# Determinantal expression and recursion for Jack polynomials

Luc Lapointe
Centre de recherches mathématiques
Université de Montréal, C.P. 6128, succ. Centre-Ville,
Montréal, Québec H3C 3J7, Canada
lapointe@crm.umontreal.ca

A. Lascoux
Institut Gaspard Monge, Université de Marne-la-Vallée
5 Bd Descartes, Champs sur Marne
77454 Marne La Vallée, Cedex, FRANCE
Alain.Lascoux@univ-mlv.fr

J. Morse
Department of Mathematics
University of Pennsylvania
209 South 33rd Street, Philadelphia, PA 19103, USA
morsej@math.upenn.edu

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**Abstract**

We describe matrices whose determinants are the Jack polynomials expanded in terms of the monomial basis. The top row of such a matrix is a list of monomial functions, the entries of the sub-diagonal are of the form $-(r\alpha + s)$, with $r$ and $s \in \mathbb{N}^+$, the entries above the sub-diagonal are non-negative integers, and below all entries are 0. The quasi-triangular nature of these matrices gives a recursion for the Jack polynomials allowing for efficient computation. A specialization of these results yields a determinantal formula for the Schur functions and a recursion for the Kostka numbers.

# 1 Introduction

The Jack polynomials $J_\lambda[x_1,\ldots,x_N;\alpha]$ form a basis for the space of $N$-variable symmetric polynomials. Here we give a matrix of which the determinant is $J_\lambda[x;\alpha]$ expanded in terms of the monomial basis. The top row of this matrix is a list of monomial functions, the entries of the sub-diagonal are of the form $-(r\alpha+s)$, with $r$ and $s\in\mathbb{N}^+$, the entries above the sub-diagonal are non-negative integers, and below all entries are 0. The quasi-triangular nature of this matrix gives a simple recursion for the Jack polynomials allowing for their rapid computation. The result here is a transformed specialization of the matrix expressing Macdonald polynomials given in [2]. However, we give a self-contained derivation of the matrix for Jack polynomials. Since the Schur functions $s_\lambda[x]$ are the specialization $\alpha=1$ in $J_\lambda[x;\alpha]$, we obtain a matrix of which the determinant gives $s_\lambda[x]$. A by-product of this result is a recursion for the Kostka numbers, the expansion coefficients of the Schur functions in terms of the monomial basis.

Partitions are weakly decreasing sequences of non-negative integers. We use the dominance order on partitions, defined $\mu\le\lambda \iff \mu_1+\cdots+\mu_i\le\lambda_1+\cdots+\lambda_i\ \forall i$. The number of non-zero parts of a partition $\lambda$ is denoted $\ell(\lambda)$. The Jack polynomials can be defined up to normalization by the conditions

$$\text{(i)}\qquad J_\lambda=\sum_{\mu\le\lambda}v_{\lambda\mu}m_\mu\,,\qquad \text{with }\ v_{\lambda\lambda}\neq 0\,,$$

$$\text{(ii)}\qquad HJ_\lambda=\left[\sum_{i=1}^{N}\left(\frac{\alpha}{2}\lambda_i^2+\frac{1}{2}(N+1-2i)\lambda_i\right)\right]J_\lambda\,, \tag{1}$$

where $H$ is the Hamiltonian of the Calogero-Sutherland model [7] defined

$$H=\frac{\alpha}{2}\sum_{i=1}^{N}\left(x_i\frac{\partial}{\partial x_i}\right)^2+\frac{1}{2}\sum_{i<j}\left(\frac{x_i+x_j}{x_i-x_j}\right)\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)\,. \tag{2}$$

A composition $\beta=(\beta_1,\ldots,\beta_n)$ is a vector of non-negative integral components and the partition rearrangement of $\beta$ is denoted $\beta^*$. The raising operator $R_{ij}^\ell$ acts on compositions by $R_{ij}^\ell\beta=(\beta_1,\ldots,\beta_i-\ell,\ldots,\beta_j+\ell,\ldots,\beta_n)$, for any $i<j$. We will use $n(k)$ to denote the number of occurrences of $k$ in $\mu$. This given, we use the following theorem [5] :

**Theorem 1.** *Given a partition $\lambda$, we have*

$$H\,m_\lambda=\left[\sum_{i=1}^{\ell(\lambda)}\left(\frac{\alpha}{2}\lambda_i^2+\frac{1}{2}(N+1-2i)\lambda_i\right)\right]m_\lambda+\sum_{\mu<\lambda}C_{\lambda\mu}\,m_\mu\,, \tag{3}$$

