is directly based on the (algebraic) Bethe–Ansatz solution of the model and thus is applicable to a large variety of correlation functions and integrable models. It allows to derive explicit expressions for the *large distance* asymptotics of correlation functions (even at finite temperature), and the inclusion of an external magnetic field poses no problem. The DFA thus nicely complements the approach of the RIMS group. In this paper we will apply the DFA to the Heisenberg XXZ and XXX chains at zero temperature in a magnetic field h, i.e. the hamiltonian

$$H = \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1) - h \sum_{j=1}^{L} \sigma_j^z, \quad -1 < \Delta \leqq 1, \tag{1.1}$$

where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\Delta = \cos(2\eta)$, $\frac{\pi}{2} < \eta \leqq \pi$. We impose periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$, where $\alpha = x, y, z$.

There are four main steps in the DFA: First the model needs to be "solved" by means of the Algebraic Bethe Ansatz. Then one uses this solution to express correlation functions in terms of determinants of Fredholm integral operators. In step three these determinants are embedded in systems of integrable integro-difference equations (IDE). This step is inspired by the work of E. Barouch, B.M. McCoy, T.T. Wu [1] and C. Tracy and B.M. McCoy [54], who pioneered the idea of describing quantum correlation functions by means of differential equations. Finally the large-distance asymptotics of the correlators is extracted from a Riemann–Hilbert problem for the IDE's. As the computations for the various steps are rather involved we will only deal with the first steps here, i.e. review the known Bethe Ansatz solution for the XXZ and XXX chains and then derive determinant representations for correlation functions.

2. A Short Review of Algebraic Bethe Ansatz

Let us review a few main features of the Algebraic Bethe Ansatz (ABA) for both XXZ and XXX Heisenberg magnets [11, 36, 47, 53, 52] in order to fix notations for things to come. The XXX case can of course be obtained by taking a certain limit of the XXZ case, but in practice this is more difficult than treating the XXX case separately from the beginning. Thus we will treat both cases on an equal footing throughout this paper.

The starting point and central object of the Quantum Inverse Scattering Method (for a comprehensive review and more references see [37, 52] is the R-matrix, which is a solution of the Yang–Baxter equation [59, 3]. For the case of the XXZ and XXX models it is of the form

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}, \tag{2.1}$$

where for XXZ

$$f(\lambda,\mu)=\frac{\sinh(\lambda-\mu+2i\eta)}{\sinh(\lambda-\mu)},\qquad g(\lambda,\mu)=\frac{i\sin(2\eta)}{\sinh(\lambda-\mu)}, \tag{2.2}$$

and for the XXX-case

$$f(\lambda,\mu)=1+\frac{i}{\lambda-\mu},\qquad g(\lambda,\mu)=\frac{i}{\lambda-\mu}. \tag{2.3}$$

The R-matrix is a linear operator on the tensor product of two two-dimensional linear spaces: $R(\mu)\in\mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$. From the R-matrix (2.1) one can construct an L-operator of a "fundamental spin model" (see e.g. [32] p. 126) by considering the matrix $R(\mu)\Pi$, where Π is the permutation matrix on $\mathbb{C}^2\otimes\mathbb{C}^2$, and then making it into an operator-valued matrix by identifying one of the linear spaces with the two-dimensional Hilbert space $\mathscr{H}_n$ of $SU(2)$-spins over the n^{th} site of a lattice of length L,

$$L_n^{xxz}(\mu)=\begin{pmatrix}\sinh(\mu-i\eta\sigma_n^z) & -i\sin(2\eta)\sigma_n^- \\ -i\sin(2\eta)\sigma_n^+ & \sinh(\mu+i\eta\sigma_n^z)\end{pmatrix},$$

$$L_n^{xxx}(\mu)=\mu-\frac{i}{2}\begin{pmatrix}\sigma_n^z & 2\sigma_n^- \\ 2\sigma_n^+ & -\sigma_n^z\end{pmatrix}. \tag{2.4}$$

The Yang–Baxter equation for R implies the following relations for the L-operator

$$R(\lambda-\mu)(L_n(\lambda)\otimes L_n(\mu))=(L_n(\mu)\otimes L_n(\lambda))R(\lambda-\mu). \tag{2.5}$$

From the ultralocal L-operator the monodromy matrix is constructed as

$$T(\mu)=L_L(\mu)L_{L-1}(\mu)\cdots L_1(\mu)=\begin{pmatrix}A(\mu) & B(\mu)\\ C(\mu) & D(\mu)\end{pmatrix}. \tag{2.6}$$

Equation (2.5) can be lifted to the level of the monodromy matrix

$$R(\lambda-\mu)(T(\lambda)\otimes T(\mu))=(T(\mu)\otimes T(\lambda))R(\lambda-\mu). \tag{2.7}$$

Below we will repeatedly use especially the following matrix elements of (2.7):

$$\begin{aligned}
&[B(\lambda),B(\mu)]=0=[C(\lambda),C(\mu)],\\
&[B(\lambda),C(\mu)]=g(\lambda,\mu)(D(\lambda)A(\mu)-D(\mu)A(\lambda)),\\
&D(\mu)B(\lambda)=f(\lambda,\mu)B(\lambda)D(\mu)+g(\mu,\lambda)B(\mu)D(\lambda),\\
&A(\mu)B(\lambda)=f(\mu,\lambda)B(\lambda)A(\mu)+g(\lambda,\mu)B(\mu)A(\lambda).
\end{aligned} \tag{2.8}$$

By tracing (2.7) over the matrix space one then finds that the *transfer matrices* $\tau(\mu)=\mathrm{tr}(T(\mu))=A(\mu)+D(\mu)$ commute for any values of spectral parameter μ, i.e. $[\tau(\mu),\tau(\nu)]=0$. From this it follows that the transfer matrix is the generating functional of an infinite number of mutually commuting conserved quantum operators (via expansion in powers of spectral parameter). One of these operators is the hamiltonian

$$H = -2i\sin(2\eta)\frac{\partial}{\partial\mu}\ln(\tau_{xxz}(\mu))\Big|_{\mu=-i\eta} - 2L\cos(2\eta) - 2hS^z\,. \tag{2.9}$$

Below we also make use of some properties of *inhomogeneous* XXX and XXZ models, which are constructed in the following way: we first note that the intertwining relation for the L-operator (2.5) still holds, if we shift both spectral parameters λ and μ by an arbitrary amount v_n, i.e.

$$R(\lambda-\mu)(L_n(\lambda-v_n)\otimes L_n(\mu-v_n)) = (L_n(\mu-v_n)\otimes L_n(\lambda-v_n))R(\lambda-\mu)\,. \tag{2.10}$$

The reason for this fact is of course that the R-matrix only depends on the difference of spectral parameters. We now can construct a monodromy matrix as

$$T_{inh}(\lambda) = L_L(\lambda-v_L)L_{L-1}(\lambda-v_{L-1})\dots L_1(\lambda-v_1) = \begin{pmatrix} \mathscr{A}(\mu) & \mathscr{B}(\mu) \\ \mathscr{C}(\mu) & \mathscr{D}(\mu) \end{pmatrix}\,. \tag{2.11}$$

The inhomogeneous monodromy matrix (2.11) obeys the same intertwining relation (2.7) as (2.6).

The ABA deals with the construction of simultaneous eigenstates of the transfer matrix and the hamiltonian. The starting point is the choice of a *reference state*, which is a trivial eigenstate of $\tau(\mu)$. In our case we make the choice $|0\rangle = |\uparrow\uparrow\uparrow\dots\uparrow\rangle = \bigotimes_{n=1}^{L}|\uparrow\rangle_n$, i.e. we choose the completely ferromagnetic state. The action of the L-operator (2.4) on $|\uparrow\rangle_n$ can be easily computed and implies the following actions of the matrix elements of the monodromy matrix for the XXZ case,

$$\begin{aligned} A(\mu)|0\rangle &= a(\mu)|0\rangle\,, \quad a(\mu) = (\sinh(\mu-i\eta))^L\,, \\ D(\mu)|0\rangle &= d(\mu)|0\rangle\,, \quad d(\mu) = (\sinh(\mu+i\eta))^L\,, \\ C(\mu)|0\rangle &= 0\,, \\ B(\mu)|0\rangle &\neq 0\,, \end{aligned} \tag{2.12}$$

whereas in the XXX case

$$a(\mu) = \left(\mu - \frac{i}{2}\right)^L, \qquad d(\mu) = \left(\mu + \frac{i}{2}\right)^L. \tag{2.13}$$

From (2.12) it follows that $B(\lambda)$ plays the role of a creation operator, i.e. one can construct a set of states of the form

$$\Psi_N(\lambda_1,\dots,\lambda_N) = \prod_{j=1}^{N} B(\lambda_j)|0\rangle\,. \tag{2.14}$$

The requirement that the states (2.14) ought to be eigenstates of the transfer matrix $\tau(\mu)$ puts constraints on the allowed values of the parameters λ_n: the set $\{\lambda_j\}$ must be a solution of the following system of coupled algebraic equations, called *Bethe* *equations* [4, 46]:

$$\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{\substack{k=1\\k\neq j}}^{N} \frac{f(\lambda_k,\lambda_j)}{f(\lambda_j,\lambda_k)}, \quad j = 1,\dots,N. \tag{2.15}$$

These equations are the basis for studying the ground state, excitation spectrum and thermodynamics of Bethe Ansatz solvable models. For the case of the XXZ model with $\Delta > -1$ (the case we are interested in here) it was proved by C.N. Yang and C.P. Yang in [56, 57] that the ground state is characterized by a set of *real* λ_j subject to the Bethe equations (2.15). Without an external magnetic field ($h = 0$) their number is $N = L/2$. In the thermodynamic limit the ground state is described by means of an integral equation for the density of spectral parameters $\rho(\lambda)$ [17, 57]

$$2\pi\rho(\lambda) - \int_{-\Lambda}^{\Lambda} d\mu\, K(\lambda,\mu)\rho(\mu) = D(\lambda), \tag{2.16}$$

where the integral kernel K and the driving term D are given by

$$K(\mu,\lambda) = \frac{\sin(4\eta)}{\sinh(\mu-\lambda+2i\eta)\sinh(\mu-\lambda-2i\eta)},$$
$$D(\lambda) = \frac{-\sin(2\eta)}{\sinh(\lambda-i\eta)\sinh(\lambda+i\eta)}. \tag{2.17}$$

For the XXX case we have [16]

$$K_{XXX}(\mu,\lambda) = -\frac{2}{(\mu-\lambda)^2+1}, \qquad D_{XXX}(\lambda) = \frac{1}{\lambda^2+\frac{1}{4}}. \tag{2.18}$$

Here Λ depends on the external magnetic field h. The physical picture of the ground state is that of a filled Fermi sea with boundaries $\pm\Lambda$. The dressed energy of a particle in the sea is given by the solution of the integral equation [49, 50]

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\mu K(\lambda,\mu)\varepsilon(\mu) = 2h - \frac{2(\sin(2\eta))^2}{\sinh(\lambda-i\eta)\sinh(\lambda+i\eta)}. \tag{2.19}$$

The requirement of the vanishing of the dressed energy at the Fermi boundary $\varepsilon(\pm\Lambda) = 0$ determines the dependence of Λ on h. For small h this relation can be found explicitly by means of a Wiener–Hopf analysis [57]. For $h \geqq h_c = (2\cos\eta)^2$ the system is in the saturated ferromagnetic state, which corresponds to $\Lambda = 0$.

