Russian Math. Surveys **55**:5 923–975

Uspekhi Mat. Nauk **55**:5 107–160 DOI 10.1070/RM2000v055n05ABEH000321

Determinantal random point fields

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Abstract. This paper contains an exposition of both recent and rather old results on determinantal random point fields. We begin with some general theorems including proofs of necessary and sufficient conditions for the existence of a determinantal random point field with Hermitian kernel and of a criterion for weak convergence of its distribution. In the second section we proceed with examples of determinantal random fields in quantum mechanics, statistical mechanics, random matrix theory, probability theory, representation theory, and ergodic theory. In connection with the theory of renewal processes, we characterize all Hermitian determinantal random point fields on $\mathbb{R}^1$ and $\mathbb{Z}^1$ with independent identically distributed spacings. In the third section we study translation-invariant determinantal random point fields and prove the mixing property for arbitrary multiplicity and the absolute continuity of the spectra. In the last section we discuss proofs of the central limit theorem for the number of particles in a growing box and of the functional central limit theorem for the empirical distribution function of spacings.

Contents

AMS 2000 *Mathematics Subject Classification.* Primary 60G60, 60K35; Secondary 15A52, 60K05, 60F05, 37A25, 82C22.

1. Definition and general properties of determinantal random point fields

Let E be a one-particle space and let X be a space of countable configurations of particles in E. In general, E can be a separable Hausdorff space; however, for our purposes it suffices to set

$$E = \prod_{j=1}^{m} E_j, \quad \text{where} \quad E_j \cong \mathbb{R}^d \text{ (or } \mathbb{Z}^d). \tag{1.1}$$

Unless otherwise stated explicitly, we always assume below that $E = \mathbb{R}^d$ because the arguments and the proofs remain valid in the case of (1.1). We assume that each configuration $\xi = (x_i)$ with $x_i \in E$ for $i \in \mathbb{Z}^1$ (or $i \in \mathbb{Z}^1_+$ for $d > 1$) is locally finite, that is, the number $\#_K(\xi) = \#(x_i \in K)$ of particles in K is finite for any compact set $K \subset E$. The particles in ξ are ordered in some (natural) way, for instance, $x_i \le x_{i+1}$ for $d = 1$. For $d > 1$ we assume that either $x_i = x_{i+1}$ or one of the following conditions holds:

$$|x_i| = \left(\sum_{j=1}^{d} \bigl(x_i^{(j)}\bigr)^2\right)^{1/2} < |x_{i+1}| = \left(\sum_{j=1}^{d} \bigl(x_{i+1}^{(j)}\bigr)^2\right)^{1/2}, \tag{1.2}$$

where $x_i = \bigl(x_i^{(1)}, \dots, x_i^{(d)}\bigr)$, or $|x_i| = |x_{i+1}|$ and there is an r, $1 \le r \le d$, such that $x_i^{(j)} \le x_{i+1}^{(j)}$, $1 \le j \le r-1$, and $x_i^{(r)} < x_{i+1}^{(r)}$.

To introduce a σ-algebra of measurable subsets of X, we first define so-called *cylinder* sets. Let $B \subset E$ be a Borel set and let $n \ge 0$. We refer to $C_n^B = \{\xi \in X : \#_B(\xi) = n\}$ as a cylinder set and define $\mathcal{B}$ as the σ-algebra generated by all cylinder sets (that is, $\mathcal{B}$ is the minimal σ-algebra containing all sets of the form C_n^B).

Definition 1. A *random point field* is a triplet $(X, \mathcal{B}, \mathsf{P})$, where P is a probability measure on $(X, \mathcal{B})$.

This definition leads to a natural question. Namely, how can one construct probability measures of this kind? The corresponding theory was developed by Lenard in [1]–[3] for an arbitrary second countable locally compact Hausdorff space E. To some extent, the arguments are based on Kolmogorov's fundamental theorem in the theory of stochastic processes [4] and become especially simple in the case of (1.1). Let t and s be two vectors in E with rational components, $t = \bigl(t^{(1)}, \dots, t^{(d)}\bigr)$ and $s = \bigl(s^{(1)}, \dots, s^{(d)}\bigr)$. By $\sqcap_{t,s}$ we denote the open parallelepiped $\bigl\{x = \bigl(x^{(1)}, \dots, x^{(d)}\bigr) \in E : x^{(j)} = t^{(j)} + \theta_j\bigl(s^{(j)} - t^{(j)}\bigr),\ 0 < \theta_j < 1,\ j = 1, \dots, d\bigr\}$, and by $\mathcal{R}$ the family of finite unions of open, closed, and semi-closed parallelepipeds with rational points t and s. Suppose that a joint distribution of non-negative integer-valued random variables η_D, $D \in \mathcal{R}$, is constructed to be finitely additive (in what follows, we identify this distribution with $\#_D$), that is,

$$\eta_D = \sum_{i=1}^{n} \eta_{D_i} \quad \text{(a.e.)} \tag{1.3}$$

for $D = \bigsqcup_{i=1}^{n} D_i$ with $D, D_i \in \mathcal{R}$ for $i = 1, \ldots, n$. From (1.3), it is immediate that this distribution is countably additive, that is,

$$\eta_D = \sum_{i=1}^{\infty} \eta_{D_i} \quad \text{(a.e.)} \tag{1.4}$$

for $D = \bigsqcup_{i=1}^{\infty} D_i$ with $D, D_i \in \mathcal{R}$ for $i = 1, 2, \ldots$. (Of course, it is essential here that η_D takes non-negative integral values only!)

Further, one can readily see that the joint distribution of the random variables $\#_D = \eta_D$, $D \in \mathcal{R}$, satisfying condition (1.3) (or (1.4)) uniquely determines a probability distribution on $(X, \mathcal{B})$.

Since it is often convenient to define a distribution of random variables via their moments, the next definition is quite reasonable.

Definition 2. A locally integrable function $\rho_k \colon E^k \to \mathbb{R}^1_+$ is called a *k-point correlation function* of a random point field $(X, \mathcal{B}, \mathsf{P})$ if, for any disjoint Borel subsets $A_1, \ldots, A_m$ of E and for any $k_i \in \mathbb{Z}^1_+$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} k_i = k$, the following formula holds:

$$\mathsf{E} \prod_{i=1}^{m} \frac{(\#_{A_i})!}{(\#_{A_i} - k_i)!} = \int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho_k(x_1, \ldots, x_k)\, dx_1 \cdots dx_k, \tag{1.5}$$

where E stands for the expectation with respect to P.

In particular, $\rho_1(x)$ is the one-particle density because

$$\mathsf{E}\#_A = \int_A \rho_1(x)\, dx$$

for any Borel set $A \subset E$. In the general case $k \geq 1$ the function $\rho_k(x_1, \ldots, x_k)$ admits the following probabilistic interpretation: if $[x_1, x_i + dx_i]$, $i = 1, \ldots, k$, are infinitesimal parallelepiped neighbourhoods of x_i, then the product $\rho_k(x_1, x_2, \ldots, x_k)\, dx_1 \cdots dx_k$ is the probability of the event that each set $[x_1, x_i + dx_i]$ contains a particle. The existence and uniqueness problem for a random point field with given correlation functions was studied in [1]–[3]. It is not surprising that Lenard's papers revealed many clear parallels to the classical moment problem [5], [6]. In particular, a random point field is uniquely determined by its correlation functions if the distribution of the random variables $\{\#_A\}$ is uniquely determined by its moments. In [1] Lenard obtained the following sufficient condition for the uniqueness:

$$\sum_{k=0}^{\infty} \left(\frac{1}{(k+j)!} \int_{A^{k+j}} \rho_{k+j}(x_1, \ldots, x_{k+j})\, dx_1 \cdots dx_{k+j} \right)^{-\frac{1}{k}} = \infty \tag{1.6}$$

for any bounded Borel set $A \subset E$ and any integer $j \geq 0$. In fact, we can readily see that if the series is divergent for $j = 0$,

$$\sum_{k=0}^{\infty} \left(\frac{1}{k!} \int_{A^k} \rho_k(x_1, \ldots, x_k)\, dx_1 \cdots dx_k \right)^{-\frac{1}{k}} = \infty, \tag{1.6$'$}$$

then so is the series (1.6) for any $j \geq 0$. In [2], [3] Lenard obtained the following necessary and sufficient condition for the existence of a random point field with prescribed correlation functions.

Theorem 1. (Lenard) *Locally integrable functions $\rho_k \colon E^k \to \mathbb{R}^1$, $k = 1, 2, \dots$, can be represented as the correlation functions of a random point field if and only if the following conditions a) and b) are satisfied*:

a) **Symmetry condition**: *Each function ρ_k is invariant under the action of the symmetric group S_k, that is,*

$$\rho_k\big(x_{\sigma(1)}, \dots, x_{\sigma(k)}\big) = \rho_k(x_1, \dots, x_k) \tag{1.7}$$

for any $\sigma \in S_k$.

b) **Positivity condition**: *For any finite set of compactly supported measurable functions $\varphi_k \colon E^k \to \mathbb{R}^1$, $k = 0, 1, \dots, N$, such that*

$$\varphi_0 + \sum_{k=1}^{N} \sum_{i_1 \neq \cdots \neq i_k} \varphi_k(x_{i_1}, \dots, x_{i_k}) \geq 0 \tag{1.8}$$

for any $\xi = (x_i) \in X$, the following inequality holds:

$$\varphi_0 + \sum_{k=1}^{N} \int_{E^k} \varphi_k(x_1, \dots, x_k) \rho_k(x_1, \dots, x_k)\, dx_1 \cdots dx_k \geq 0. \tag{1.9}$$

The necessity of conditions a) and b) is elementary because both conditions have obvious probabilistic interpretations. In particular, the positivity condition means that the expectation must be non-negative for any non-negative random variable of a certain class. The proof of the sufficiency of conditions a) and b) is much more elaborate and is based on an analogue of the Riesz representation theorem and on the Riesz–Krein extension theorem (a distant relative of the Hahn–Banach theorem). It should be noted once more that Lenard's results hold for an arbitrary second countable locally compact Hausdorff space E.