*where if there exists some* $i < j$, *and* $1 \le \ell \le \lfloor \frac{\lambda_i - \lambda_j}{2} \rfloor$ *such that* $\left(R^{\ell}_{ij}\lambda\right)^* = \mu$, *then*

$$C_{\lambda\mu} = \begin{cases} (\lambda_i - \lambda_j)\binom{n(\mu_i)}{2} & \text{if} \quad \mu_i = \mu_j \\ (\lambda_i - \lambda_j) n(\mu_i) n(\mu_j) & \text{if} \quad \mu_i \neq \mu_j \end{cases} \tag{4}$$

*and otherwise* $C_{\lambda\mu} = 0$.

*Example 1: with* $N = 5$,

$$\begin{array}{ll} H\, m_4 = (8+8\alpha) m_4 + 4\, m_{3,1} + 4\, m_{2,2} & H\, m_{2,1,1} = (5+3\alpha)\, m_{3,1} + 12\, m_{1,1,1,1} \\ H\, m_{3,1} = (7+5\alpha)\, m_{3,1} + 2\, m_{2,2} + 6\, m_{2,1,1} & H\, m_{1,1,1,1} = (2+2\alpha)\, m_{1,1,1,1} \\ H\, m_{2,2} = (6+4\alpha)\, m_{2,2} + 2\, m_{2,1,1} & \end{array}$$

## 2 Determinant for the Jack polynomials

We can obtain non-vanishing determinants which are eigenfunctions of the Hamiltonian $H$ by using the triangular action of $H$ on the monomial basis.

**Theorem 2.** *If* $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)} = \mu$ *is a linear ordering of all partitions* $\le \mu$, *then the Jack polynomial* $J_\mu$ *is proportional to the following determinant;*

$$J_\mu \doteq \begin{vmatrix} m_{\mu^{(1)}} & m_{\mu^{(2)}} & \ldots & \ldots & m_{\mu^{(n-1)}} & m_{\mu^{(n)}} \\ d_{\mu^{(1)}} - d_{\mu^{(n)}} & C_{\mu^{(2)}\mu^{(1)}} & \ldots & \ldots & C_{\mu^{(n-1)}\mu^{(1)}} & C_{\mu^{(n)}\mu^{(1)}} \\ 0 & d_{\mu^{(2)}} - d_{\mu^{(n)}} & & & \ldots & C_{\mu^{(n)}\mu^{(2)}} \\ \vdots & 0 & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \ldots & 0 & d_{\mu^{(n-1)}} - d_{\mu^{(n)}} & C_{\mu^{(n)}\mu^{(n-1)}} \end{vmatrix} \tag{5}$$

*where* $d_\lambda$ *denotes the eigenvalue in 1(ii) and* $C_{\mu^{(i)}\mu^{(j)}}$ *is defined by (4).*

Note that in the case $\mu = (a)$, the matrix $J_a$ contains all possible $C_{\mu^{(i)}\mu^{(j)}}$. Therefore, the matrices corresponding to $J_1, J_2, \ldots$ determine the entries off the sub-diagonal for all other matrices. Further, the sub-diagonal entries $d_{\mu^{(i)}} - d_{\mu^{(n)}}$ do not depend on the number of variables $N$, when $N \ge \ell(\mu)$, since for any partitions $\mu$ and $\lambda$ where $\ell(\mu) \ge \ell(\lambda)$,

$$d_\mu - d_\lambda = \sum_{i=1}^{\ell(\mu)} \left( \frac{\alpha}{2}(\mu_i^2 - \lambda_i^2) - i(\mu_i - \lambda_i) \right) . \tag{6}$$

It is also easily checked that if $\mu < \lambda$, then $d_\mu - d_\lambda = -(r + s\alpha)$ for some $r, s \in \mathbb{N}^+$, showing that the sub-diagonal entries are of this form.

*Example 2: The entries of the matrix for $J_4$ are obtained using the action of $H$ on $m_{\mu^{(i)}}$, where $\mu^{(1)}=(1,1,1,1)$, $\mu^{(2)}=(2,1,1)$, $\mu^{(3)}=(2,2)$, $\mu^{(4)}=(3,1)$, $\mu^{(5)}=(4)$, given in Example 1;*

$$J_4 \doteq \begin{vmatrix} m_{1,1,1,1} & m_{2,1,1} & m_{2,2} & m_{3,1} & m_4 \\ -6-6\alpha & 12 & 0 & 0 & 0 \\ 0 & -3-5\alpha & 2 & 6 & 0 \\ 0 & 0 & -2-4\alpha & 2 & 4 \\ 0 & 0 & 0 & -1-3\alpha & 4 \end{vmatrix} \tag{7}$$

We also obtain a determinantal expression for the Schur functions in terms of monomials using Theorem 2 since $s_\lambda[x]$ is the specialization $J_\lambda[x;1]$.