3. Two-Site Generalized Model

For the evaluation of correlation functions the so-called "two-site generalized model" has proven an extremely useful tool. From the mathematical point of view this is simply the application of the co-product associated with the algebra defined by (2.7). The main idea is to divide the chain of length L into two parts and associate a monodromy matrix with both sub-chains, i.e.

$$T(\mu) = T(2,\mu)T(1,\mu), \qquad T(i,\mu) = \begin{pmatrix} A_i(\mu) & B_i(\mu) \\ C_i(\mu) & D_i(\mu) \end{pmatrix} \quad (i = 1,2). \tag{3.1}$$

In terms of L-operators the monodromy matrices are given by

$$T(2,\mu) = L_L(\mu)L_{L-1}(\mu)\dots L_n(\mu)\,,$$
$$T(1,\mu) = L_{n-1}(\mu)L_{n-2}(\mu)\dots L_1(\mu)\,. \tag{3.2}$$

By construction it is clear that both monodromy matrices $T(i,\mu)$ fulfill the same intertwining relation (2.7) as the complete monodromy matrix $T(\mu)$. Similarly the reference state for the complete chain is decomposed into a direct product of reference states $|0\rangle_i$ for the two sub-chains $|0\rangle = |0\rangle_2 \otimes |0\rangle_1$. The resulting structure can be summarized as

$$A_i(\mu)|0\rangle_i = a_i(\mu)|0\rangle, \qquad D_i(\mu)|0\rangle_i = d_i(\mu)|0\rangle_i\,,$$
$$C(\mu)|0\rangle_i = 0, \qquad B_i(\mu)|0\rangle_i \neq 0\,, \tag{3.3}$$

where the eigenvalues a and d in (2.12) are given by $a(\mu) = a_2(\mu)a_1(\mu)$ and $d(\mu) = d_2(\mu)d_1(\mu)$. The creation operators $B(\mu)$ for the complete chain are decomposed as $B(\mu) = A_2(\mu)\otimes B_1(\mu) + B_2(\mu)\otimes D_1(\mu)$, which implies that eigenstates of the transfer matrix can be represented as

$$\prod_{j=1}^{N} B(\lambda_j)|0\rangle = \sum_{I,II}\prod_{j\in I}^{n_1}\prod_{k\in II}^{n_2} a_2(\lambda_j^I)d_1(\lambda_k^{II})$$
$$\times f(\lambda_j^I,\lambda_k^{II})(B_2(\lambda_k^{II})|0\rangle_2)\otimes(B_1(\lambda_j^I)|0\rangle_1)\,, \tag{3.4}$$

where the sum is over all partitions $\{\lambda_j^I\}\cup\{\lambda_k^{II}\}$ of the set $\{\lambda_i\}$ with card$\{\lambda^I\} = n_1$, card$\{\lambda^{II}\} = n_2 = N - n_1$. A similar equation holds for dual states

$$\langle 0|\prod_{j=1}^{N} C(\lambda_j) = \sum_{I,II}\prod_{j\in I}^{n_1}\prod_{k\in II}^{n_2} d_2(\lambda_j^I)a_1(\lambda_k^{II})$$
$$\times f(\lambda_k^{II},\lambda_j^I)({}_1\langle 0|C_1(\lambda_j^I))\otimes({}_2\langle 0|C_2(\lambda_k^{II}))\,. \tag{3.5}$$

4. Reduction of Correlators to Scalar Products

In this section we reduce the problem of evaluating correlators of the form $\langle\sigma_j^z\sigma_k^z\rangle$ (where $\langle\rangle$ denotes the normalized zero temperature vacuum expectation value, i.e. the expectation value with respect to the antiferromagnetic ground state described by (2.16)–(2.19)) to the computation of certain scalar products between states given by the Algebraic Bethe Ansatz. We start by noting that due to translational invariance it is sufficient to consider the correlator $G(m) = \langle\sigma_m^z\sigma_1^z\rangle$. In terms of the operators $q_j = \frac{1}{2}(1-\sigma_j^z)$ the correlator takes the form

$$G(m) = 4\langle q_m q_1\rangle - 4\langle q_1\rangle + 1\,, \tag{4.1}$$

where we have again used translational invariance. The quantity $\langle q_1\rangle$ is nothing but the density of down spins in the ground state and can thus be reexpressed as $\langle q_1\rangle = \int_{-\Lambda}^{\Lambda} d\Lambda\rho(\Lambda)$. The first term in (4.1) is expressed in terms of the quantity

$Q_1(m) = \sum_{j=1}^m q_j = \sum_{j=1}^m \sigma_j^- \sigma_j^+$ as follows:

$$\langle q_m q_1 \rangle = \frac{1}{2} \hat{\Delta} \langle (Q_1(m))^2 \rangle ,$$

where $\hat{\Delta}$ (not to be confused with the inhomogeneity Δ in the XXZ hamiltonian) is the lattice laplacian $\hat{\Delta} f(j) = f(j) + f(j-2) - 2f(j-1)$. Putting everything together we obtain

$$G(m) = 2\hat{\Delta} \langle (Q_1(m))^2 \rangle + 1 - 4 \int_{-\Lambda}^{\Lambda} d\lambda\, \rho(\lambda) \ . \tag{4.2}$$

The only nontrivial quantity to determine is thus $\langle (Q_1(m))^2 \rangle = \frac{\partial^2}{\partial \alpha^2} \langle \exp(\alpha Q_1 (m)) \rangle |_{\alpha=0}$. We will now use the two-site generalized model to express the "generating functional"

$$F(\alpha, m) := \langle \exp(\alpha Q_1(m)) \rangle := \frac{\langle 0 | \prod_{j=1}^N C(\lambda_j) \exp(\alpha \sum_{l=1}^m \sigma_l^- \sigma_l^+) \prod_{k=1}^N B(\lambda_k) | 0 \rangle}{\langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle} , \tag{4.3}$$

in terms of scalar products: we take the first sub-chain to contain sites 1 to m and the second one sites $m+1$ to L. We note that $Q_1(m)$ now acts only on the first sub-chain and simply counts the number of down spins. Using (3.4) and (3.5) in (4.3) we obtain

$$\begin{aligned} F(\alpha, m) = \frac{1}{\sigma_N} \sum {}_1\langle 0 | \prod_{I_C} C_1(\lambda_{I_C}^C) \prod_{I_B} B_1(\lambda_{I_B}^B) \, | 0 \rangle_1 \\ {}_2\langle 0 | \prod_{II_C} C_2(\lambda_{II_C}^C) \prod_{II_B} B_2(\lambda_{II_B}^B) | 0 \rangle_2 \ e^{\alpha n_1} \prod_{I_B, I_C} a_2(\lambda_{I_B}^B) d_2(\lambda_{I_C}^C) \prod_{II_B, II_C} a_1(\lambda_{II_C}^C) d_1(\lambda_{II_B}^B) \\ \times \prod_{I_B, II_B} f(\lambda_{I_B}^B, \lambda_{II_B}^B) \prod_{I_C, II_C} f(\lambda_{II_C}^C, \lambda_{I_C}^C) , \end{aligned} \tag{4.4}$$

where the sum is over all partitions

$$\{\lambda_{I_B}^B\} \cup \{\lambda_{II_B}^B\} = \{\lambda\}, \quad \{\lambda_{I_B}^B\} \cap \{\lambda_{II_B}^B\} = \emptyset,$$

$$\{\lambda_{I_C}^C\} \cup \{\lambda_{II_C}^C\} = \{\lambda\}, \quad \{\lambda_{I_C}^C\} \cap \{\lambda_{II_C}^C\} = \emptyset$$

of the set $\{\lambda\}$ with $\mathrm{card}\{\lambda_{I_B}\} = \mathrm{card}\{\lambda_{I_C}\} = n_1$, $\mathrm{card}\{\lambda_{II_C}\} = \mathrm{card}\{\lambda_{II_B}\} = N - n_1$ and

$$\sigma_N = \langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle \ . \tag{4.5}$$

Note that due to (3.3) and (2.7) (for $B_i(\mu), C_i(\mu)$) we only need to consider partitions such that the size of partitions I_B and I_C (and II_B and II_C) are the same. In the following section we will show that scalar products of the form appearing in (4.4) can be expressed as determinants and then use this fact to obtain a determinant representation for $F(\alpha, m)$.