One can obtain a somewhat weaker (but as before hopelessly ineffective!) version of the positivity condition by taking upper approximations of the functions φ_k by step functions. By $\mathcal{P}_k$ we denote the class of polynomials in k variables that take non-negative values on $\underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Since the polynomials

$$\left\{ \prod_{i=1}^{k} \prod_{j=0}^{m_i - 1} (x_i - j),\ m_i \geq 0 \right\}$$

form a linear basis in the vector space of all polynomials in k variables, it follows that any polynomial $q(x_1, \dots, x_k) \in \mathcal{P}_k$ can be represented in the form

$$q(x_1, \dots, x_k) = \sum_{m_1, \dots, m_k \geq 0} a_{m_1, \dots, m_k} \cdot \prod_{i=1}^{k} \prod_{j=0}^{m_i - 1} (x_i - j). \tag{1.10}$$

Positivity* condition. For any $q \in \mathcal{P}_k$ and any bounded Borel sets $A_1, \dots, A_k \subset E$, $k \geq 1$, the following inequality holds:

$$a_{0, \dots, 0} + \sum_{m \geq 1} \sum_{m_1 + \cdots + m_k = m} a_{m_1, \dots, m_k} \times \int_{\prod_{i=1}^{k} A_i^{m_i}} \rho_m(x_1, \dots, x_m)\, dx_1 \cdots dx_m \geq 0. \tag{1.11}$$

Indeed, the left-hand side in (1.11) is equal to

$$\mathsf{E}q(\#_{A_1},\dots,\#_{A_k}) = \mathsf{E}\Bigg[a_{0,\dots,0} + \sum_{m\ge 1}\sum_{m_1+\dots+m_k=m} a_{m_1,\dots,m_k} \times \sum_{i_1\ne\dots\ne i_m} \chi_{A_1^{m_1}\times\dots\times A_k^{m_k}}(x_{i_1},\dots,x_{i_m})\Bigg]. \quad (1.12)$$

It is useful to note that, in a sense, the positivity* condition is similar to the condition on the moments of an integer-valued non-negative random variable.

In this paper we study a special class of random point fields that was introduced by Macchi in [7] (see also [8]). Let $K\colon L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ be a non-negative operator that is locally of trace class. The last condition means that the operator $K\cdot\chi_B$ is of trace class for any bounded Borel set $B\subset\mathbb{R}^d$, where $\chi_B(x)$ is the characteristic function of B. Thus,

$$K\ge 0, \qquad \mathrm{Tr}(\chi_B K\chi_B) < +\infty. \quad (1.13)$$

An integral kernel $K(x,y)$ of the operator K is defined up to a set of Lebesgue measure zero in $\mathbb{R}^d\times\mathbb{R}^d$. For our purposes, it is convenient to choose $K(x,y)$ so that

$$\mathrm{Tr}\big((\chi_B K\chi_B)^n\big) = \int_{B^n} K(x_1,x_2)\cdot K(x_2,x_3)\cdot\ldots\cdot K(x_n,x_1)\,dx_1\cdots dx_n \quad (1.14)$$

for an arbitrary bounded measurable set B and any positive integer n. Lemmas 1 and 2 below show that such a choice is indeed possible.

Lemma 1 ([9], [10], Remark 3.4). *Let K be a trace class operator on $L^2(\mathbb{R}^d)$. Then an integral kernel of K can be chosen so that the function $M(x,y)\equiv K(x,x+y)$ is continuous as a function of y with range in $L^1(\mathbb{R}^d)$. Furthermore, if $m(y) = \int M(x,y)\,dx$, then $\mathrm{Tr}\,K = m(0) = \int K(x,x)\,dx$.*

Proof. We present the proof only for the case in which K is non-negative. The general case is quite similar. Let $\{\lambda_j\}_{j\ge1}$ be the set of non-zero eigenvalues of K and let $\{\varphi_j\}_{j\ge1}$ be the set of corresponding eigenfunctions. The canonical form of K (regarded as a self-adjoint compact operator) is

$$K = \sum_{j\ge1}\lambda_j\cdot(\varphi_j,\,\cdot\,)\cdot\varphi_j. \quad (1.15)$$

Let us choose $y\in\mathbb{R}^d$ and regard $M(x,y)=\sum_{j=1}^{\infty}\lambda_j\cdot\varphi_j(x)\cdot\overline{\varphi_j(x+y)}$ as a function of x. Since

$$\|\varphi_j(\,\cdot\,)\cdot\overline{\varphi_j(\,\cdot\,+y)}\,\|_1 = \int_{\mathbb{R}^d}|\varphi_j(x)\cdot\overline{\varphi_j(x+y)}|\,dx \le \|\varphi_j\|_2\cdot\|\varphi_j\|_2 = 1,$$

it follows that the series defining $M(\,\cdot\,,y)$ is convergent in $L^1(\mathbb{R}^d)$ for any y and that $\|M(\,\cdot\,,y)\|_1 \le \sum_{j=1}^{\infty}\lambda_j = \mathrm{Tr}\,K < +\infty$. Let us consider now the function

$K(x,y) \equiv M(x,y-x)$; it is well defined a.e. in $\mathbb{R}^d \times \mathbb{R}^d$ and gives a kernel for K. The L^1-continuity of $M(\cdot\,,y)$ follows from the inequality

$$\Big\| \sum_{j\ge 1} \lambda_j \big(\varphi_j(\cdot)\overline{\varphi_j(\cdot+y_1)} - \varphi_j(\cdot)\cdot\overline{\varphi_j(\cdot+y_2)}\big)\Big\|_1$$
$$\le \sum_{j=1}^{N} \lambda_j \cdot \|\varphi_j\|_2 \cdot \|\varphi_j(\cdot+y_1) - \varphi_j(\cdot+y_2)\|_2 + \sum_{j\ge N} \lambda_j.$$

Choosing N sufficiently large, we obtain $\sum_{j\ge N}\lambda_j < \frac{\varepsilon}{2}$. Choosing y_1 close enough to y_2 to make

$$\|\varphi_j(\cdot) - \varphi_j(\cdot + y_2 - y_1)\|_2 \le \frac{\varepsilon}{2\sum_{j=1}^N \lambda_j}$$

for any j, $1 \le j \le N$, we see that the first term is also less than $\frac{\varepsilon}{2}$.

Using Lemma 1, we obtain the following assertion.

Lemma 2. *Let K be a non-negative operator on $L^2(\mathbb{R}^d)$ that is locally of trace class. Then an integral kernel of K can be chosen so that the function*

$$M_B^{(k)}(x,y) = \big(\underbrace{(K\cdot\chi_B)\cdot\ldots\cdot(K\cdot\chi_B)}_{\longleftarrow\ k \text{ times}\ \longrightarrow}\big)(x,x+y)$$

is continuous as a function of y with range in $L^2(B)$ for any bounded Borel set $B \subset \mathbb{R}^d$ and any positive integer k. Furthermore,

$$\operatorname{Tr}(\chi_B K \chi_B)^k = \int_{B^k} K(x_1,x_2)\cdot\ldots\cdot K(x_k,x_1)\,dx_1\cdots dx_k.$$

Proof. Let $K_n = \chi_{[-n,n]^d} K \chi_{[-n,n]^d}$. By Lemma 1, one can choose a kernel $K_n(x,y)$ such that $K_n(\cdot\,,\cdot+y)$ is a continuous function of y with respect to the $L^1([-n,n]^d)$-norm. We write $M_n(x,y) = K_n(x,x+y)$. Since $K_{n+1}(x,y) = K_n(x,y)$ for almost all $(x,y) \in [-n,n]^d \times [-n,n]^d$, it follows that $M_{n+1}(x,y) = M_n(x,y)$ x-a.e. in the interval $|x| \le n - |y|$ for almost all y with $|y| \le n$. The L^1-continuity of $M_{n+1}(\cdot\,,y)$ and $M_n(\cdot\,,y)$ enables us to replace "for almost all y with $|y| \le n$" by "for all y with $|y| \le n$". Therefore, for any y the limit values of $M_n(x,y)$ agree x-a.e. We denote these limit values by $M(x,y)$. The function $M(\cdot\,,y)$ inherits local L^1-continuity from $\{M_n(\cdot\,,y)\}$. Furthermore, let B be a bounded Borel subset of $\mathbb{R}^d$ as above. Then by setting $K_B = \chi_B K \chi_B$ we obtain

$$\int \Big| \underbrace{(K_B\cdot\ldots\cdot K_B)}_{\longleftarrow\ k\text{ times}\ \longrightarrow}(x,x+y_1) - \underbrace{(K_B\cdot\ldots\cdot K_B)}_{\longleftarrow\ k\text{ times}\ \longrightarrow}(x,x+y_2)\Big|\,dx$$
$$\le \int |K_B(x,x_1)|\cdot\ldots\cdot|K_B(x_{k-2},x_{k-1})|$$
$$\times |K_B(x_{k-1},x+y_1) - K_B(x_{k-1},x+y_2)|\,dx_1\cdots dx_{k-1}\,dx$$
$$\le \int \Big(\prod_{i=1}^{k-2} K_B(x_i,x_i)\Big)\cdot K_B(x,x)^{1/2}\cdot K_B(x_{k-1},x_{k-1})^{1/2}$$
$$\times |K_B(x_{k-1},x+y_1) - K_B(x_{k-1},x+y_2)|\,dx_1\cdots dx_{k-1}\,dx \quad (1.16)$$