**Corollary 3.** *Given a partition $\mu$, the specialization $\alpha=1$ in the determinant (5) is proportional to the Schur function $s_\mu$.*

*Example 3: $s_4$ can be obtained by specializing $\alpha = 1$ in the matrix (7).*

$$s_4 \doteq \begin{vmatrix} m_{1,1,1,1} & m_{2,1,1} & m_{2,2} & m_{3,1} & m_4 \\ -12 & 12 & 0 & 0 & 0 \\ 0 & -8 & 2 & 6 & 0 \\ 0 & 0 & -6 & 2 & 4 \\ 0 & 0 & 0 & -4 & 4 \end{vmatrix} \tag{8}$$

**Proof of Theorem 2.** We have from (6) that $d_\lambda \neq d_\mu$ for $\lambda < \mu$ implying that the sub-diagonal entries, $d_{\mu^{(i)}} - d_\mu$, of determinantal expression (5) are non-zero. Since the coefficient of $m_{\mu^{(n)}} = m_\mu$ is the product of the sub-diagonal elements, this coefficient does not vanish and by the construction of $J_\mu$, Property 1(i) is satisfied. It thus suffices to check that $(H - d_\mu)\, J_\mu = 0$. Since $H$ acts non-trivially only on the first row of the determinant $J_\mu$, the first row of $(H - d_\mu)\, J_\mu$ is obtained from Theorem 1 and expression (5) gives rows $2,\ldots,n$.

$$\big(H-d_\mu\big)J_\mu=\left|\begin{array}{ccc} \ldots & d_{\mu^{(j)}}m_{\mu^{(j)}}+\sum_{i<j}C_{\mu^{(j)}\mu^{(i)}}m_{\mu^{(i)}}-d_\mu m_{\mu^{(j)}} & \ldots \\ \ldots & C_{\mu^{(j)}\mu^{(1)}} & \ldots \\ & \vdots & \\ \ldots & C_{\mu^{(j)}\mu^{(j-1)}} & \ldots \\ \ldots & d_{\mu^{(j)}}-d_\mu & \ldots \\ \ldots & 0 & \ldots \\ & \vdots & \end{array}\right|$$

$m_{\mu^{(n)}}$ appears only in the first row, column $n$, with coefficient $d_\mu-d_\mu=0$. Further, we have that the first row is the linear combination: $m_{\mu^{(1)}}\mathrm{row}_2+m_{\mu^{(2)}}\mathrm{row}_3+\cdots+m_{\mu^{(n-1)}}\mathrm{row}_n$, and thus the determinant must vanish.

# 3 Recursion for quasi-triangular matrices

We use the determinantal expressions for Jack and Schur polynomials to obtain recursive formulas. First we will give a recursion for $J_\lambda[x;\alpha]$ providing an efficient method for computing the Jack polynomials and we will finish our note by giving a recursive definition for the Kostka numbers. These results follow from a general property of quasi-triangular determinants [3, 8];

**Property 4.** *Any quasi-triangular determinant of the form*

$$D\;=\;\begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ -a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & -a_{32} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & -a_{n,n-1} & a_{n,n} \end{vmatrix} \tag{9}$$

*has the expansion* $D=\sum_{i=1}^n c_ib_i$ , *where* $c_n=a_{21}a_{32}\cdots a_{n,n-1}$ *and*

$$c_i=\frac{1}{a_{i+1,i}}\sum_{j=i+1}^{n}a_{i+1,j}\,c_j\quad\text{for all }\;i\in\{1,2,\ldots,n-1\}\,. \tag{10}$$