A particularly simple correlator to compute within this approach is the "Ferromagnetic String Formation Probability" (FSFP). This correlation function is defined as

$$P(m) = \left\langle \prod_{j=1}^{m} \frac{1}{2}(\sigma_j^z + 1) \right\rangle ,$$

and physically corresponds to the probability to find a string of ferromagnetically ordered adjacent spins in the (antiferromagnetic) ground state. It can be obtained from $F(\alpha, m)$ in the limit $\alpha \to -\infty$,

$$P(m) = \lim_{\alpha \to -\infty} F(\alpha, m) .$$

5. Scalar Products

We now turn to the investigation of scalar products of the form

$$S_N = \langle 0 | \prod_{j=1}^{N} C(\lambda_j^C) \prod_{k=1}^{N} B(\lambda_k^B) | 0 \rangle . \tag{5.1}$$

Following [30, 31] we will show how to represent (5.1) as a determinant. Here we *do not* assume that the sets of spectral parameters $\{\lambda^B\}$ and $\{\lambda^C\}$ are the same, and we also *do not* impose the Bethe equations (2.15), because our goal is to determine the scalar products occurring in (4.4). From (2.8) and (2.12) it follows that scalar products can be represented as [29]

$$S_N = \sum_{A,D} \prod_{j=1}^{N} a(\lambda_j^A) \prod_{k=1}^{N} d(\lambda_k^D) K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^A\} & \{\lambda^D\} \end{pmatrix} , \tag{5.2}$$

where the sum is over all partitions of $\{\lambda^C\} \cup \{\lambda^B\}$ into two sets $\{\lambda^A\}$ and $\{\lambda^D\}$. The coefficients K_N are functions of the λ_j and are completely determined by the intertwining relation (2.7). In particular the K_N's are identical for the homogeneous model (2.6) and the *inhomogeneous* model (2.11), i.e. the K_N's are independent of the inhomogeneities $\{\nu_n\}$ and also do not depend on the lattice length L as long as $N < L$. The reason for this is that the intertwining relations for the matrix elements $\mathscr{A}(\mu)$, $\mathscr{B}(\mu)$, $\mathscr{C}(\mu)$ and $\mathscr{D}(\mu)$ of (2.11) are the same as the ones for the matrix elements $A(\mu)$, $B(\mu)$, $C(\mu)$, $D(\mu)$ of (2.6) (see Sect. 2 above). We will exploit this fact by considering special inhomogeneous models for which all terms but one in the sum in (5.2) vanish, and then represent this term as a determinant. The basic tool for representing scalar products as determinants in a theorem due to Izergin, Coker and Korepin [24, 25], which deals with determinant representations for the *partition functions* of inhomogeneous XXZ and XXX models constructed according to (2.11):

Theorem 1. *Consider an inhomogeneous XXZ chain of even length N with inhomogeneities ν_j, $j = 1 \dots N$. Let $|0\rangle$ and $|\bar{0}\rangle$ be the ferromagnetic reference states with all spins up and down respectively. Let $\mathscr{B}(\mu)$ and $\mathscr{C}(\mu)$ be the creation/annihilation operators over the reference state $|0\rangle$. Then the following determinant* *representations hold for the XXZ magnet*:

$$\langle\bar{0}|\prod_{j=1}^{N}\mathscr{B}(\lambda_j)|0\rangle = \langle 0|\prod_{j=1}^{N}\mathscr{C}(\lambda_j)|\bar{0}\rangle$$
$$= (-1)^N\prod_{\alpha=1}^{N}\prod_{k=1}^{N}\sinh(\lambda_\alpha - \nu_k - i\eta)\sinh(\lambda_\alpha - \nu_k + i\eta)$$
$$\times\left(\prod_{1\leqq\alpha<\beta\leqq N}\sinh(\lambda_\alpha-\lambda_\beta)\prod_{1\leqq k<l\leqq N}\sinh(\nu_l-\nu_k)\right)^{-1}\det(\mathscr{M}), \tag{5.3}$$

where

$$\mathscr{M}_{\alpha k} = \frac{i\sin(2\eta)}{\sinh(\lambda_\alpha - \nu_k - i\eta)\sinh(\lambda_\alpha - \nu_k + i\eta)}. \tag{5.4}$$

A similar representation holds for the XXX magnet.

Let us now derive explicit expressions for the coefficients K_N. It will be convenient to work with the following sets of spectral parameters

$$\{\lambda^{AC}\} = \{\lambda^A\}\cap\{\lambda^C\},\qquad \{\lambda^{DC}\} = \{\lambda^D\}\cap\{\lambda^C\},$$
$$\{\lambda^{AB}\} = \{\lambda^A\}\cap\{\lambda^B\},\qquad \{\lambda^{DB}\} = \{\lambda^D\}\cap\{\lambda^B\},$$

with cardinalities

$$n = \mathrm{card}\{\lambda^{DC}\} = \mathrm{card}\{\lambda^{AB}\},$$
$$N - n = \mathrm{card}\{\lambda^{AC}\} = \mathrm{card}\{\lambda^{DB}\}.$$

The partition with $n = 0$ is characterized by $\{\lambda^{AC}\} = \{\lambda^C\}$, $\{\lambda^{DB}\} = \{\lambda^B\}$, $\{\lambda^{AB}\} = \emptyset = \{\lambda^{DC}\}$. The corresponding coefficient $K_N\begin{pmatrix}\{\lambda^C\} & \{\lambda^B\}\\ \{\lambda^C\} & \{\lambda^B\}\end{pmatrix}$ is called *highest coefficient.*

Lemma 1. *For highest coefficients the following determinant representation holds*:

$$K_N\begin{pmatrix}\{\lambda^C\} & \{\lambda^B\}\\ \{\lambda^C\} & \{\lambda^B\}\end{pmatrix} = \left(\prod_{j>k} g(\lambda_j^B,\lambda_k^B)g(\lambda_k^C,\lambda_j^C)\right)\prod_{j,k} h(\lambda_j^C,\lambda_k^B)\det(M_C^B),$$

$$h(\mu,\nu) = \frac{f(\mu,\nu)}{g(\mu,\nu)},\qquad (M_C^B)_{jk} = \frac{g(\lambda_j^C,\lambda_k^B)}{h(\lambda_j^C,\lambda_k^B)} = t(\lambda_j^C,\lambda_k^B). \tag{5.5}$$

For the XXZ magnet we find

$$h(\lambda,\mu) = \frac{\sinh(\lambda-\mu+2i\eta)}{i\sin(2\eta)},\qquad t(\lambda,\mu) = \frac{(i\sin(2\eta))^2}{\sinh(\lambda-\mu+2i\eta)\sinh(\lambda-\mu)}, \tag{5.6}$$

and in the XXX case we have instead

$$h(\lambda,\mu) = 1 - i(\lambda-\mu),\quad t(\lambda,\mu) = -\frac{1}{(\lambda-\mu)(\lambda-\mu+i)}. \tag{5.7}$$

Proof. We will carry out the proof for the XXZ case, the XXX case is similar. Consider an inhomogeneous XXZ model on a lattice of length N with inhomogeneities $v_j = \lambda_j^C + i\eta$. We have $a(\lambda) = \prod_{j=1}^N \sinh(\lambda - \lambda_j^C - 2i\eta)$ and $d(\lambda) = \prod_{j=1}^N \sinh(\lambda - \lambda_j^C)$. Inspection of (5.2) yields that in this situation only one term in the sum of the r.h.s of (5.2) survives, namely the one with $\{\lambda^D\} = \{\lambda^B\}$. Thus for this special scalar product we obtain

$$S_N|_{v_j=\lambda_j^C+i\eta} = K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^C\} & \{\lambda^B\} \end{pmatrix} \prod_{j,k} \sinh(\lambda_j^C - \lambda_k^C - 2i\eta) \prod_{m,l} \sinh(\lambda_m^B - \lambda_l^C) . \quad (5.8)$$

On the other hand $B(\lambda)$ flips one spin, and as we have chosen N to be the length of the lattice we find that $\prod_{j=1}^N B(\lambda_j)|0\rangle$ is proportional to the ferromagnetic state with all spins flipped, and thus orthogonal to all states in a basis other than $|\bar{0}\rangle$. Thus

$$S_N|_{v_j=\lambda_j^C+i\eta} = \langle 0| \prod_{j=1}^N C(\lambda_j^C)|\bar{0}\rangle\langle\bar{0}| \prod_{k=1}^N B(\lambda_k^B)|0\rangle .$$

By Theorem 1 both factors can be represented as determinants. By direct computation we find for one of the factors

$$\langle 0| \prod_{j=1}^N C(\lambda_j^C)|\bar{0}\rangle = \prod_{j,k} \sinh(\lambda_k^C - \lambda_j^C - 2i\eta) .$$

Using the determinant representation given by Theorem 1 on the other factor we arrive at (5.5). □

Lemma 2. *Arbitrary coefficients K_N are expressed in terms of highest coefficients as follows:*

$$K_N \begin{pmatrix} \{\lambda^C\}\{\lambda^B\} \\ \{\lambda^A\}\{\lambda^D\} \end{pmatrix} = \left(\prod_{j\in AC} \prod_{k\in DC} f(\lambda_j^{AC}, \lambda_k^{DC}) \right) \left(\prod_{l\in AB} \prod_{m\in DB} f(\lambda_l^{AB}, \lambda_m^{DB}) \right)$$
$$\times K_n \begin{pmatrix} \{\lambda^{AB}\} & \{\lambda^{DC}\} \\ \{\lambda^{AB}\} & \{\lambda^{DC}\} \end{pmatrix} K_{N-n} \begin{pmatrix} \{\lambda^{AC}\} & \{\lambda^{DB}\} \\ \{\lambda^{AC}\} & \{\lambda^{DB}\} \end{pmatrix} . \quad (5.9)$$

Proof. We again will only treat the XXZ case explicitly, the XXX case being very similar. Consider an inhomogeneous XXZ model with inhomogeneities$\{v_j\} = \{\lambda_j^{AB} + i\eta\} \cup \{\lambda_j^{AC} + i\eta\}$. Now only the term proportional to $K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^A\} & \{\lambda^D\} \end{pmatrix}$ in the sum on the r.h.s of. (5.2) survives. Proceeding as in the proof of Lemma 1 above we arrive at (5.9). □

Combining the results of Lemmas 1 and 2 with (5.2) we arrive at the following expression for general scalar products of XXZ and XXX magnets:

$$S_N = \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \sum \mathrm{sgn}(P_C)\, \mathrm{sgn}(P_B) \prod_{j,k} h(\lambda_j^{AB}, \lambda_k^{DC}) \prod_{l,m} h(\lambda_l^{AC}, \lambda_m^{DB})$$
$$\times \prod_{l,k} h(\lambda_l^{AC}, \lambda_k^{DC}) \prod_{j,m} h(\lambda_j^{AB}, \lambda_m^{DB}) \det(M_{DC}^{AB}) \det(M_{DB}^{AC}) , \quad (5.10)$$

where P_C is the permutation $\{\lambda_1^{AC},\ldots,\lambda_n^{AC},\lambda_1^{DC},\ldots,\lambda_{N-n}^{DC}\}$ of $\{\lambda_1^C,\ldots,\lambda_N^C\}$, P_B is the permutation $\{\lambda_1^{DB},\ldots,\lambda_n^{DB},\lambda_1^{AB},\ldots,\lambda_{N-n}^{AB}\}$ of $\{\lambda_1^B,\ldots,\lambda_N^B\}$, $\mathrm{sgn}(P)$ is the sign of the permutation P, and

$$(M_{DC}^{AB})_{jk} = t(\lambda_j^{AB},\lambda_k^{DC})d(\lambda_k^{DC})a(\lambda_j^{AB}), \qquad t(\lambda,\mu) = \frac{(g(\lambda,\mu))^2}{f(\lambda,\mu)} .$$

Note that (5.10) is formally the same as the corresponding expression for the delta-function Bose gas (see [32] p. 213), only the functions $f(\lambda,\mu)$, $g(\lambda,\mu)$ (and thus also h and t), $a(\lambda)$ and $d(\lambda)$ are different.