(in the last inequality we used the fact that $K_B(x,y)$ is positive definite). Integrating with respect to $x_1, \dots, x_{k-2}$, we obtain

$$(\mathrm{Tr}\, K_B)^{k-2} \cdot \int K_B(x,x)^{1/2} \cdot K_B(x_{k-1}, x_{k-1})^{1/2} \times |K_B(x_{k-1}, x+y_1) - K_B(x_{k-1}, x+y_2)|\, dx\, dx_{k-1}. \quad (1.17)$$

We note that the integrand in (1.17) is bounded above by the quantity

$$\begin{aligned} &K_B(x,x)^{1/2} \cdot K_B(x_{k-1},x_{k-1})^{1/2} \cdot \Big(K_B(x_{k-1},x_{k-1})^{1/2} \cdot K_B(x+y_1,x+y_1)^{1/2} \\ &\qquad + K_B(x_{k-1},x_{k-1})^{1/2} \cdot K_B(x+y_2,x+y_2)^{1/2} \Big) \\ &\le K_B(x_{k-1},x_{k-1}) \cdot \Big(\frac{1}{2} K_B(x,x) + \frac{1}{2} K_B(x+y_1,x+y_1) \\ &\qquad + \frac{1}{2} K_B(x,x) + \frac{1}{2} K_B(x+y_2,x+y_2) \Big) \\ &= K_B(x_{k-1},x_{k-1}) \cdot K_B(x,x) + \frac{1}{2} K_B(x_{k-1},x_{k-1}) \cdot K_B(x+y_1,x+y_1) \\ &\qquad + \frac{1}{2} K_B(x_{k-1},x_{k-1}) \cdot K_B(x+y_2,x+y_2). \end{aligned} \quad (1.18)$$

Let us choose an arbitrarily large N. The integral in (1.17) can be represented as the sum of two integrals, the first over the set

$$\big\{(x, x_{k-1}) : K_B(x,x)^{1/2} \cdot K_B(x_{k-1},x_{k-1})^{1/2} \le N \big\}$$

and the second over the complement of this set. The first integral is not greater than

$$(\mathrm{Tr}\, K_B)^{k-2} \cdot N \cdot \int \big| K_B(x_{k-1}, x_{k-1} + (x - x_{k-1}) + y_1) - K_B(x_{k-1}, x_{k-1} + (x - x_{k-1}) + y_2) \big|\, dx_{k-1}\, dx. \quad (1.19)$$

Since $K_B(\,\cdot\,, \,\cdot + y)$ is uniformly continuous with respect to the L^1-norm as y varies over a compact set B, it follows that (1.19) tends to zero as $y_1 \to y_2$. The second integral is bounded above by the value of the expression on the right-hand side of (1.18) integrated over the set

$$\big\{(x, x_{k-1}) : K_B(x,x)^{1/2} \cdot K_B(x_{k-1},x_{k-1})^{1/2} > N \big\}. \quad (1.20)$$

For sufficiently large N the Lebesgue measure of the set in (1.20) is arbitrarily small. Since $K_B(x,x) \cdot K_B(x_{k-1},x_{k-1})$ is integrable, the second integral also tends to zero. By Remark 2 below, (1.14) follows from the L^1-continuity of the kernel near the diagonal.

Definition 3. A random point field on E is said to be *determinantal* (or *fermion*) if its n-point correlation functions are of the form

$$\rho_n(x_1,\dots,x_n)=\det\bigl(K(x_i,x_j)\bigr)_{1\le i\le n}. \tag{1.21}$$

In the case $E=\bigsqcup_{j=1}^m E_j$ with $E_j\cong\mathbb{R}^d$ the definition takes the following form. Let K be an integral operator of trace class that acts on $\underbrace{L^2(\mathbb{R}^d)\oplus\cdots\oplus L^2(\mathbb{R}^d)}_{\longleftarrow\ m \text{ times}\ \longrightarrow}$. Then K has a matrix-valued kernel $\bigl(K_{rs}(x,y)\bigr)_{1\le r,s\le m}$, $x,y\in\mathbb{R}^d$.

Definition 3′. A random point field on E is said to be *determinantal* (or *fermion*) if its n-point correlation functions are of the form

$$\rho_n(x_{11},x_{12},\dots,x_{1i_1},\dots,x_{m1},x_{m2},\dots,x_{mi_m}) \\ =\det\bigl(K_{rs}(x_{ri},x_{sj})\bigr)_{\substack{1\le j\le i_s,\ s=1,\dots,m,\\ 1\le i\le i_r,\ r=1,\dots,m}} \tag{1.21′}$$

where $n=i_1+i_2+\cdots+i_m$, $x_{ri}\in E_r$, $1\le r\le m$, $1\le i\le i_r$.

Remark 1. If the kernel is Hermitian-symmetric and if the n-point correlation functions are non-negative, then the kernel $K(x,y)$ is non-negative definite, and hence K is a non-negative operator. However, it should be noted that there are determinantal random point fields corresponding to non-Hermitian kernels (see the remark after (1.36) and examples in §§ 2.2 and 2.5).

Remark 2. Condition (1.13) holds for all continuous kernels that are non-negative definite (see [11], § III.10 and [12], vol. 3, § XI.4). In the general situation (in which $K(x,x)$ is locally integrable), it follows from the assumption that $K(x,y)$ is non-negative definite that K_B is a Hilbert–Schmidt operator, and we can apply the Gokhberg–Krein theorem [11], § III.10, Theorem 10.1, which claims that a non-negative Hilbert–Schmidt operator A is of trace class if and only if

$$\varlimsup_{h\to0}\frac{1}{(2h)^{2d}}\int\prod_{j=1}^d\bigl[2h-|x^j-y^j|\bigr]_+A(x,y)\,dx\,dy<\infty, \tag{1.22}$$

where $t_+=\max(t,0)$, $x=(x^1,\dots,x^d)$, and $y=(y^1,\dots,y^d)$, and if this condition holds, then the trace $\operatorname{Tr}A$ is given by (1.22). One can readily see that the L^1-continuity of $A(\,\cdot\,,\,\cdot+y)$ implies that $\operatorname{Tr}A=\int A(x,x)\,dx$. By the classical Fredholm formula (see, for instance, [13], Chapter 3), the following formula holds for a trace-class operator A with continuous kernel (in the usual sense):

$$\operatorname{Tr}\bigl(\wedge^n(A)\bigr)=\frac{1}{n!}\int\det\bigl(A(x_i,x_j)\bigr)_{1\le i,j\le n}dx_1\cdots dx_n. \tag{1.23}$$

The kernel $K(x,y)$ can be discontinuous in general. However, it follows from (1.14) and Lidskii's theorem (see, for instance, [12], vol. IV, § XIII.17 or [13], Theorem 3.7) that

$$\int_{B^n}K(x_1,x_2)\cdot\ldots\cdot K(x_n,x_1)\,dx_1\cdots dx_n=\sum_{j=1}^\infty\lambda_j^n(K_B), \tag{1.24}$$

$$\operatorname{Tr}\bigl(\wedge^n(K_B)\bigr)=\sum_{j_1<\cdots<j_n}\lambda_{j_1}(K_B)\cdot\ldots\cdot\lambda_{j_n}(K_B). \tag{1.25}$$

Combining (1.24) and (1.25), we obtain

$$\operatorname{Tr}\bigl(\wedge^n(K_B)\bigr) = \frac{1}{n!}\int_{B^n} \det\bigl(K(x_i,x_j)\bigr)_{1\le i,j\le n}\, dx_1\cdots dx_n. \tag{1.26}$$

Since it also follows from the $L^1(B)$-continuity of $K(\,\cdot\,,\,\cdot+y)$ that

$$\operatorname{Tr}(K\cdot\chi_{B_1})\cdot\ldots\cdot(K\cdot\chi_{B_n}) = \int_{B_1\times\cdots\times B_n} K(x_1,x_2)\cdot\ldots\cdot K(x_n,x_1)\,dx_1\cdots dx_n$$

(the proof is similar to that of Lemma 2), we also have

$$\begin{aligned}&\operatorname{Tr}\bigl((K\cdot\chi_{B_1})\wedge\cdots\wedge(K\cdot\chi_{B_n})\bigr)\\ &\quad = \frac{1}{n!}\int \det\bigl(K(x_i,x_j)\cdot\chi_{B_j}(x_j)\bigr)_{1\le i,j\le n}\,dx_1\cdots dx_n.\end{aligned} \tag{1.27}$$

Definition 4. Let the kernel K satisfy the conditions of Lemma 2. We say that this kernel defines a *determinantal random point field* $(X,\mathcal{B},\mathsf{P})$ if (1.21) holds.

Theorem 2. *Let (X,B,P) be a determinantal random point field with kernel K. For finitely many disjoint bounded Borel sets $B_j\subset E$, $j=1,\ldots,n$, the generating function of the probability distribution of $\#_{B_j}=\#\{x_i\in B_j\}$ is given by*

$$\mathsf{E}\prod_{j=1}^n z_j^{\#_{B_j}} = \det\Bigl(\operatorname{Id}+\chi_B\sum_{j=1}^n (z_j-1)\cdot K\cdot\chi_{B_j}\Bigr). \tag{1.28}$$

Remark 3. The formula (1.28) means that two entire functions coincide. The right-hand side of (1.28) is well defined as the Fredholm determinant of a trace-class operator (see, for instance, [12], vol. IV, § XIII.17 or [13], § 3).