**Proof.** Given linearly independent $n$-vectors $\mathbf{b}$ and $\mathbf{a}^{(i)}$, $i \in \{2,\ldots,n\}$, let $\mathbf{c}$ be a vector orthogonal to $\mathbf{a}^{(2)},\ldots,\mathbf{a}^{(n)}$. This implies that the determinant of a matrix with row vectors $\mathbf{b},\mathbf{a}^{(2)},\ldots,\mathbf{a}^{(n)}$ is the scalar product $(\mathbf{c},\mathbf{b}) = \sum_{i=1}^n c_i b_i$, up to a normalization. In the particular case of matrices with the form given in (9), that is with $\mathbf{a}^{(i)} = (0,\ldots,0,-a_{i,i-1},a_{i,i},\ldots,a_{i,n})$, $i \in \{2,\ldots,n\}$, we can see that $(\mathbf{c},\mathbf{a}^{(i)}) = 0$ if the components of $\mathbf{c}$ satisfy recursion (10). Since we have immediately that the coefficient of $b_n$ in (9) is $c_n = a_{21}\cdots a_{n,n-1}$, ensuring that $\sum_{i=1}^n c_i b_i$ is properly normalized, Property 4 is thus proven.

Notice that we can freely multiply the rows of matrix (9) by non-zero constants and still preserve recursion (10). This implies that to obtain a matrix proportional to determinant (9), one would simply multiply the value of $c_n$ by the proportionality constant.

Since the determinantal expression (5) for the Jack polynomials is of the form that appears in Property 4, we may compute the Jack polynomials, in any normalization, using recursion (10).

<u>*Example:*</u> *Recall that* $v_{(4);(4)} = (1+\alpha)(1+2\alpha)(1+3\alpha)$ *in the normalization associated to the positivity of Jack polynomials [4, 6, 1]. Thus, using*

$$J_4 \doteq \begin{vmatrix} m_{1,1,1,1} & m_{2,1,1} & m_{2,2} & m_{3,1} & m_4 \\ -6-6\alpha & 12 & 0 & 0 & 0 \\ 0 & -3-5\alpha & 2 & 6 & 0 \\ 0 & 0 & -2-4\alpha & 2 & 4 \\ 0 & 0 & 0 & -1-3\alpha & 4 \end{vmatrix}, \tag{11}$$

*and initial condition* $c_5 = (1+\alpha)(1+2\alpha)(1+3\alpha)$*, we get*

$$\begin{aligned} c_4 &= \frac{1}{1+3\alpha}\,(4c_5) = 4(1+\alpha)(1+2\alpha)\,, \quad & c_2 &= \frac{1}{3+5\alpha}\,(2c_3+6c_4) = 12(1+\alpha)\,, \\ c_3 &= \frac{1}{2+4\alpha}\,(2c_4+4c_5) = 6(1+\alpha)^2\,, \quad & c_1 &= \frac{1}{6+6\alpha}\,(12c_2) = 24\,, \end{aligned} \tag{12}$$

*which gives that*

$$\begin{aligned} J_4 &= (1+\alpha)(1+2\alpha)(1+3\alpha)m_4 + 4(1+\alpha)(1+2\alpha)m_{3,1} \\ &\qquad + 6(1+\alpha)^2 m_{2,2} + 12(1+\alpha)m_{2,1,1} + 24 m_{1,1,1,1}\,. \end{aligned} \tag{13}$$

If we let $\mu^{(1)},\mu^{(2)},\ldots,\mu^{(n)} = \mu$ be a linear ordering of all partitions $\le \mu$ and recall that the Kostka numbers are the coefficients $K_{\mu\mu^{(i)}}$ in

$$s_\mu[x] = \sum_{i=1}^n K_{\mu\mu^{(i)}} m_{\mu^{(i)}}[x] \quad \text{where} \quad K_{\mu\mu^{(n)}} = K_{\mu\mu} = 1\,, \tag{14}$$

we can use Property 4 to obtain a recursion for the $K_{\mu\mu^{(i)}}$.

**Corollary 5.** *Let* $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)} = \mu$ *be a linear ordering of all partitions* $\leq \mu$. $K_{\mu\mu^{(i)}}$ *is defined recursively, with initial condition* $K_{\mu\mu} = 1$, *by*

$$K_{\mu\mu^{(i)}} = \frac{1}{g_{\mu^{(i)}} - g_\mu} \sum_{j=i+1}^{n} C_{\mu^{(i+1)}\mu^{(j)}} K_{\mu\mu^{(j)}} \quad \text{for all } \; i \in \{1, 2, \ldots, n-1\}, \tag{15}$$

*where* $C_{\mu^{(i+1)}\mu^{(j)}}$ *is given in (4) and where* $g_{\mu^{(i)}} - g_\mu$ *is the specialization* $\alpha = 1$ *of* $d_{\mu^{(i)}} - d_\mu$ *introduced in (4).*

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