6. Dual Fields

The most important step in the DFA follows next: we introduce *dual quantum fields* in order to simplify (5.10) and obtain a manageable expression for scalar products. This step was first carried out in [30] for the delta-function Bose gas. The XXX and XXZ cases of interest here can be treated very similarly, so that we will be brief in our discussion. The fundamental observation is that the r.h.s. in (5.10) looks like the determinant of the *sum* of two matrices:

Lemma 3. *Let A and B be two $N \times N$ matrices over $\mathbb{C}$. Then the determinant of their sum can be decomposed as follows*:

$$\det(A+B) = \sum_{P_r,P_c} \mathrm{sgn}(P_r)\,\mathrm{sgn}(P_c)\det(A_{P_rP_c})\det(B_{P_rP_c}) . \tag{6.1}$$

Here P_r and P_c are partitions of the N rows and columns into two subsets $\mathscr{R},\bar{\mathscr{R}}$ and $\mathscr{C},\bar{\mathscr{C}}$ of cardinalities n (for $\mathscr{R},\mathscr{C}$) and $N-n$ (for $\bar{\mathscr{R}},\bar{\mathscr{C}}$) respectively, $A_{P_rP_c}$ is the $n\times n$ matrix obtained from A by removing all $\bar{\mathscr{R}}$-rows and $\bar{\mathscr{C}}$-columns, and $B_{P_rP_c}$ is the $N-n\times N-n$ matrix obtained from B by removing all $\mathscr{R}$-rows and $\mathscr{C}$-columns. Finally $\mathrm{sgn}(P_r)$ *is the parity of the permutation obtained from $(1,\ldots,N)$ by moving all $\mathscr{R}$-rows to the front.*

Proof. See [32] p. 221 ff.

Comparison of (5.10) with Lemma 3 shows that one does not get the $h(\lambda,\mu)$-factors by simply taking the determinant of the sum of the matrices M_{jk}. This leads to the introduction of two *dual quantum fields* $\Phi_A(\lambda)$ and $\Phi_D(\lambda)$ which are represented as sums of "momenta" P_A and "coordinates" Q_A as follows:

$$\Phi_A(\lambda) = Q_A(\lambda) + P_D(\lambda), \qquad \Phi_D(\lambda) = Q_D(\lambda) + P_A(\lambda) ,$$

$$[P_D(\lambda),Q_D(\mu)] = \ln(h(\lambda,\mu)), \qquad [P_A(\lambda),Q_A(\mu)] = \ln(h(\mu,\lambda)) . \tag{6.2}$$

All other commutators of P's and Q's vanish. A very important property of the fields Φ is that they commute for different values of spectral parameters

$$[\Phi_A(\lambda),\Phi_D(\mu)] = 0 = [\Phi_A(\lambda),\Phi_A(\mu)] = [\Phi_D(\lambda),\Phi_D(\mu)] .$$

The dual quantum fields act on a bosonic Fock space with reference states $|0)$ and $(0|$ defined by

$$P_a(\lambda)|0) = 0, \quad (0|Q_a(\lambda) = 0, \quad a = A,D, \ (0|0) = 1 . \tag{6.3}$$

Using the dual fields it is now possible to recast (5.10) as a determinant of the sum of two matrices.

Theorem 2. *Scalar products for the Heisenberg XXZ and XXX magnets can be represented as determinants in the following way*:

$$S_N = \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B)(0|\det S|0)\,,$$

$$S_{jk} = t(\lambda_j^C, \lambda_k^B) a(\lambda_j^C) d(\lambda_k^B) \exp(\Phi_A(\lambda_j^C) + \Phi_D(\lambda_k^B))$$
$$+ t(\lambda_k^B, \lambda_j^C) d(\lambda_j^C) a(\lambda_k^B) \exp(\Phi_D(\lambda_j^C) + \Phi_A(\lambda_k^B))\,. \tag{6.4}$$

Proof. Using Lemma 3 to expand the determinant in (6.4) we arrive at

$$(0|\det S|0) = \sum \operatorname{sgn}(P_C)\operatorname{sgn}(P_B)\det(M_{DC}^{AB})\det(M_{DB}^{AC})$$
$$\times(0|\exp\left(\sum_{j=1}^{n} \Phi_A(\lambda_j^{AC}) + \Phi_D(\lambda_j^{DB}) + \sum_{k=1}^{N-n} \Phi_A(\lambda_k^{AB}) + \Phi_D(\lambda_k^{DC})\right)|0)\,. \tag{6.5}$$

Evaluating the expectation value of the dual quantum fields by means of (6.2) and (6.3) we arrive at (5.10). □

It is possible to further simplify (6.4) by eliminating one dual field: we define a new dual vacuum $(\tilde{0}|$ according to

$$(\tilde{0}| = (0|\exp\left(\sum_{j=1}^{N} P_D(\lambda_j^C) + P_A(\lambda_j^B)\right)\,, \quad (0|0) = 1\,, \tag{6.6}$$

and a new dual field

$$\varphi(\lambda) = p(\lambda) + q(\lambda), \quad q(\lambda) = Q_A(\lambda) - Q_D(\lambda) - (\tilde{0}|Q_A(\lambda) - Q_D(\lambda)|0)\,,$$
$$p(\lambda) = P_D(\lambda) - P_A(\lambda), \quad (\tilde{0}|q(\lambda) = 0 = p(\lambda)|0)\,,$$
$$[p(\lambda), q(\mu)] = -\ln(h(\lambda,\mu)h(\mu,\lambda)),$$
$$[p(\lambda), p(\mu)] = 0 = [q(\lambda), q(\mu)], [\varphi(\lambda), \varphi(\mu)] = 0\,. \tag{6.7}$$

In terms of this field we obtain the following determinant representation:

$$S_N = \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \prod_{j=1}^{N} a(\lambda_j^C) d(\lambda_j^B) \prod_{j,k} h(\lambda_j^C, \lambda_k^B)(\tilde{0}|\det S|0)\,,$$
$$S_{jk} = t(\lambda_j^C, \lambda_k^B) + t(\lambda_k^B, \lambda_j^C)\frac{r(\lambda_k^B)}{r(\lambda_j^C)}\exp(\varphi(\lambda_k^B) - \varphi(\lambda_j^C))$$
$$\times \prod_{m=1}^{N} \frac{h(\lambda_k^B, \lambda_m^B) h(\lambda_m^C, \lambda_j^C)}{h(\lambda_m^C, \lambda_k^B) h(\lambda_j^C, \lambda_m^B)}\,, \tag{6.8}$$

where $r(\lambda) = \frac{a(\lambda)}{d(\lambda)}$. Now we have all the machinery ready to tackle the problem of representing (4.4) as a determinant.

7. On Norms

In this section we will have a closer look at norms of Bethe wave functions. These were first conjectured in [14] (see also [15]). This conjecture was generalized and proved in [29], so that the answers are already known. Here we will consider norms as special cases of scalar products in order to build up some machinery needed below for further analysis of (6.4). We will treat the XXZ case in detail and quote the results for XXX. In order to study norms we ought to set $\{\lambda^C\} = \{\lambda^B\}$ in (6.4) and then impose the Bethe equations (2.15). Immediately some problems arise as the diagonal elements of the matrix S in (6.4) become ill-defined ("$\frac{0}{0}$") and have to be investigated more carefully. The off-diagonal matrix elements are easily dealt with. The Bethe equations (2.15) together with the antisymmetry property $g(\lambda,\mu) = -g(\mu,\lambda)$ imply

$$r(\lambda_k)\prod_{\substack{j=1\\ j\neq k}}^{N}\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)} = (-1)^{N-1}, \quad k=1,\dots,N\,. \tag{7.1}$$

Thus we obtain

$$S_{jk} = t(\lambda_j,\lambda_k) + t(\lambda_k,\lambda_j)\exp(\varphi(\lambda_k)-\varphi(\lambda_j)), \quad j\neq k\,. \tag{7.2}$$

To obtain the diagonal matrix elements we take the limit $\lambda_j \longrightarrow \lambda_k$ in the matrix S and use l'Hospital's rule (here we have to make use of the explicit expressions for the functions f, g, a, d, etc for the XXZ case)

$$S_{jj} = i\sin(2\eta)\frac{\partial}{\partial\lambda}\left[\ln(r(\lambda)) + \sum_{n=1}^{N}\ln\left(\frac{h(\lambda,\lambda_n)}{h(\lambda_n,\lambda)}\right)\right]\Bigg|_{\lambda=\lambda_j}$$
$$-\,2\,\cos(2\eta) + i\sin(2\eta)\frac{\partial\varphi}{\partial\lambda}\bigg|_{\lambda=\lambda_j}\,. \tag{7.3}$$

To obtain this expression we also have made use of the Bethe equations (7.1). We observe that the last two terms in (7.3) are precisely what one obtains when taking the limit $\lambda_j \longrightarrow \lambda_k$ in (7.2). Putting everything together we find the following expression for the norm (4.5):

$$\sigma_N = \langle 0|\prod_{j=1}^{N} C(\lambda_j)\prod_{k=1}^{N} B(\lambda_k)|0\rangle = \prod_{j\neq k} f(\lambda_j,\lambda_k)\prod_{j=1}^{N} a(\lambda_j)d(\lambda_j)(\tilde{0}|\det\mathcal{N}|0)\,,$$

$$\mathcal{N}_{jk} = t(\lambda_j,\lambda_k) + t(\lambda_k,\lambda_j)\exp(\varphi(\lambda_k)-\varphi(\lambda_j))$$
$$+\,i\sin(2\eta)\delta_{jk}\frac{\partial}{\partial\lambda}\left[\ln(r(\lambda)) + \sum_{n=1}^{N}\ln\left(\frac{h(\lambda,\lambda_n)}{h(\lambda_n,\lambda)}\right)\right]_{\lambda=\lambda_j}\,, \tag{7.4}$$

where we now interpret the first two terms in $\mathcal{N}$ in the sense of l'Hospital for the diagonal elements. For the case of the XXX magnet we have to replace $\sin(2\eta)$ by 1 and use the functions f, g, h, t following from (2.3). There is one further simplification: from [29] it follows that the expectation value of the dual field part in (7.4) is such that the dual fields can be simply set equal to zero, i.e. we can replace $(\tilde{0}|\det \mathcal{N}|0)$ by $\det \mathcal{N}'$, where $\mathcal{N}'$ is obtained from $\mathcal{N}$ by dropping the $\exp(\varphi)$-terms. Then a further simplification takes place as

$$t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) = -\sin(2\eta) K(\lambda_j, \lambda_k),$$

where K is defined in (2.17). This is summarized in the following theorem due to Korepin [29]:

Theorem 3. *Norms for the Heisenberg XXZ and XXX magnets can be represented as determinants in the following way:*

$$\langle 0| \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k) |0\rangle = \prod_{j \neq k} f(\lambda_j, \lambda_k) \prod_{j=1}^{N} a(\lambda_j) d(\lambda_j) \det \mathcal{N}' . \tag{7.5}$$

For the XXZ case the matrix $\mathcal{N}'$ is given by

$$\mathcal{N}'_{jk} = \sin(2\eta) \left(-K(\lambda_j, \lambda_k) + i\, \delta_{jk} \frac{\partial}{\partial \lambda_j} \left[\ln(r(\lambda_j)) + \sum_{n=1}^{N} \ln \left(\frac{h(\lambda_j, \lambda_n)}{h(\lambda_n, \lambda_j)} \right) \right] \right)$$

$$= \sin(2\eta) \left(-K(\lambda_j, \lambda_k) + \delta_{jk} \left[i \frac{\partial}{\partial \lambda_j} \ln(r(\lambda_j)) + \sum_{n=1}^{N} K(\lambda_j, \lambda_n) \right] \right) , \tag{7.6}$$

where $K(\lambda, \mu)$ and $h(\lambda, \mu)$ are defined in (2.17) *and* (5.6) *respectively. For the XXX case we have instead*

$$\mathcal{N}'_{jk} = -\frac{2}{(\lambda_j - \lambda_k)^2 + 1} + \delta_{jk} \left[i \frac{\partial}{\partial \lambda_j} \ln(r(\lambda_j)) + \sum_{n=1}^{N} \frac{2}{1 + (\lambda_j - \lambda_n)^2} \right] . \tag{7.7}$$

Proof. See [29].

8. Correlators on the Finite Chain

Let us now come back to the generating functional for correlators (4.4). We will now use the machinery built up in the last few sections to express $F(\alpha, m)$ as a determinant. We will proceed in two steps: we first will analyse (4.4) *without* using that $\{\lambda^B\} = \{\lambda^C\} = \{\lambda\}$ and *without* imposing the Bethe-equations (2.15). In the second step we will then impose these two constraints. Using (6.8) we can represent the scalar products in the two-site generalized models in (4.4) as determinants.

Lemma 4.

$$F(\alpha, m) = \frac{1}{\sigma_N} \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \sum \mathrm{sgn}(P_C) \mathrm{sgn}(P_B) \prod_{I_B, II_B} h(\lambda_I^B, \lambda_{II}^B)$$

$$\times \prod_{I_C, II_C} h(\lambda_{II}^C, \lambda_I^C) (0| \det_n s_1(\{\lambda_I^C\}, \{\lambda_I^B\}) \det_{N-n} s_2(\{\lambda_{II}^C\}, \{\lambda_{II}^B\}) |0) ,$$

$$(s_1(\{\lambda^C\}, \{\lambda^B\}))_{jk} = \exp(\alpha) d_2(\lambda_j^C) a_2(\lambda_k^B) (\bar{s}_1(\{\lambda^C\}, \{\lambda^B\}))_{jk} ,$$

$$(s_2(\{\lambda^C\}, \{\lambda^B\}))_{jk} = a_1(\lambda_j^C) d_1(\lambda_k^B) (\bar{s}_2(\{\lambda^C\}, \{\lambda^B\}))_{jk} , \tag{8.1}$$

where

$$(\bar{s}_\gamma(\{\lambda^C\},\{\lambda^B\}))_{jk} = t(\lambda_j^C,\lambda_k^B)a_\gamma(\lambda_j^C)d_\gamma(\lambda_k^B)\exp(\Phi_{A_\gamma}(\lambda_j^C)+\Phi_{D_\gamma}(\lambda_k^B))$$
$$+ t(\lambda_k^B,\lambda_j^C)d_\gamma(\lambda_j^C)a_\gamma(\lambda_k^B)\exp(\Phi_{D_\gamma}(\lambda_j^C)+\Phi_{A_\gamma}(\lambda_k^B)). \quad (8.2)$$

Here the dual fields are defined according to

$$\Phi_{A_\gamma}(\lambda) = Q_{A_\gamma}(\lambda)+P_{D_\gamma}(\lambda), \qquad \Phi_{D_\gamma}(\lambda) = Q_{D_\gamma}(\lambda)+P_{A_\gamma}(\lambda),$$
$$[P_{D_\gamma}(\lambda),Q_{D_\beta}(\mu)] = \delta_{\gamma\beta}\ln(h(\lambda,\mu)), \qquad [P_{A_\gamma}(\lambda),Q_{A_\beta}(\mu)] = \delta_{\gamma\beta}\ln(h(\mu,\lambda)). \quad (8.3)$$

All other commutators vanish. The reference state $|0)$ *and its dual* $(0|$ *are annihilated by all momenta/coordinates respectively*

$$P_a(\lambda)|0) = 0, \quad (0|Q_a(\lambda) = 0, \quad a = A_\gamma, D_\gamma, \ (0|0) = 1. \quad (8.4)$$

Proof. We use (6.4) to express both ${}_1\langle 0|\prod_{I_C} C_1(\lambda_{I_C}^C)\prod_{I_B} B_1(\lambda_{I_B}^B)|0\rangle_1$ and ${}_2\langle 0|\prod_{II_C} C_2(\lambda_{II_C}^C)\prod_{II_B} B_2(\lambda_{II_B}^B)|0\rangle_2$ as determinants. We are led to introduce two sets of dual fields (one for each scalar product) $\Phi_{A_\gamma}(\lambda),\Phi_{D_\gamma}(\lambda)$, $\gamma = 1,2$ with commutation relations given by (8.3). The two kinds of dual fields are completely independent of each other (all commutators between momenta/coordinates of different sets vanish). The representation (8.1)–(8.2) is now obtained by direct computation, where the $\mathrm{sgn}(P_B)\mathrm{sgn}(P_C)$ arises upon taking the factor $\prod_{j>k} g(\lambda_j^C,\lambda_k^C)g(\lambda_k^B,\lambda_j^B)$ in front of the sum due to $g(\lambda,\mu) = -g(\mu,\lambda)$. □

We now observe that (8.1) is basically of the same structure as (5.10). Thus, in analogy with (6.4), we can introduce new dual quantum fields and reexpress $F(\alpha,m)$ as *a single* determinant.

Lemma 5. *Consider the set of four commuting dual quantum fields*

$$\psi_{D_1}(\lambda) = \mathcal{Q}_{D_1}(\lambda)+\mathcal{P}_{A_2}(\lambda), \qquad \psi_{A_1}(\lambda) = \mathcal{Q}_{A_1}(\lambda)+\mathcal{P}_{D_2}(\lambda),$$
$$\psi_{D_2}(\lambda) = \mathcal{Q}_{D_2}(\lambda)+\mathcal{P}_{A_1}(\lambda), \qquad \psi_{A_2}(\lambda) = \mathcal{Q}_{A_2}(\lambda)+\mathcal{P}_{D_1}(\lambda),$$

with commutation relations of the momenta/coordinates given by

$$[\mathcal{P}_{D_\gamma}(\lambda),\mathcal{Q}_{D_\beta}(\mu)] = \delta_{\gamma\beta}\ln(h(\lambda,\mu)), \quad [\mathcal{P}_{A_\gamma}(\lambda),\mathcal{Q}_{A_\beta}(\mu)] = \delta_{\gamma\beta}\ln(h(\mu,\lambda)).$$

All other commutators vanish. The action of the dual fields on the dual reference states is given by $\mathcal{P}_a(\lambda)|0) = 0$, $(0|\mathcal{Q}_a(\lambda) = 0$, $a = A_1,A_2,D_1,D_2$. *Then the following determinant representation holds*

$$F(\alpha,m) = \frac{1}{\sigma_N}\prod_{j>k} g(\lambda_j^C,\lambda_k^C)g(\lambda_k^B,\lambda_j^B)(0|\det\mathcal{M}|0),$$

$$\mathcal{M}_{jk} = (s_1(\{\lambda^C\},\{\lambda^B\}))_{jk}\exp(\psi_{D_1}(\lambda_j^C)+\psi_{A_1}(\lambda_k^B))$$
$$+ (s_2(\{\lambda^C\},\{\lambda^B\}))_{jk}\exp(\psi_{A_2}(\lambda_j^C)+\psi_{D_2}(\lambda_k^B)), \quad (8.5)$$

where $(s_\gamma)_{jk}$ *are given by* (8.1).

Proof. The proof is analogous to the one for Theorem 2, only the expectation value of dual quantum fields is slightly different.

So far we have not used the fact that we are dealing with expectation values of Bethe states, i.e. we have neither used the fact that $\{\lambda^C\} = \{\lambda^B\} = \{\lambda\}$ nor imposed the Bethe equations (2.15). In the next step we will impose these constraints. The discussion will be reminiscent of Sect. 7 above. The result is summarized in the following.

Theorem 4. *The generating functional $F(\alpha, m)$ can be represented as a ratio of determinants in the following way:*

$$F(\alpha,m) = \frac{(\tilde{0}|\det \mathscr{G}|0)}{\det \mathscr{N}'},$$

$$\begin{aligned}\mathscr{G}_{jk} &= t(\lambda_j,\lambda_k) + t(\lambda_k,\lambda_j)\frac{r_1(\lambda_j)}{r_1(\lambda_k)}\exp(\varphi_2(\lambda_k) - \varphi_2(\lambda_j)) \\ &+ \exp(\alpha + \varphi_4(\lambda_k) - \varphi_3(\lambda_j)) \\ &\times \left[t(\lambda_k,\lambda_j) + t(\lambda_j,\lambda_k)\frac{r_1(\lambda_j)}{r_1(\lambda_k)}\exp(\varphi_1(\lambda_j) - \varphi_1(\lambda_k))\right] \\ &+ \delta_{jk}\omega\left(LD(\lambda_j) + \sum_n K(\lambda_j,\lambda_n)\right), \end{aligned} \tag{8.6}$$

where $r_1(\lambda) = a_1(\lambda)/d_1(\lambda)$, $K(\lambda,\mu)$ *and* $D(\lambda)$ *are defined in* (2.17), (2.18), $\omega = \sin(2\eta)$ *for XXZ and* $\omega = -1$ *for XXX, and the commuting dual fields* φ_a *are defined according to*

$$\varphi_a(\lambda) = p_a(\lambda) + q_a(\lambda), \quad (\tilde{0}|q_a(\lambda) = 0 = p_a(\lambda)|0), \quad (\tilde{0}|0) = 1, \ a = 1\ldots 4,$$

$$[q_b(\mu), p_a(\lambda)] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \ln(h(\lambda,\mu)) + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \ln(h(\mu,\lambda)),$$
$$a, b = 1\ldots 4. \tag{8.7}$$

Here all terms not proportional to δ_{jk} in $\mathscr{G}_{jk}$ are understood in the sense of l'Hospital for the diagonal elements.