We recall that

$$\mathsf{E}\prod_{j=1}^n z^{\#_{B_j}} = \sum_{k_1,\ldots,k_n=0}^{\infty} \mathsf{P}\bigl(\#_{B_j}=k_j,\ j=1,\ldots,n\bigr)\cdot\prod_{j=1}^n z_j^{k_j} \tag{1.29}$$

and

$$\begin{aligned}&\det\Bigl(\operatorname{Id}+\chi_B\sum_{j=1}^n (z_j-1)\cdot K\cdot\chi_{B_j}\Bigr)\\ &= 1+\sum_{m=1}^{\infty}\sum_{j_1,\ldots,j_m=1}^{n}\prod_{\ell=1}^m (z_{j_\ell}-1)\cdot\operatorname{Tr}\bigl(\chi_B\cdot K\cdot\chi_{B_{j_1}}\wedge\cdots\wedge\chi_B\cdot K\cdot\chi_{B_{j_m}}\bigr)\end{aligned} \tag{1.30}$$

by definition.

Proof of Theorem 2. The Taylor expansion of the generating function in a neighbourhood of the point $(z_1,\ldots,z_n)=(1,\ldots,1)$ is given by

$$\mathsf{E}\prod_{j=1}^n z_j^{\#_{B_j}} = 1+\sum_{m=1}^{\infty}\sum_{m_1+\cdots+m_n=m}\mathsf{E}\prod_{j=1}^n \frac{(\#_{B_j})!}{(\#_{B_j}-m_j)!\,(m_j)!}\cdot\prod_{j=1}^n (z_j-1)^{m_j}. \tag{1.31}$$

The radius of convergence of the series (1.30) is infinite because

$$\operatorname{Tr}\bigl(K\cdot\chi_{B_{j_1}}\wedge\cdots\wedge K\cdot\chi_{B_{j_m}}\bigr)\le\frac{1}{m!}\operatorname{Tr}(K\cdot\chi_B)^m,\quad\text{where}\quad B=\bigsqcup_{j=1}^{n}B_j.\tag{1.32}$$

Therefore, it suffices to show that the coefficients in the series (1.30) and (1.31) coincide. In the case $n=1$ this follows from (1.5), (1.21), and (1.26). Using (1.27) instead of (1.26), we can prove the assertion for $n\ge 1$ as well.

Remark 4. Theorem 2 is well known in the theory of random point fields (see [8], p. 140, Exercise 5.4.9) and in random matrix theory [14].

As noted above, if an operator K defines a determinantal random point field, then it is non-negative because its correlation functions are. It also follows from Theorem 2 (see (1.28)) that K is bounded above by the identity operator, that is, $K\le 1$. Indeed, suppose the contrary, and let $\|K\|>1$. Then there is a bounded Borel subset $B\subset E$ such that $\|K_B\|>1+\frac{\|K\|-1}{2}>1$. Let $\lambda_1(K_B)\ge\lambda_2(K_B)\ge\lambda_3(K_B)\ge\cdots$ be the eigenvalues of K_B. We choose $0<z_0<1$ so that $1+(z_0-1)\cdot\lambda_1(K_B)=0$. Then $\mathsf{E}z_0^{\#_B}=\sum_{k=1}^{\infty}\mathsf{P}(\#_B=k)z_0^k=\det(\operatorname{Id}+(z_0-1)\cdot K_B)=$ (by Theorem XIII.106 in [12]) $=\prod_{j\ge1}(1+(z_0-1)\cdot\lambda_j(K_B))=0$. Therefore, $\mathsf{P}(\#_B=k)=0$ for any k, a contradiction. On the other hand, we assume that $0\le K\le 1$ and that (1.28) defines something which is a distribution (we hope) of non-negative integer-valued random variables $\{\#_B\}$.

Lemma 3. *Let $0\le K\le 1$ and let K be a locally trace-class operator. Then* (1.28) *defines a distribution of non-negative integer-valued random variables $\{\#_B\}$ such that*

$$\#_B=\sum_{i=1}^{n}\#_{B_i}\quad\text{(a.e.)}\tag{1.33}$$

for $B=\bigsqcup_{i=1}^{n}B_i$.

We must prove three assertions. Let us show first that (1.28) defines some finite-dimensional distributions, second, that the finite-dimensional distributions satisfy the additivity property (1.33), and, third, that the finite-dimensional distributions are consistent, and therefore we can apply Kolmogorov's fundamental theorem to prove the existence of a distribution of $\{\#_B\}$. Since the Fredholm determinant in (1.28) is equal to 1 for $z_i=1,\ i=1,\dots,n$, it follows that the first statement will be established provided that the coefficients of the Taylor expansion of the Fredholm determinant at $z_i=0,\ i=1,\dots,n$, are non-negative. Let $0\le z_i\le 1$, $i=1,\dots,n$; we assume first that $\|K\|<1$ (in the case $\|K\|=1$, the assertion will be obtained below by passage to the limit). Let $B=\bigsqcup_{i=1}^{n}B_i$. Then $\|K_B\|<1$, and $(\operatorname{Id}-K_B)^{-1}$ is a bounded linear operator such that the difference $(\operatorname{Id}-K_B)^{-1}-\operatorname{Id}=K_B\cdot(\operatorname{Id}-K_B)^{-1}$ is of trace class. Applying Theorem XIII in [12], vol. 4, p. 105, we obtain

$$\begin{aligned}
&\det\Bigl(\mathrm{Id}+\chi_B\sum_{j=1}^{n}(z_j-1)\cdot K\cdot\chi_{B_j}\Bigr)\\
&\quad=\det\Bigl((\mathrm{Id}-K_B)\cdot\Bigl(\mathrm{Id}+\sum_{j=1}^{n}z_j\cdot(\mathrm{Id}-K_B)^{-1}\cdot\chi_B\cdot K\cdot\chi_{B_j}\Bigr)\Bigr)\\
&\quad=\det(\mathrm{Id}-K_B)\cdot\det\Bigl(\mathrm{Id}+\sum_{j=1}^{n}z_j\cdot(\mathrm{Id}-K_B)^{-1}\cdot\chi_B\cdot K\cdot\chi_{B_j}\Bigr)\\
&\quad=\det(\mathrm{Id}-K_B)\cdot\sum_{k=1}^{\infty}\mathrm{Tr}\Bigl(\wedge^k\Bigl(\sum_{j=1}^{n}z_j\cdot(\mathrm{Id}-K_B)^{-1}\cdot\chi_B\cdot K\cdot\chi_{B_j}\Bigr)\Bigr)\\
&\quad=\sum_{k_1,\dots,k_n\ge 0}\frac{(k_1+\cdots+k_n)!}{k_1!\cdots k_n!}\prod_{j=1}^{n}z_j^{k_j}\cdot\det(\mathrm{Id}-K_B)\\
&\qquad\qquad\times\mathrm{Tr}\bigl(\wedge_{j=1}^{n}\bigl(\wedge^{k_j}\bigl(\chi_{B_j}\cdot(\mathrm{Id}-K_B)^{-1}\cdot\chi_B\cdot K\cdot\chi_{B_j}\bigr)\bigr)\bigr). \qquad (1.34)
\end{aligned}$$

It follows from (1.34) that up to positive factors the Taylor coefficients are traces of exterior products of non-negative operators, and therefore are non-negative. Hence, (1.28) defines some finite-dimensional distributions.

The second assertion is a simple consequence of Theorem 2 because

$$\mathsf{E}\prod_{i=1}^{n}z^{\#_{B_i}}=\det\Bigl(\mathrm{Id}+\chi_B\sum_{i=1}^{n}(z-1)K\chi_{B_i}\Bigr)=\det(\mathrm{Id}+(z-1)\cdot K_B)=\mathsf{E}z^{\#_B},$$

and hence $\#_B=\sum_{i=1}^{n}\#_{B_i}$ (a.e.). The formula (1.28) determines the finite-dimensional distributions of $\#_{B_i}$ for disjoint compact sets B_i. If the intersections of distinct sets B_i are non-empty, then we represent B_i as unions $\sqcup C_{k_i}$ of disjoint sets $\{C_k\}$, determine the distributions of $\#_{C_k}$, and then apply the additivity property (1.33) to determine the distributions of the random variables $\#_{B_i}$.

Let us prove now that the distributions of the random variables $\#_{B_i}$ are consistent. By the additivity property (1.33), it suffices to prove the desired consistency for disjoint sets $B_1,\dots,B_{n+1}$. The last assertion obviously follows from the relation

$$\begin{aligned}
&\det\Bigl(\mathrm{Id}+\chi_B\sum_{j=1}^{n}(z_j-1)\cdot K\cdot\chi_{B_j}+\chi_B(1-1)\cdot K\cdot\chi_{B_{n+1}}\Bigr)\\
&\quad=\det\Bigl(\mathrm{Id}+\chi_B\sum_{j=1}^{n}(z_j-1)\cdot K\cdot\chi_{B_j}\Bigr).
\end{aligned}$$

This completes the proof of Lemma 3 in the case $\|K\|<1$. We now assume that $\|K\|=1$. Let $K^{(\varepsilon)}:=K\cdot(1-\varepsilon)$ for $\varepsilon>0$, and let $\#_B^{(\varepsilon)}$ be the random variables corresponding to the kernel $K^{(\varepsilon)}$. Since $\|K^{(\varepsilon)}\|<1$, the above arguments prove Lemma 3 for $K^{(\varepsilon)}$. We can readily see that

$$\mathsf{E}\prod_{i=1}^{n}z_i^{\#_{B_i}^{(\varepsilon)}}=1+\sum_{m=1}^{\infty}\sum_{m_1+\cdots+m_n=m}\mathsf{E}\prod_{j=1}^{n}\frac{(\#_{B_j^{(\varepsilon)}})!}{(\#_{B_j^{(\varepsilon)}}-m_j)!\cdot(m_j)!}\cdot\prod_{j=1}^{n}(z_1-1)^{m_j}$$

is uniformly convergent to $\mathsf{E}\prod_{i=1}^{n} z_i^{\#_{B_i}}$ (together with all derivatives) on any compact set as $\varepsilon \to 0$. This proves Lemma 3.

The above results imply the following assertion.

Theorem 3. *Let K be a Hermitian, locally trace-class operator on $L^2(E)$. Then K determines a determinantal random point field on E if and only if $0 \le K \le 1$. If this random point field exists, then it is unique.*

A necessary and sufficient condition for the existence of the field was established above. Uniqueness follows from the general criterion (1.6′) because

$$\frac{1}{k!}\int_{A^k} \rho_k(x_1,\dots,x_k)\,dx_1\cdots dx_k = \mathrm{Tr}(\wedge^k(K_A)) \le \frac{\mathrm{Tr}(K_A)^k}{k!} \le \frac{1}{k!}\,.$$

Let us consider an arbitrary bounded Borel set $B \subset E$. Then $\mathrm{Tr}(K_B) = \mathsf{E}\#_B < \infty$, and the number of particles in B is finite with probability 1. Let us write $X = \bigsqcup_{0\le k<\infty} C_k^B$, where $C_k^B = \{\xi \in X : \#_B(\xi) = k\}$ as above. By Lemma 1, we can choose a kernel of the operator $\chi_B \cdot K \cdot \chi_B$ so that

$$(\chi_B \cdot K \cdot \chi_B)(x, x+y) = \sum_{i=1}^{\infty} \lambda_i(B)\cdot \varphi_i(x)\cdot \overline{\varphi_i(x+y)}$$

is a continuous function of y in the $L^1(B)$-norm. We first assume that $K_B < 1$. Then

$$L_B(x, x+y) = \sum_{i=1}^{\infty} \frac{\lambda_i(B)}{1-\lambda_i(B)}\cdot \varphi_i(x)\cdot \overline{\varphi_i(x+y)} \tag{1.35}$$

is also a continuous function of y with respect to the $L^1(B)$-norm, and it defines a kernel of the operator $L_B = (\mathrm{Id} - K_B)^{-1}K_B$. Taking infinitesimal parallelepipeds B_j in (1.34), we can show that for each set C_k^B the distribution of k particles $x_1 \le x_2 \le \cdots \le x_k$ in B is absolutely continuous with respect to Lebesgue measure. We denote the corresponding density by $p_k(x_1,\dots,x_k)$ and obtain

$$p_k(x_1,\dots,x_k) = \det(\mathrm{Id} - K_B)\cdot \det\bigl(L_B(x_i,x_j)\bigr)_{1\le i,j\le k}. \tag{1.36}$$

(It should be noted that (1.36) can be non-negative even for a non-Hermitian kernel K; we can readily see that such kernels K must have non-negative minors.) It follows from the definition of k-point correlation functions that

$$\rho_k(x_1,\dots,x_k) = \sum_{j=1}^{\infty} \frac{1}{j!}\int_{B^j} p_{k+j}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+j})\,dx_{k+1}\cdots dx_{k+j}. \tag{1.37}$$

This system of equations can be inverted as follows:

$$p_k(x_1,\dots,x_k) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\int_{B^j} \rho_{k+j}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+j})\,dx_{k+1}\cdots dx_{k+j}. \tag{1.38}$$

The functions $p_k(x_1,\ldots,x_k)$ are referred to as Janossy probability densities (see [8], p. 122) or as exclusion probability densities [7]. We easily see that

$$\sum_{j=0}^{\infty} \frac{1}{j!} \int_{B_j} p_j(x_1,\ldots,x_j)\, dx_1 \cdots dx_j = 1. \tag{1.39}$$

The right-hand side of (1.36) still makes sense for $\|K_B\| = \lambda_1(B) = 1$ (and thus the probability densities $p_k(x_1,\ldots,x_k)$ are also well defined in this case). Indeed, the determinant $\det(\mathrm{Id} - K_B) = \prod_{j=1}^{\infty}(1-\lambda_j(B))$ can be regarded as a function of λ_1, and this function has a first-order zero at the point $\lambda_1 = 1$. We claim that $\det(L(x_i,x_j))_{1\le i,j\le k}$ has a pole also of first order at $\lambda_1 = 1$. To prove this, we write $L = \tilde{L} + \tilde{\tilde{L}}$, where

$$\tilde{L}_{i,j} = \frac{\lambda_1(B)}{1-\lambda_1(B)} \cdot \varphi_1(x_i) \cdot \overline{\varphi_1(x_j)}, \qquad \tilde{\tilde{L}} = \sum_{\ell\ge 2} \frac{\lambda_\ell(B)}{1-\lambda_\ell(B)} \cdot \varphi_\ell(x_i) \cdot \overline{\varphi_\ell(x_j)}.$$

Then

$$\det\bigl(L(x_i,x_j)\bigr)_{1\le i,j\le k} = \wedge^k\bigl(L(x_i,x_j)_{1\le i,j\le k}\bigr),$$

and we can use the fact that $\operatorname{rank}(\tilde{L}) = 1$. If 1 is a multiple eigenvalue of K_B, say, if $\lambda_1(B) = \lambda_2(B) = \cdots = \lambda_m(B) = 1 > \lambda_{m+1}(B)$, the we can set

$$\tilde{L}_{i,j} = \sum_{\ell=1}^{m} \frac{\lambda_\ell(B)}{1-\lambda_\ell(B)}\, \varphi_\ell(x_i) \cdot \overline{\varphi_\ell(x_j)}$$

and repeat the above reasoning.

Remark 5. Following Macchi, we say that a random point field is regular if, for any Borel $B \subset E$ satisfying $\#_B < \infty$ (P-a.e.), the generating function $\mathsf{E}z^{\#_B}$ is entire. It follows from our results (see also Theorem 4 below) that any determinantal random point field is regular.

Remark 6. In fact, in [7], Theorem 12, p. 113 (see also [8], p. 138), Macchi claimed in fact that the condition $0 \le K < 1$ is necessary and sufficient for an integral operator K that is locally of trace class to determine a regular random point field which is fermion (= determinantal in our terminology). As follows from Theorem 3 above, this condition is sufficient but not necessary (as claimed in our Theorem 3, a necessary and sufficient condition is that $0 \le K \le 1$). For completeness of our exposition, it should be noted that Macchi studied the case of a continuous kernel $K(x,y)$ with $\operatorname{Tr} K < \infty$.

Remark 7. (1.36) was established in [7], p. 113 (see also [8], p. 138 and [14], p. 820).

In concluding § 1 we prove two general results concerning determinantal random point fields.

Theorem 4.

a) *The probability that the total number of particles is finite is either* 0 *or* 1*, depending on whether* $\operatorname{Tr} K$ *is finite or not.*

b) *The number of particles is less than or equal to* n *with probability* 1 *if and only if* K *is a finite-rank operator with* $\operatorname{rank}(K) \le n$.

c) *The number of particles is* n *with probability* 1 *if and only if* K *is an orthogonal projection of* $\operatorname{rank}(K) = n$.

d) *For any determinantal random point field, the event that no two particles coincide has probability* 1.

e) *For the results of the theorem to be valid for* $B \subset E$, K *must be replaced by* K_B.

Proof.

a) One implication is obvious. Indeed, if $\operatorname{Tr} K = \mathsf{E}\#_E < +\infty$, then $\#_E < +\infty$ with probability 1. We assume that $\operatorname{Tr} K = +\infty$ and consider a monotone absorbing family of compact sets $\{B_j\}_{j=1}^{\infty}$ (that is, compact subsets such that $B_i \subset B_{i+1}$ and $\bigcup_{i=1}^{\infty} B_i = E$). Then $\operatorname{Tr} K_{B_j} \underset{j\to\infty}{\longrightarrow} +\infty$. Choose an arbitrarily large N. By the construction of $\{B_j\}$,

$$\mathsf{P}(\#_E \le N) = \lim_{j\to\infty} \mathsf{P}(\#_{B_j} \le N).$$

Since

$$\mathsf{P}(\#_{B_j} \le N) \le 2^N \cdot \mathsf{E}2^{-\#_{B_j}} = 2^N \cdot \det\left(\mathrm{Id} - \frac{1}{2} \cdot K_{B_j}\right) \le 2^N \cdot e^{-\frac{1}{2}\operatorname{Tr}(K_{B_j})} \underset{j\to\infty}{\longrightarrow} 0,$$

we obtain the desired assertion.

b) If $\operatorname{rank}(K) = n$, then, representing the kernel in the form $K(x,y) = \sum_{i=1}^{n} \lambda_i \cdot \varphi_i(x) \cdot \overline{\varphi_i(y)}$ (a.e.) and setting $\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{1\le i,j\le n}$, we see that $\rho_m(x_1, \ldots x_m) = 0$ (a.e.) for any $m > n$. Therefore,

$$\mathsf{E}\#_E \cdot (\#_E - 1) \cdot \ldots \cdot (\#_E - n) = \int \rho_{n+1}(x_1, \ldots, x_{n+1})\, dx_1 \cdots dx_{n+1} = 0,$$

and hence $\#_E \le n$ with probability 1.

Conversely, if $\#_E \le n$ (a.e.), then

$$\int_{B^{n+1}} \rho_{n+1}(x_1, \ldots, x_{n+1})\, dx_1 \ldots dx_{n+1} = 0$$

for an arbitrary bounded Borel $B \subset E$, and therefore $\operatorname{Tr}(\wedge^{n+1}(K_B)) = 0$. Since $K \ge 0$, it follows that $\operatorname{rank}(K_B) \le n$ for an arbitrary compact set B, and hence $\operatorname{rank}(K) \le n$.

c) This assertion follows from b) and from the formula

$$\mathsf{D}(\#_E) = \operatorname{Tr}(K - K^2) = \prod_{i=1}^{n} \lambda_i \cdot (1 - \lambda_i).$$

d) Let $B_n = [-n, n]^d$. It suffices to show that for any n the probability is 1 that no two particles interior to B_n can coincide. Let ε be arbitrarily small. Then

$$\begin{aligned} \mathsf{P}\bigl\{\exists\, i \ne j : x_i = x_j \in B_n\bigr\} &\le \mathsf{P}\bigl\{\exists\, i \ne j : |x_i - x_j| < \varepsilon,\ x_i \in B_n,\ x_j \in B_n\bigr\} \\ &\le \int_{B_n} \left(\int_{|x-y|<\varepsilon} \rho_2(x,y)\, dx \right) dy. \end{aligned}$$

Since $\rho_2(x,y)$ is locally integrable, it follows that the last integral can be made arbitrarily small as $\varepsilon \to 0$.