Proof. We start by defining a new dual vacuum $(\tilde{0}|$ and a new set of dual fields according to

$$(\tilde{0}| = (0|\exp\left(\sum_{j=1}^{N} P_{D_2}(\lambda_j) + P_{A_2}(\lambda_j) + \mathscr{P}_{D_1}(\lambda_j) + \mathscr{P}_{A_1}(\lambda_j)\right), \quad (\tilde{0}|0) = 1, \tag{8.8}$$

$$\begin{aligned}\phi_1(\lambda) &= \Phi_{A_1}(\lambda) - \Phi_{D_1}(\lambda), \quad \phi_2(\lambda) = \Phi_{A_2}(\lambda) - \Phi_{D_2}(\lambda), \\ \phi_3(\lambda) &= \psi_{A_2}(\lambda) - \psi_{D_1}(\lambda) - \Phi_{D_1}(\lambda) + \Phi_{A_2}(\lambda), \\ \phi_4(\lambda) &= \psi_{A_1}(\lambda) - \psi_{D_2}(\lambda) + \Phi_{A_1}(\lambda) - \Phi_{D_2}(\lambda). \end{aligned} \tag{8.9}$$

The fields $\phi_a(\lambda)$ can be decomposed into momenta $\mathring{p}_a$ and coordinates $\mathring{q}_a$ (by using (8.3) and the definitions of ψ_a given in Lemma 5), which are found to obey the commutation relations (8.7). By straightforward rewriting of (8.5) in terms of the new fields and the new dual reference state we obtain

$$F(\alpha, m) = \frac{1}{\sigma_N} \prod_{j>k} f(\lambda_j, \lambda_k) f(\lambda_k, \lambda_j) \prod_{j=1}^{N} a(\lambda_j) d(\lambda_j) (\tilde{0}| \det \bar{\mathcal{M}} |0) ,$$

$$\bar{\mathcal{M}}_{jk} = t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} \exp(\phi_2(\lambda_k) - \phi_2(\lambda_j))$$

$$+ \frac{r(\lambda_k)}{r(\lambda_j)} \exp(\alpha + \phi_4(\lambda_k) - \phi_3(\lambda_j))$$

$$\times \left[t(\lambda_k, \lambda_j) + t(\lambda_j, \lambda_k) \frac{r_1(\lambda_j)}{r_1(\lambda_k)} \exp(\phi_1(\lambda_j) - \phi_1(\lambda_k)) \right] . \qquad (8.10)$$

Here we have used that

$$(0| \exp \left(\sum_{j=1}^{N} \psi_{A_2}(\lambda_j) + \psi_{D_2}(\lambda_j) + \Phi_{A_2}(\lambda_j) + \Phi_{D_2}(\lambda_j) \right) = \prod_{j,k} h(\lambda_j, \lambda_k) (\tilde{0}| .$$

It is found that whereas $\mathring{p}_a(\lambda)|0) = 0$, the coordinates $\mathring{q}_a(\lambda)$ of $\phi_a(\lambda)$ do not annihilate the new dual reference state $(\tilde{0}|$. Therefore we "shift" $\phi_a(\lambda)$ by subtracting their vacuum expectation values in analogy with (6.7),

$$\varphi_a(\lambda) = \phi_a(\lambda) - (\tilde{0}|\phi_a(\lambda)|0) = p_a(\lambda) + q_a(\lambda), \quad a = 1, \ldots 4 . \qquad (8.11)$$

By construction the p's and q's have the same commutation relations (8.7) as the momenta/coordinates $\mathring{p}_a(\lambda)$ and $\mathring{q}_a(\lambda)$ of the $\phi_a(\lambda)$'s. Furthermore, $p_a(\lambda)|0) = 0$ and $(\tilde{0}|q_a(\lambda) = 0$ for $a = 1 \ldots 4$. The shifts are found to be

$$\kappa_a(\lambda) = (\tilde{0}|\phi_a(\lambda)|0) = (1 - \delta_{a1}) \sum_j \ln \left(\frac{h(\lambda, \lambda_j)}{h(\lambda_j, \lambda)} \right) .$$

If we replace the fields ϕ_a in (8.10) by the fields φ_a we pick up additional factors due to the shifts

$$F(\alpha, m) = \frac{1}{\sigma_N} \prod_{j \neq k} f(\lambda_j, \lambda_k) \prod_{j=1}^{N} a(\lambda_j) d(\lambda_j) (\tilde{0}| \det \mathcal{G} |0) ,$$

$$\mathcal{G}_{jk} = t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} e^{\varphi_2(\lambda_k) - \varphi_2(\lambda_j)} e^{\kappa_2(\lambda_k) - \kappa_2(\lambda_j)}$$

$$+ \frac{r(\lambda_k)}{r(\lambda_j)} e^{\alpha + \varphi_4(\lambda_k) - \varphi_3(\lambda_j)} e^{\kappa_4(\lambda_k) - \kappa_3(\lambda_j)}$$

$$\times \left[t(\lambda_k, \lambda_j) + t(\lambda_j, \lambda_k) \frac{r_1(\lambda_j)}{r_1(\lambda_k)} \exp(\varphi_1(\lambda_j) - \varphi_1(\lambda_k)) \right] . \qquad (8.12)$$

The off-diagonal matrix elements of G can be further simplified by simply imposing the Bethe equations. Rewriting the Bethe equations (2.15) as

$$r_2(\lambda_k)\prod_{\substack{j=1\\ j\neq k}}^{N}\frac{h(\lambda_k,\lambda_j)}{h(\lambda_j,\lambda_k)}=\frac{(-1)^{N-1}}{r_1(\lambda_k)}\,,$$

$$\frac{1}{r_2(\lambda_k)}\prod_{\substack{j=1\\ j\neq k}}^{N}\frac{h(\lambda_j,\lambda_k)}{h(\lambda_k,\lambda_j)}=(-1)^{N-1}r_1(\lambda_k),\quad k=1,\dots,N\,, \tag{8.13}$$

we find that the additional factors take the form

$$\exp\left(\kappa_2(\lambda_k)-\kappa_2(\lambda_j)\right)\frac{r_2(\lambda_k)}{r_2(\lambda_j)}=\frac{r_1(\lambda_j)}{r_1(\lambda_k)},$$

$$\exp\left(\kappa_4(\lambda_k)-\kappa_3(\lambda_j)\right)\frac{r(\lambda_k)}{r(\lambda_j)}=1\,.$$

Inserting this into (8.10) we arrive at (8.6) *without* the term proportional to δ_{jk}, i.e. we have proved (8.6) for the off-diagonal matrix elements. To get the diagonal matrix elements we have to investigate the limit $\lambda_j \rightarrow \lambda_k$ of (8.12) in detail. In the limit $\lambda_j \rightarrow \lambda_k$ the sum of the first two terms in G_{jk} and the expression in brackets are both of the form "$\frac{0}{0}$". By using l'Hospital's rule we find analogously to Sect. 7 above,

$$\lim_{\lambda_j\to\lambda_k}\left(t(\lambda_j,\lambda_k)+t(\lambda_k,\lambda_j)\frac{r_2(\lambda_k)}{r_2(\lambda_j)}e^{\varphi_2(\lambda_k)-\varphi_2(\lambda_j)}e^{\kappa_2(\lambda_k)-\kappa_2(\lambda_j)}\right)$$

$$=-2\cosh(2i\eta)+\sinh(2i\eta)\left.\frac{\partial\varphi_2(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_j}$$

$$+\sinh(2i\eta)\frac{\partial}{\partial\lambda_j}\left[\ln(r_2(\lambda_j))+\sum_n\ln\left(\frac{h(\lambda_j,\lambda_n)}{h(\lambda_n,\lambda_j)}\right)\right]\,,$$

$$\lim_{\lambda_j\to\lambda_k}\left(t(\lambda_j,\lambda_k)+t(\lambda_k,\lambda_j)\frac{r_1(\lambda_j)}{r_1(\lambda_k)}e^{\varphi_2(\lambda_k)-\varphi_2(\lambda_j)}\right)$$

$$=-2\cosh(2i\eta)+\sinh(2i\eta)\left(\frac{\partial\varphi_2(\lambda_j)}{\partial\lambda_j}+m\frac{\sinh(2i\eta)}{\sinh(\lambda_j+i\eta)\sinh(\lambda_j-i\eta)}\right)\,.$$

Using these expressions we find that the diagonal terms of G in (8.12) are equal to the diagonal terms of G in (8.6) if we keep in mind that the first two lines of G_{jk} in (8.6) are interpreted à la l'Hospital for $j=k$. Last but not least we insert the expression (7.5) for σ_N in the resulting expression and arrive at (8.6). This completes the proof of the theorem. □

Theorem 4 states the determinant representation for the generating functional $F(\alpha,m)$ on a *finite* chain of length L. As always in Bethe Ansatz solvable models significant simplifications take place if we take the thermodynamic limit $L\rightarrow\infty$. This is done in the next section.

9. Thermodynamic Limit

The results of taking the thermodynamic limit of (8.6) and the main results of this paper are summarized in the following.

Theorem 5. *In the thermodynamic limit the generating functional $F(\alpha, m)$ for the case of the XXZ magnet can be represented as a ratio of determinants of Fredholm integral operators $\left(id + \frac{1}{2\pi}\widehat{V}\right)$ and $\left(id - \frac{1}{2\pi}\widehat{K}\right)$ in the following way:*

$$F(\alpha, m) = \frac{(\tilde{0}|\det(id + \frac{1}{2\pi}\widehat{V})|0)}{\det(id - \frac{1}{2\pi}\widehat{K})} . \tag{9.1}$$

Here $(\tilde{0}|$ and $|0)$ are the vacua of the dual bosonic Fock space defined in (8.7) *and the integral operators act on functions f defined on the interval $[-\Lambda, \Lambda]$ according to*

$$\left(id - \frac{1}{2\pi}\widehat{K}\right) * f\Bigg|_{\lambda} = f(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) f(\mu) ,$$

$$\left(id + \frac{1}{2\pi}\widehat{V}\right) * f\Bigg|_{\lambda} = f(\lambda) + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\mu V(\lambda, \mu) f(\mu) ,$$

where the kernel $K(\lambda, \mu)$ is defined in (2.17) *and the kernel of $\widehat{V}$ is given by*

$$\begin{aligned} V(\lambda, \mu) = \frac{-\sin(2\eta)}{\sinh(\lambda - \mu)} \Bigg\{ & \frac{1}{\sinh(\lambda - \mu + 2i\eta)} - \frac{e_2^{-1}(\lambda) e_2(\mu)}{\sinh(\mu - \lambda + 2i\eta)} \\ & + \exp(\alpha + \varphi_4(\mu) - \varphi_3(\lambda)) \\ & \times \left(\frac{-1}{\sinh(\mu - \lambda + 2i\eta)} + \frac{e_1^{-1}(\mu) e_1(\lambda)}{\sinh(\lambda - \mu + 2i\eta)} \right) \Bigg\} , \end{aligned} \tag{9.2}$$

$$e_2(\lambda) = \left(\frac{\sinh(\lambda + i\eta)}{\sinh(\lambda - i\eta)}\right)^m \exp(\varphi_2(\lambda)), \qquad e_1(\lambda) = \left(\frac{\sinh(\lambda - i\eta)}{\sinh(\lambda + i\eta)}\right)^m \exp(\varphi_1(\lambda)) . \tag{9.3}$$

The dual fields $\varphi_a(\lambda)$ are defined in (8.7), *with $h(\lambda, \mu)$ given in* (5.6).