The next result gives a criterion for weak convergence of the distributions of determinantal random point fields.

Theorem 5. *Let* P *and* P_n, $n = 1, 2, \dots$, *be probability measures on* $(X, \mathcal{B})$ *corresponding to determinantal random point fields determined by Hermitian kernels* K *and* K_n. *Let* K_n *converge to* K *in the weak operator topology and let*

$$\mathrm{Tr}(\chi_B K_n \chi_B) \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}(\chi_B K \chi_B)$$

for any bounded Borel $B \subset E$. *Then the probability measures* P_n *converge weakly to* P *on cylinder sets.*

Proof of Theorem 5. It follows from [13], Theorem 2.20, p. 40, that the assumptions of our theorem imply the relation

$$\mathrm{Tr}\,|(K_n - K)_B| = \|(K_n - K)_B\|_1 \underset{n\to\infty}{\longrightarrow} 0. \tag{1.40}$$

As a consequence of (1.40), we obtain

$$\mathrm{Tr}(K_n \cdot \chi_{B_1} \cdot \ldots \cdot K_n \cdot \chi_{B_m}) \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}(K \cdot \chi_{B_1} \cdot \ldots \cdot K \cdot \chi_{B_m}) \tag{1.41}$$

for any compact sets $B_1, \dots, B_m$.

It follows from (1.26) and (1.27) that the joint moments of $\{\#_B\}$ with respect to the measure P_n are convergent to the joint moments with respect to P. Since the moments of $\#_B$ define the distribution of $\#_B$ uniquely in the case of determinantal random point fields, we can readily prove also the convergence of the probability distributions on cylinder sets (exercise for the reader).

The remaining sections of the paper are organized as follows. § 2 is devoted to various examples of determinantal random point fields arising in quantum mechanics, statistical mechanics, random matrix theory, representation theory, and probability theory. In § 3 we discuss ergodic properties of translation-invariant determinantal random point fields. We indicate the special role of the sine kernel $K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$. In § 4 we discuss the central limit theorem for counting functions and the functional central limit theorem for the empirical distribution function of spacings.

It is a great pleasure to thank Ya. Sinai for encouraging me to write this paper, B. Simon for explaning the result of Lemma 1, G. Olshanski for many valuable remarks, and A. Borodin, B. Khoruzhenko, R. Killip, and Yu. Kondratiev for useful conversations.

2. Examples of determinantal random point fields

2.1. Fermion gas. Let $H = -\frac{d^2}{dx^2} + V(x)$ be a Schrödinger operator with discrete spectrum acting on $L^2(E)$ and let $\{\varphi_\ell\}_{\ell=0}^{\infty}$ be an orthonormal basis of eigenfunctions, $H\varphi_\ell = \lambda_\ell \cdot \varphi_\ell$, where $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$. Let us consider the nth exterior power $\wedge^n(H)$ of H acting from $\wedge^n(L^2(E))$ to $\wedge^n(L^2(E))$, where $\wedge^n(L^2(E)) = A_n L^2(E^n))$ is the space of square-integrable antisymmetric functions of n variables and $\wedge^n(H) = \sum_{i=1}^n(-\frac{d^2}{dx_i^2} + V(x_i))$. In quantum mechanics, $\wedge^n(H)$ describes a Fermi gas with n particles. The ground state of the Fermi gas is given by

$$\begin{aligned}\psi(x_1,\ldots,x_n) &= \frac{1}{\sqrt{n!}}\sum_{\sigma\in S_n}(-1)^\sigma\prod_{i=1}^n\varphi_{i-1}(x_{\sigma(i)})\\ &= \frac{1}{\sqrt{n!}}\det\bigl(\varphi_{i-1}(x_j)\bigr)_{1\le i,j\le n}.\end{aligned}\tag{2.1}$$

(It should be noted that $\psi(x_1,\ldots,x_n)$ coincides, up to a sign $\varepsilon(x_1,\ldots,x_n)$, with the ground state of the operator $\sum_{i=1}^n(-\frac{d^2}{dx_i^2}+V(x_i))$ acting on $S_nL^2(E^n)$ with the boundary conditions $\psi\big|_{x_i=x_j}=0$.) According to the principal postulate of quantum mechanics, the squared absolute value of the ground state defines the probability distribution of n particles. In our case,

$$\begin{aligned}p(x_1,\ldots,x_n) &= |\psi(x_1,\ldots,x_n)|^2\\ &= \frac{1}{n!}\det\bigl(\varphi_{i-1}(x_j)\bigr)_{1\le i,j\le n}\cdot\det\bigl(\overline{\varphi_{j-1}(x_i)}\bigr)_{1\le i,j\le n}\\ &= \frac{1}{n!}\det\bigl(K_n(x_i,x_j)\bigr)_{1\le i,j\le n},\end{aligned}\tag{2.2}$$

where $K_n(x,y) = \sum_{i=0}^{n-1}\varphi_{i-1}(x)\overline{\varphi_{i-1}(y)}$ stands for the kernel of the orthogonal projection onto the subspace spanned by the first n eigenfunctions of H. We claim that (2.2) defines a determinantal random point field. Indeed, the k-point correlation functions are given by

$$\begin{aligned}\rho_k^{(n)}(x_1,\ldots,x_n) &= \frac{n!}{(n-k)!}\int p_n(x_1,\ldots,x_n)\,dx_{k+1}\cdots dx_n\\ &= \det\bigl(K_n(x_i,x_j)\bigr)_{1\le i,j\le k}.\end{aligned}\tag{2.3}$$

The last equality in (2.3) follows from the next lemma, which is well known in random matrix theory.

Lemma 4 ([15], p. 89). *Let $(E,d\mu)$ be a measurable space and let a kernel $K\colon E^2\to\mathbb{R}^1$ satisfy the conditions*

$$\int_E K(x,y)\cdot K(y,z)\,d\mu(y) = K(x,z),\tag{2.4}$$

$$\int_E K(x,x)\,d\mu(x) = \text{const}\,.\tag{2.5}$$

Then

$$\int_E \det\bigl(K(x_i,x_j)\bigr)_{1\le i,j\le n}d\mu(x_n) = (\text{const}-n+1)\cdot\det\bigl(K(x_i,x_j)\bigr)_{1\le i,j\le n-1}.\tag{2.6}$$

Let us consider two special cases in more detail. The first concerns the harmonic oscillator.

a) $H = -\frac{d^2}{dx^2} + x^2$, $E = \mathbb{R}^1$. In this case, the functions

$$\varphi_\ell(x) = \frac{(-1)^\ell}{\pi^{\frac{1}{4}} \cdot (2^\ell \cdot \ell!)^{1/2}} \exp\left(\frac{x^2}{2}\right) \frac{d^\ell}{dx^\ell}\big(\exp(-x^2)\big) \tag{2.7}$$

are known as Weber–Hermite functions. To pass to the thermodynamic limit as $n \to \infty$, we make a proper rescaling

$$x_i = \frac{\pi}{(2n)^{1/2}}\, y_i, \qquad i = 1, \dots, n. \tag{2.8}$$

Then it follows from the Christoffel–Darboux formula and from the Plancherel–Rotach asymptotics of the Hermite polynomials [16] that the kernel

$$K_n(x_1, x_2) = \sum_{\ell=0}^{n-1} \varphi_\ell(x_1)\varphi_\ell(x_2) = \left(\frac{n}{2}\right)^{1/2} \left[\frac{\varphi_n(x_1)\varphi_{n-1}(x_2) - \varphi_n(x_2)\varphi_{n-1}(x_1)}{x_1 - x_2}\right]$$

has a limit as $n \to +\infty$,

$$K_n(x_1, x_2) \underset{n\to\infty}{\longrightarrow} K(y_1, y_2) = \frac{\sin \pi(y_1 - y_2)}{\pi(y_1 - y_2)}\,. \tag{2.9}$$

The convergence of the kernels implies convergence of the k-point correlation functions, which, in turn, implies weak convergence of the distribution

$$\left(\frac{\pi}{(2n)^{1/2}}\right)^n p_n\left(\frac{\pi}{(2n)^{1/2}} y_1, \dots, \frac{\pi}{(2n)^{1/2}} y_n\right) dy_1 \cdots dy_n$$

to the translation-invariant determinantal random point field with the 'sine kernel' $K(y_1, y_2) = \dfrac{\sin \pi(y_1 - y_2)}{\pi(y_1 - y_2)}$.

b) For another example we take $E = S^1 = \{z = e^{i\theta}, 0 \le \theta < 2\pi\}$ and $H = -\frac{d^2}{d\theta^2}$. Then

$$\varphi_\ell(\theta) = \frac{1}{\sqrt{2\pi}}\, e^{i\ell\theta},$$

$$\begin{aligned} p_n(\theta_1, \dots, \theta_n) &= \frac{1}{n!} \det\left(\sum_{\ell=0}^{n-1} \frac{1}{2\pi} e^{i\ell(\theta_j - \theta_k)}\right)_{1 \le j,k \le n} \\ &= \frac{1}{n!} \det\big(K_n(\theta_i, \theta_j)\big)_{1 \le i,j \le n}, \end{aligned} \tag{2.10}$$

where

$$K_n(\theta_1, \theta_2) = \frac{1}{2\pi} \frac{\sin\left(\frac{n}{2} \cdot (\theta_2 - \theta_1)\right)}{\sin\left(\frac{\theta_2 - \theta_1}{2}\right)}\,. \tag{2.11}$$

After the rescaling $\frac{n}{2\pi}\theta_i = y_i$, $i = 1, \dots, n$, the rescaled correlation functions have the same limit as in (2.9), in particular,

$$\lim_{n\to\infty} \frac{2\pi}{n} K_n\left(\frac{2\pi}{n} y_1, \frac{2\pi}{n} y_2\right) = \frac{\sin \pi(y_2 - y_1)}{\pi(y_2 - y_1)}\,.$$

For additional information, we refer the reader to [17]–[22].