Proof. We begin by taking the thermodynamic limit for the norm σ_N (7.5). We first write $\mathcal{N}'$ as the product of two matrices:

$$\mathcal{N}'_{jk} = \sin(2\eta) \sum_m I_{jm} J_{mk}, \qquad I_{jm} = \delta_{jm} - \frac{K_{jm}}{\theta_m}, \qquad J_{jm} = \delta_{jm}\theta_m ,$$

where $\theta_m = LD(\lambda_m) + \sum_n K(\lambda_m, \lambda_n)$. Here D and K are defined in (2.16)–(2.17). The determinant of $\mathcal{N}'$ is the product of the determinants of I and J. Next we use that the set of roots $\{\lambda_j\}$ describes the ground state and the roots thus obey the equations

$$2\pi L\rho(\lambda_j) - \sum_{k=1}^{N} K(\lambda_j, \lambda_k) = LD(\lambda_j), \quad j = 1 \ldots N ,$$

which is the discrete version of (2.16). Here $\rho(\lambda_j) = \frac{1}{L(\lambda_{j+1}-\lambda_j)}$, which becomes $\rho(\lambda)$ defined by (2.16) in the thermodynamic limit. We thus can rewrite $\theta_m = 2\pi L\rho(\lambda_m)$, which leads to

$$\det J = \prod_{j=1}^{N} 2\pi L\rho(\lambda_j)\,. \tag{9.4}$$

In the thermodynamic limit the matrix I turns into an integral operator $\widehat{I} = id - \frac{1}{2\pi}\widehat{K}$,

$$\widehat{I} * f\,|_\lambda = f(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\mu K(\lambda,\mu) f(\mu)\,,$$

where K is the kernel of $\widehat{K}$ defined by (2.17).

The matrix G_{jk} in (8.6) is treated in a very similar way. We rewrite it as a product

$$G_{jk} = \sin(2\eta)\sum_m W_{jm}J_{mk}\,,$$

where $J_{jm} = \delta_{jm}2\pi L\rho(\lambda_m)$ is the same as above, and

$$\begin{aligned} W_{jk} = \delta_{jk} + \frac{1}{\sin(2\eta)\theta_k}&\Bigg\{t(\lambda_j,\lambda_k) + t(\lambda_k,\lambda_j)\frac{r_1(\lambda_j)}{r_1(\lambda_k)}\exp\left(\varphi_2(\lambda_k) - \varphi_2(\lambda_j)\right) \\ &+ \exp\left(\alpha + \varphi_4(\lambda_k) - \varphi_3(\lambda_j)\right) \\ &\times\left[t(\lambda_k,\lambda_j) + t(\lambda_j,\lambda_k)\frac{r_1(\lambda_j)}{r_1(\lambda_k)}\exp\left(\varphi_1(\lambda_j) - \varphi_1(\lambda_k)\right)\right]\Bigg\}\,. \end{aligned}$$

In the thermodynamic limit the matrix W_{jk} turns into an integral operator $\widehat{W} = id + \frac{1}{2\pi}\widehat{V}$, with kernel $V(\lambda,\mu)$ defined by (9.2). Thus we obtain (9.1) in the thermodynamic limit. □

For the XXX chain a determinant representation is obtained in an analogous way. The result is found to be

$$F_{XXX}(\alpha,m) = \frac{(\tilde{0}|\det(1 + \frac{1}{2\pi}\widehat{V}_{XXX})|0)}{\det(1 - \frac{1}{2\pi}\widehat{K}_{XXX})}\,, \tag{9.5}$$

where $\widehat{K}_{XXX}$ and $\widehat{V}_{XXX}$ are integral operators with kernels $K(\lambda,\mu)$ from (2.18) and

$$\begin{aligned} V(\lambda,\mu) = \frac{1}{\lambda-\mu}&\left\{\frac{1}{\lambda-\mu+i} - \frac{e_2^{-1}(\lambda)e_2(\mu)}{\mu-\lambda+i} + \exp\left(\alpha + \varphi_4(\mu) - \varphi_3(\lambda)\right)\right. \\ &\left.\times\left(\frac{-1}{\mu-\lambda+i} + \frac{e_1^{-1}(\mu)e_1(\lambda)}{\lambda-\mu+i}\right)\right\}\,, \end{aligned} \tag{9.6}$$

$$e_2(\lambda) = \left(\frac{\lambda+\frac{i}{2}}{\lambda-\frac{i}{2}}\right)^m \exp\left(\varphi_2(\lambda)\right), \quad e_1(\lambda) = \left(\frac{\lambda-\frac{i}{2}}{\lambda+\frac{i}{2}}\right)^m \exp\left(\varphi_1(\lambda)\right).$$

The dual fields $\varphi_a(\lambda)$ are again defined in (8.7), but now $h(\lambda,\mu) = 1 - i(\lambda - \mu)$.

The Ferromagnetic String Formation Probability can be easily obtained from (9.1) and (9.5) by setting $\alpha = -\infty$, which corresponds to dropping the second line in the expressions for the kernel of $\widehat{V}$ in (9.2) and (9.6). For the XXX case this exactly reproduces the result of [33].

10. Some Limiting Cases

It is quite straightforward to evaluate the determinants in (9.1) for strong magnetic fields $h \sim h_c = (2\cos\eta)^2$, in which the ground state is very close to the ferromagnetic vacuum and $\Lambda \ll 1$. The near asymptotics ($m \ll (\pi/2\Lambda)\tan\eta$) of the FSFP for the XXZ case follows to be

$$P(m) = 1 - \left(\frac{2\Lambda}{\pi\sin\eta}\cos\eta\right) m \, .$$

Using (2.16) and (2.19) this reproduces the obvious result

$$P(m) = 1 - \frac{1}{2}(1 - \langle\sigma_j^z\rangle)m = 1 - \frac{m}{\pi}\sqrt{h_c - h}, \;\; h \to h_c, \quad h < h_c \, .$$

Another interesting limiting case (which allows to make contact with known results) is the $XX0$ free fermionic limit of the XXZ model [42], where $\eta = \frac{3\pi}{4}$ in (9.2) and (2.17). Correlation functions for this case have been previously considered by various authors [43, 44, 55, 8, 22]. In [8, 22] a determinant representation for $F(\alpha, m)$ was found, which does not involve dual quantum fields. Taking the free fermionic limit of (8.7) we obtain

$$[q_b(\mu), p_a(\lambda)] = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} \ln(\cosh(\lambda - \mu)) \, .$$

Thus we can choose $\varphi_3(\lambda) = \varphi_4(\lambda)$ and reduce the number of dual fields to 3. Furthermore we have $\sin(4\eta) = 0$ and therefore $K(\lambda, \mu) = 0$. The determinant formula then reads

$$F_{XX}(\alpha, m) = (0|\det\left(id + \frac{1}{2\pi}\widehat{V}\right)|0),$$

$$V_{XX}(\lambda, \mu) = \frac{2i}{\sinh 2(\lambda - \mu)} \times \left\{1 - e_2^{-1}(\lambda)e_2(\mu) - e^{\alpha + \varphi_3(\mu) - \varphi_3(\lambda)}(1 - e_1^{-1}(\mu)e_1(\lambda))\right\} \, , \quad (10.1)$$

where now

$$e_2(\lambda) = \left(\frac{e^{2\lambda} - i}{e^{2\lambda} + i}\right)^m \exp(\varphi_2(\lambda)), \qquad e_1(\lambda) = \left(\frac{e^{2\lambda} + i}{e^{2\lambda} - i}\right)^m \exp(\varphi_1(\lambda)) \, .$$

This expression should be compared with the result obtained in [8, 9] by means of direct integration of the coordinate-space ground state wave function (which is of free fermionic form). After transforming the latter expression to the notation used in the present paper the result of [8, 9] reads

$$F_{XX}(\alpha, m) = \det\left(id + \frac{1}{2\pi}\widehat{U}\right) . \tag{10.2}$$

Here $\widehat{U}$ is the integral operator defined in terms of the kernel

$$U(\lambda, \mu) = +\frac{i(1-e^{\alpha})}{\sinh(\lambda-\mu)}\left\{1 - \left(\frac{e^{2\lambda}+i}{e^{2\lambda}-i}\,\frac{e^{2\mu}-i}{e^{2\mu}+i}\right)^{m}\right\} . \tag{10.3}$$

The representation (10.2)–(10.3) can be reobtained in the present formulation by making use of a determinant representation for scalar products in the XX0 case due to N. Slavnov [48].

Theorem (N. Slavnov). *Scalar products in the Heisenberg XX0 model can be represented as determinants in the following way:*

$$\begin{aligned} S_N &= \prod_{j>k}^{N} f(\lambda_j^C, \lambda_k^C) f(\lambda_k^B, \lambda_j^B) \det n, \\ n_{jk} &= \langle 0|C(\lambda_j^C) B(\lambda_k^B)|0\rangle = g(\lambda_j^C, \lambda_k^B)(a(\lambda_j^C) d(\lambda_k^B) - d(\lambda_j^C) a(\lambda_k^B)) , \end{aligned} \tag{10.4}$$

where $f(\lambda, \mu) = -i\coth(\lambda - \mu)$, $g(\lambda, \mu) = \frac{-i}{\sinh(\lambda-\mu)}$, $a(\lambda) = (\sinh(\lambda - \frac{3\pi i}{4}))^L$ *and* $d(\lambda) = (\sinh(\lambda + \frac{3\pi i}{4}))^L$.