2.2. Coulomb gas with $\beta = 2$. Examples a) and b) in § 2.1 can be interpreted as the equilibrium distributions of n unit charges confined to the one-dimensional line (Example 2.1a)) or to the unit circle (Example 2.1b)) repelling one another according to the Coulomb law of two-dimensional electrostatics. Representing the potential energy in the form

$$H(z_1, \dots, z_n) = -\sum_{1\le i<j\le n} \log|z_i - z_j| + \sum_{i=1}^{n} V(z_i),$$

where V is an external potential, we see that the Boltzmann factor

$$\frac{1}{Z}\exp\bigl(-\beta H(z_1, \dots, z_n)\bigr), \qquad \beta = 2,$$

is exactly the function $p_n(z_1, \dots, z_n)$ in Example 2.1a) for the case in which $V(z) = \frac{1}{2}z^2$, and $p_n(\theta_1, \dots, \theta_n)$ in Example 2.1b) for $V(z) = 0$, where $z_j = e^{i\theta_j}$ for $j = 1, \dots, n$.

The one-component two-dimensional Coulomb gas (a two-dimensional one-component plasma) has been studied in a number of papers including [23]–[28]. This topic is closely related to the theory of non-Hermitian Gaussian random matrices (to be discussed in § 2.3d). The two-component two-dimensional Coulomb gas (that is, a system of positively and negatively charged particles) has been studied in [29]–[34]. Let us begin with a neutral system formed by n positively charged and n negatively charged particles. Denoting the complex coordinates of these particles by u_j and v_j, $j = 1, \dots, n$, we represent the Boltzmann factor for $\beta = 2$ in the form

$$\exp\biggl(2\sum_{1\le i<j\le n}\bigl(\log|u_i - u_j| + \log|v_i - v_j|\bigr) - 2\sum \log|u_i - v_j|\biggr)$$
$$= \frac{\prod_{1\le i<j\le n}|u_i - u_j|^2 \cdot |v_i - v_j|^2}{\prod_{i,j}|u_i - v_j|^2} = \left|\det\left(\frac{1}{u_i - v_j}\right)_{1\le i,j\le n}\right|^2.$$

In the discrete case, one allows the positive particles to occupy only the sites of the sublattice $\gamma\cdot\mathbb{Z}^2$ and the negative particles to occupy only the sites of the sublattice $\gamma\cdot\bigl(\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})\bigr)$. The grand canonical ensemble is defined by the partition function, which has the following form for $\gamma = 1$:

$$Z = 1 + \sum_{u,v}\lambda_+(u)\lambda_-(v)\cdot\frac{1}{|u - v|^2}$$
$$+ \left(\frac{1}{2!}\right)^2 \sum_{u_1,u_2,v_1,v_2}\lambda_+(u_1)\lambda_+(u_2)\lambda_+(v_1)\lambda_+(v_2)\cdot\left|\det\left(\frac{1}{u_i - v_j}\right)_{1\le i,j\le 2}\right|^2 + \cdots,$$

where $\lambda_+(u) = e^{-V(u)}$ and $\lambda_-(u) = e^{V(u)}$ are the fugacities (V stands for the external potential). One can rewrite the last formula as follows:

$$Z = \det\biggl(\mathrm{Id} + \left(\lambda_+\frac{1+\sigma_z}{2} + \lambda_-\frac{1-\sigma_z}{2}\right)$$
$$\times\left(\frac{\sigma_x + i\sigma_y}{2}\cdot\frac{1}{z - z'} + \frac{\sigma_x - i\sigma_y}{2}\cdot\frac{1}{\bar{z} - \overline{z'}}\right)\biggr),$$

where σ_x, σ_y, and σ_z are the 2×2 Pauli matrices. In particular, we see that the grand canonical ensemble is a discrete fermion random point field. (The appearance of a matrix-valued kernel reflects the fact that $E = \mathbb{Z}^2 \bigsqcup (\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}))$.) Passing to the continuous limit ($\gamma = 0$), one can see that for $k \geq 2$ the k-point correlation functions have finite limits, and the limit kernel K can be expressed in terms of the Green function of the Dirac differential operator, namely,

$$K = \left(m_+ \cdot \frac{1+\sigma_z}{2} + m_- \cdot \frac{1-\sigma_z}{2} \right) \times \left(\sigma_x \partial_x + \sigma_y \partial_y + m_+ \cdot \frac{1+\sigma_z}{2} + m_- \cdot \frac{1-\sigma_z}{2} \right)^{-1},$$

where m_+ and m_- are the rescaled fugacities. In the special case $m_+ = m_- \equiv \text{const}$ (that is, $V \equiv 0$), the kernel $K = \begin{pmatrix} K++, & K+- \\ K-+, & K-- \end{pmatrix}$ can be expressed in terms of modified Bessel functions (for details, see, for instance, [32]).

2.3. Random matrix models.

a) Unitary invariant ensembles of random matrices. The probability distribution in Example 2.1a) (formulae (2.2) and (2.7)) admits another interpretation. This distribution is known in random matrix theory as the distribution of eigenvalues in the Gaussian unitary ensemble (G.U.E.). We recall the definition of the G.U.E. Let us consider the space of $n \times n$ Hermitian matrices

$$\left\{ A = (A_{ij})_{1 \leq i,j \leq n},\ \mathrm{Re}(A_{ij}) = \mathrm{Re}(A_{ji}),\ \mathrm{Im}(A_{ij}) = -\mathrm{Im}(A_{ji}) \right\}.$$

A G.U.E. random matrix is defined by its probability distribution

$$\mathsf{P}(dA) = \mathrm{const}_n \cdot \exp(-\mathrm{Tr}\, A^2)\, dA, \tag{2.12}$$

where dA stands for the flat (Lebesgue) measure, that is,

$$dA = \prod_{i<j} d\,\mathrm{Re}(A_{ij})\, d\,\mathrm{Im}(A_{ij}) \prod_{k=1}^{n} dA_{kk}.$$

The definition of a G.U.E. random matrix is equivalent to the assumption that the set $\left\{ \mathrm{Re}(A_{ij}),\ \mathrm{Im}(A_{ij}),\ 1 \leq i < j \leq n, A_{kk},\ 1 \leq k \leq n \right\}$ is formed by mutually independent random variables and $\mathrm{Re}(A_{ij}) \sim N(0, \frac{1}{4})$, $\mathrm{Im}(A_{ij}) \sim N(0, \frac{1}{4})$, and $A_{kk} \sim N(0, \frac{1}{2})$. The eigenvalues of a random Hermitian matrix are real-valued random variables. For the derivation of their joint distribution, we refer the reader to [35], §§ 5.3–5.4, and [15], Chapters 3 and 5. It turns out that the density of the joint distribution with respect to the Lebesgue measure is given exactly by (2.2) and (2.7).

We note that the distribution of a G.U.E. random matrix is invariant under any unitary transformation $A \to UAU^{-1}$, $U \in U(n)$. A natural generalization of (2.12) that preserves the unitary invariance is given by the formula

$$\mathsf{P}(dA) = \mathrm{const}_n \cdot \exp(-2 \cdot \mathrm{Tr}\, V(A))\, dA, \tag{2.13}$$

where $V(x)$ can be, for instance, a polynomial of even degree with positive leading coefficient (see [35], §5). The derivation of the formula for the joint distribution of the eigenvalues is very similar to the G.U.E. case. The density $p_n(\lambda_1, \dots, \lambda_n)$ is given by (2.2), where $\{\varphi_\ell(x) \cdot e^{-V(x)}\}_{\ell=0}^{n-1}$ are the first n orthonormal polynomials with respect to the weight $\exp(-2V(x))$. In this case, $K_n(x, y)$ is again a kernel of a projection, and hence satisfies the conditions of Lemma 4.

b) Random unitary matrices. Let us consider the group $U(n)$ of $n \times n$ unitary matrices. There is a unique translation-invariant probability measure on $U(n)$ (see [36]), the so-called Haar measure, and we denote it by μ_{Haar}. The probability density of the induced distribution of the eigenvalues is given by the formula

$$p_n(\theta_1, \dots, \theta_n) = (2\pi)^{-n} \cdot \frac{1}{n!} \cdot \prod_{1 \le k < \ell \le n} |e^{i\theta_k} - e^{i\theta_\ell}|^2,$$

which coincides with (2.10) and (2.11) (see [15], Chapters 9 and 10 and [17]–[19]). In the last formula we used the notation

$$\lambda_1 = e^{i\theta_1}, \ \dots, \ \lambda_n = e^{i\theta_n}.$$

If one starts from a probability measure $\text{const}_n \cdot e^{-\operatorname{Tr} V(U)} d\mu_{\text{Haar}}(U)$ on the unitary group (instead of Haar measure) and replaces the monomials $\frac{1}{\sqrt{2\pi}} e^{i\ell\theta}$ by $\psi_\ell(\theta) \cdot e^{-\frac{1}{2}V(\theta)}$, where $\{\psi_\ell\}_{\ell=0}^{n-1}$ are the first n orthonormal polynomials in $e^{i\theta}$ with respect to the weight $e^{-V(\theta)}\, d\theta$, then one still obtains an analogue of the formula (2.10) for the k-point correlation functions.

c) Random orthogonal and symplectic matrices. The distribution of the eigenvalues of a random orthogonal or symplectic matrix (distributed in accordance with Haar measure) has the form of a determinantal random point field with fixed number of particles. For convenience of the reader, we present below a table of kernels occurring in the ensembles of random matrices for classical compact groups.