Proof. Slavnov's proof is based on the following identity for the $N \times N$ matrix A with entries $A_{jk} = \frac{1}{\sinh(x_j - y_k)}$, where x_j and y_k are arbitrary complex numbers

$$\det A = \left(\prod_{j>k}^{N} \sinh(x_j - x_k)\sinh(y_k - y_j)\right)\left(\prod_{j,k}^{N}\sinh(x_j - y_k)\right)^{-1} . \tag{10.5}$$

This equality can be proved by induction over N. Equation (10.4) is obtained by using (10.5) for the determinants of the matrices M_{DC}^{AB} and M_{DB}^{AC} in expression (5.10) for scalar products, and then using Lemma 3 to express the sum over partitions in (5.10) as the determinant of the sum of two matrices. In the last step no dual quantum fields need to be introduced. □

Using the new determinant representation (10.4) for scalar products in the XX0 case in the expression (4.4) we arrive at

$$\begin{aligned} F(\alpha, m) = \frac{1}{\sigma_N} \prod_{j>k}^{N} f(\lambda_j^C, \lambda_k^C) f(\lambda_k^B, \lambda_j^B) \sum_{I,II} \operatorname{sgn}(P_C)\operatorname{sgn}(P_B)\det(X(\{\lambda_I^C\}, \{\lambda_I^B\})) \\ \times \det(Y(\{\lambda_{II}^C\}, \{\lambda_{II}^B\})) , \end{aligned} \tag{10.6}$$

where P_γ, $\gamma = B, C$ are the permutations $\{\lambda^\gamma_{I,1}, \dots \lambda^\gamma_{I,n}, \lambda^\gamma_{II,1}, \dots \lambda^\gamma_{II,N-n}\}$ of $\{\lambda^\gamma_1, \dots, \lambda^\gamma_N\}$ and where

$$X(\{\lambda^C_I\}, \{\lambda^B_I\})_{jk} = e^\alpha a_2(\lambda^B_{I,k}) d_2(\lambda^C_{I,j}) g(\lambda^C_{I,j}, \lambda^B_{I,k})(a_1(\lambda^C_{I,j}) d_1(\lambda^B_{I,k}) - d_1(\lambda^C_{I,j}) a_1(\lambda^B_{I,k}))\,,$$

$$Y(\{\lambda^C_{II}\}, \{\lambda^B_{II}\})_{jk} = d_1(\lambda^B_{II,k}) a_1(\lambda^C_{II,j}) g(\lambda^C_{II,j}, \lambda^B_{II,k})(a_2(\lambda^C_{II,j}) d_2(\lambda^B_{II,k}) - d_2(\lambda^C_{II,j}) a_2(\lambda^B_{II,k}))\,.$$

Now we can apply Lemma 3 to (10.6) and express $F(\alpha, m)$ as a single determinant without dual quantum fields, and then take $\{\lambda^C\} = \{\lambda^B\} = \{\lambda\}$. We obtain

$$F(\alpha, m) = \frac{\det \mathcal{M}}{\det \mathcal{N}}\,,$$

$$\mathcal{M}_{jk} = (1 - e^\alpha) g(\lambda_j, \lambda_k) \left(1 - \frac{r_1(\lambda_j)}{r_1(\lambda_k)}\right) - \delta_{jk} \frac{L}{\sinh(\lambda_j + \frac{3\pi i}{4}) \sinh(\lambda_j - \frac{3\pi i}{4})}\,,$$

$$\mathcal{N}_{jk} = -\delta_{jk} \frac{L}{\sinh(\lambda_j + \frac{3\pi i}{4}) \sinh(\lambda_j - \frac{3\pi i}{4})}\,,$$

where $r_1(\lambda) = \left(\frac{\sinh(\lambda - \frac{3\pi i}{4})}{\sinh(\lambda + \frac{3\pi i}{4})}\right)^m$. In the thermodynamic limit (cf. Sect. 9) this turns into (10.2) and (10.3). This establishes the equivalence of the two determinant representations with and without dual fields in the XX0 model.

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References

1. Barouch, E., McCoy, B.M., Wu, T.T.: Phys. Rev. Lett. **31**, 1409 (1973)
2. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. New York: Academic Press, 1982
3. Baxter, R.J.: Ann. Phys. (N.Y.) **70**, 323 (1972)
4. Bethe, H.: Z. Phys. **71**, 205 (1931)
5. Bonner, J.C., Fisher, M.E.: Phys. Rev. **135**, 640 (1964)
6. Bougourzi, A. Weston, R.: N-point Correlation Functions of the Spin-1 XXZ Model. Preprint hep-th/9307124
7. Bougourzi, A. Weston, R.: The Dynamical Correlation Function of the XXZ Model. Preprint hep-th/9407062
8. Colomo, F., Izergin, A.G., Korepin, V.E., Tognetti, V.: Phys. Lett. **169A**, 243 (1992)
9. Colomo, F., Izergin, A.G., Korepin, V.E., Tognetti, V.: Theor. Math. Phys. **94**, 19 (1993)
10. Davies, B., Foda, O., Jimbo, M., Miwa, T., Nakayashiki, A.: Commun. Math. Phys. **151**, 89 (1993)
11. Faddeev, L.D., Takhtajan, L.A.: Sov. Sci. Rev. Math. **C1**, 107 (1981)

12. Foda, O., Jimbo, M., Miki, K., Miwa, T., Nakayashiki, A.: J. Math. Phys. **35**, 13 (1994)
13. Gaudin, M.: La Fonction d'Onde de Bethe. Paris: Masson, 1983
14. Gaudin, M.: Internal report, Centre d'Études Nucléaires de Saclay, CEA-N-1559, (1), (1972)
15. Gaudin, M., McCoy, B.M., Wu, T.T.: Phys. Rev. **D23**, 417 (1981)
16. Griffiths, R.B.: Phys. Rev. **133**, 768 (1964)
17. Hulthen, L.: Arkiv Mat. Astron. Fysik **26A** (1938)
18. Its, A.R., Izergin, A.G., Korepin, V.E.: Commun. Math. Phys. **130**, 471 (1990)
19. Its, A.R., Izergin, A.G., Korepin, V.E.: Physica **D53**, 187 (1991)
20. Its, A.R., Izergin, A.G., Korepin, V.E.: Phys. Lett. **141A**, 121 (1989)
21. Its, A.R., Izergin, A.G., Korepin, V.E., Slavnov, N.: Int. J. Mod. Phys. **B4**, 1003 (1990)
22. Its, A.R., Izergin, A.G., Korepin, V.E., Slavnov, N.: Phys. Rev. Lett. **70** 1704 (1993)
23. Its, A.R., Izergin, A.G., Korepin, V.E., Varzugin G.G.: Physica **D54**, 351 (1992)
24. Izergin, A.G.: Sov. Phys. Dokl. **32**, 878 (1987)
25. Izergin, A.G., Coker, D., Korepin, V.E.: J. Physics **A25**, 4315 (1992)
26. Jimbo, M., Miwa, T., Mori, Y., Sato, M.: Physica **1D**, 80 (1980)
27. Jimbo, M., Miki, K., Miwa, T., Nakayashiki, A.: Phys. Lett. **168A**, 256 (1992)
28. Jimbo, M., Miwa, T., Nakayashiki, A.: J. Physics **A26**, 2199 (1993)
29. Korepin, V.E.: Commun. Math. Phys. **86**, 391 (1982)
30. Korepin, V.E.: Commun. Math. Phys. **113**, 177 (1987)
31. Korepin, V.E.: Funct. Analysis Appl. **23**, 15 (1989)
32. Korepin, V.E., Izergin, G., Bogoliubov, N.M.: Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz. Cambridge: Cambridge University Press, 1993
33. Korepin, V.E., Izergin, A.G., Eßler, F.H.L., Uglov, D.: Phys. Lett. **190A**, 182 (1994)
34. Korepin, V.E., Slavnov, N.: Commun. Math. Phys. **129**, 103 (1990)
35. Korepin, V.E., Slavnov, N.: Commun. Math. Phys. **136**, 633 (1991)
36. Kulish, P.P., Reshetikhin, N.Yu., Sklyanin, E.K.: Lett. Math. Phys. **5**, 393 (1981)
37. Kulish, P.P., Sklyanin, E.K.: J. Sov. Math. **19**, 1596 (1982)
38. Lenard, A.: J. Math. Phys. **5**, 930 (1964)
39. Lenard, A.: J. Math. Phys. **7**, 1268 (1966)
40. Lieb, E.H., Liniger, W.: Phys. Rev. **130**, 1605 (1963)
41. Lieb, E.H.: Phys. Rev. **130**, 1616 (1963)
42. Lieb, E.H., Schultz, T. Mattis, D.: Ann. Phys. (N.Y.) **16**, 406 (1961)
43. McCoy, B.M., Perk, J., Shrock, R.: Nucl. Phys. **B220**, 269 (1983)
44. McCoy, B.M., Perk, J. Shrock, R.: Nucl. Phys. **B220**, 35 (1983)
45. Nakayashiki, A.: Some integral formulas to the solution of the sl_2 dKZ equation with level −4. Preprint
46. Orbach, R.: Phys. Rev. **112**, 309 (1958)
47. Sklyanin, E.K., Takhtajan, L.A., Faddeev, L.D.: Theor. Mat. Phys. **40**, 688 (1979)
48. Slavnov, N.: To be published
49. Takahashi, M.: Prog. Theor. Phys. **46**, 401 (1971)
50. Takahashi, M., Suzuki, M.: Prog. Theor. Phys. **48**, 2187 (1972)
51. Takahashi, M.: Prog. Theor. Phys. **50**, 1519 (1973)
52. Takhtajan, L.A.: Exactly Solvable Problems in Condensed Matter and Relativistic field Theory. Shastry, B.S., Jha, S.S., Singh, V. (eds.) Lecture Notes in Physics, v. **242**, Berlin Heidelberg, New York: Springer, 1985, p. 175
53. Takhtajan, L.A., Faddeev, L.D.: Rus. Math. Sur **34**, 11 (1979)
54. Tracy, C., McCoy, B.M.: Phys. Rev. Lett. **31**, 1500 (1973)
55. Vaidya, H.G., Tracy, C.: Phys. Lett. **68A**, 378 (1978)
56. Yang, C.N., Yang, C.P.: Phys. Rev. **150**, 321 (1966)
57. Yang, C.N., Yang, C.P.: Phys. Rev. **150**, 327 (1966)
58. Yang, C.N., Yang, C.P.: Phys. Rev. **151**, 258 (1966)
59. Yang, C.N., Yang, C.P.: Phys. Rev. **19**, 1312 (1967)

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