	$K_n(x,y)$	
$U(n)$	$\frac{1}{2\pi} \cdot \frac{\sin\left(\frac{n}{2}\cdot(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)}$;	$E = [0, 2\pi]$
$SO(2n)$	$\frac{1}{2\pi} \cdot \left(\frac{\sin\left(\frac{2n-1}{2}\cdot(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)} + \frac{\sin\left(\frac{2n-1}{2}\cdot(x+y)\right)}{\sin\left(\frac{x+y}{2}\right)} \right)$;	$E = [0, \pi]$
$SO(2n+1)$	$\frac{1}{2\pi} \cdot \left(\frac{\sin(n\cdot(x-y))}{\sin\left(\frac{x-y}{2}\right)} - \frac{\sin(n\cdot(x+y))}{\sin\left(\frac{x+y}{2}\right)} \right)$;	$E = [0, \pi]$
$Sp(n)$	$\frac{1}{2\pi} \cdot \left(\frac{\sin(\frac{2n+1}{2}\cdot(x-y))}{\sin\left(\frac{x-y}{2}\right)} - \frac{\sin(\frac{2n+1}{2}\cdot(x+y))}{\sin\left(\frac{x+y}{2}\right)} \right)$;	$E = [0, \pi]$
$O_-(2n+2)$	the same as for $Sp(n)$	

For additional information, we refer the reader to [37]–[42].

d) Complex non-Hermitian Gaussian random matrices. In [23], Ginibre considered the ensemble of complex non-Hermitian random $n \times n$ matrices whose $2n^2$ parameters $\{\operatorname{Re} A_{ij},\ \operatorname{Im} A_{ij},\ 1 \le i,j \le n\}$ are independent Gaussian random variables with mean zero and variance $\frac{1}{2}$. The joint probability distribution of the matrix entries is then given by the formula

$$\begin{aligned} \mathsf{P}(dA) &= \mathrm{const}_n \cdot \exp\bigl(-\operatorname{Tr}(A^* \cdot A)\bigr)\, dA, \\ dA &= \prod_{1 \le j,k \le n} d \operatorname{Re} A_{jk} \cdot d \operatorname{Im} A_{jk}. \end{aligned} \tag{2.14}$$

An equivalent definition of (2.14) is that $A = \tilde{A} + i \cdot \tilde{\tilde{A}}$, where $\tilde{A}$ and $\tilde{\tilde{A}}$ are matrices belonging to two independent Gaussian unitary ensembles. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are complex random variables. It was shown that their joint distribution is given by a determinantal random point field on $\mathbb{R}^2$ with fixed number of particles ($\# = n$) and with the correlation functions

$$\rho_k^{(n)}(z_1, \ldots, z_k) = \det\bigl(K_n(z_j, \bar{z}_m)\bigr)_{1 \le j,m \le n}, \tag{2.15}$$

where

$$K_n(z_1, \bar{z}_2) = \frac{1}{\pi} \exp\biggl(-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2}\biggr) \cdot \sum_{\ell=0}^{n-1} \frac{z_1^\ell \bar{z}_2^\ell}{\ell!}\,.$$

We mention in passing that the kernel $K_n(z_1, \overline{z}_2)$ is convergent to the kernel

$$K(z_1, \bar{z}_2) = \frac{1}{\pi} \exp\biggl(-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + z_1 \cdot \bar{z}_2\biggr), \tag{2.16}$$

which defines the limit random point field. The problem of generalizing (2.14) was studied in [43]–[47]. Let $A = \tilde{A} + i \cdot v \cdot \tilde{\tilde{A}}$, where $\tilde{A}$ and $\tilde{\tilde{A}}$ are two independent G.U.E. matrices as above and v is a real parameter (it suffices to consider the case in which $0 \le v \le 1$). Let us introduce a new parameter $\tau = \frac{1-v^2}{1+v^2}$. The distribution of the matrix entries is

$$\mathsf{P}(dA) = \mathrm{const}_n \cdot \exp\biggl(-\frac{1}{1-\tau^2} \operatorname{Tr}\bigl(A^*A - \tau \operatorname{Re}(A^2)\bigr)\biggr)\, dA. \tag{2.17}$$

This formula induces the distribution of the eigenvalues,

$$\begin{aligned} p_n(z_1, \ldots, z_n) \prod_{j=1}^n dz_j\, d\bar{z}_j &= \mathrm{const}_n \cdot \exp\biggl[-\frac{1}{1-\tau^2} \cdot \sum_{j=1}^n \biggl(|z_j|^2 - \frac{\tau}{2}(z_j^2 + \bar{z}_j^2)\biggr)\biggr] \\ &\quad \times \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{j=1}^n dz_j\, d\bar{z}_j. \end{aligned} \tag{2.18}$$

It should be noted that (2.18) was obtained also in the papers [27] and [28] as the Boltzmann factor for the two-dimensional one-component plasma. For the calculation of the corresponding correlation functions, we refer the reader to [27], [28], [46], and [47]. In these calculations the crucial role is played by the introduction of the orthonormal polynomials in the complex plane with weight

$$w^2(z) = \exp\Big[-\frac{1}{1-\tau^2}\Big(|z|^2 - \frac{\tau}{2}(z^2+\bar{z}^2)\Big)\Big]. \tag{2.19}$$

These orthonormal polynomials can be represented in terms of Hermite polynomials,

$$\psi_\ell(z) = \frac{\tau^{\frac{\ell}{2}}}{\pi^{1/2}\cdot(\ell!)^{\frac{1}{2}}\cdot(1-\tau^2)^{\frac{1}{4}}}\, H_\ell\left(\frac{z}{\sqrt{\tau}}\right), \qquad \ell = 0,1,\dots, \tag{2.20}$$

where

$$\sum_{n=0}^{\infty} H_n(z)\cdot\frac{t^n}{n!} = \exp\Big(zt - \frac{t^2}{2}\Big).$$

We also note that $\psi_\ell(z) = \big(\pi^{1/2}\cdot(\ell!)^{1/2}\big)^{-1}\cdot z^\ell$ for $\tau = 0$ (this case corresponds to the Ginibre ensemble). The corresponding formula for the correlation functions (for arbitrary τ) is a generalization of (2.15),

$$\begin{aligned}\rho_k^{(n)} &= \det\big(K_n(z_i,\bar{z}_j)\big)_{1\le i,j\le k},\\ K_n(z_1,\bar{z}_2) &= w(z_1)w(\bar{z}_2)\cdot\sum_{\ell=0}^{n-1}\psi_\ell(z_1)\psi_\ell(\bar{z}_2).\end{aligned} \tag{2.21}$$

In the limit as $n\to\infty$ the kernel $K_n(z_1,\overline{z}_2)$ is convergent to

$$\begin{aligned}K(z_1,\bar{z}_2) &= \lim_{n\to\infty} K_n(z_1,\bar{z}_2)\\ &= \frac{1}{\pi(1-\tau^2)}\exp\Big(-\frac{1}{1-\tau^2}\Big(\frac{|z_1|^2}{2}+\frac{|z_2|^2}{2}-z_1\bar{z}_2\Big)\Big).\end{aligned} \tag{2.22}$$

We note that the last formula differs from (2.16) only by the trivial change of coordinates $z\to z\cdot\sqrt{1-\tau^2}$. A special regime (known in the physics literature as the regime of *weak non-Hermiticity*) was discovered for the model (2.17) by Fyodorov, Khoruzhenko, and Sommers in [46] and [47]. Let

$$\begin{aligned}\mathrm{Re}(z_1) &= n^{1/2}\cdot x + n^{-\frac{1}{2}}x_1,\\ \mathrm{Re}(z_2) &= n^{1/2}\cdot x + n^{-\frac{1}{2}}x_2,\\ \mathrm{Im}(z_1) &= n^{-\frac{1}{2}}\cdot y_1,\\ \mathrm{Im}(z_2) &= n^{-\frac{1}{2}}\cdot y_2.\end{aligned}$$

We assume that the parameters x, x_1, x_2, y_1, and y_2 are fixed and consider the limit as $n\to\infty$ under the assumption that $\lim_{n\to\infty} n\cdot(1-\tau) = \frac{\alpha^2}{2}$. Then

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}K_n(z_1,z_2) &= \frac{1}{\pi\alpha}\cdot\exp\Big[-\frac{y_1^2+y_2^2}{\alpha^2} + i\cdot x\cdot\frac{y_1-y_2}{2}\Big]\\ &\quad\times g_\alpha\Big(\frac{y_1+y_2}{2} - i\cdot\frac{x_1-x_2}{2}\Big),\end{aligned} \tag{2.23}$$

where

$$g_\alpha(y) = \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \frac{du}{\sqrt{2\pi}} \exp\left[-\frac{\alpha^2 u^2}{2} - 2uy\right] \tag{2.24}$$

(for $x > 2$, the limit in (2.23) vanishes). The formulae (2.23) and (2.24) define a determinantal random point field on $\mathbb{R}^2$ distinct from (2.16).

e) Positive Hermitian random matrices. Following Bronk [48], we define the Laguerre ensemble of positive Hermitian $n \times n$ matrices. Any positive Hermitian matrix M can be represented in the form $M = A^*A$, where A is a complex matrix. The probability distribution of a random matrix M is given by

$$\mathrm{const}_n \cdot \exp(-\operatorname{Tr} A^*A) \cdot [\det(A^*A)]^\alpha \, dA, \tag{2.25}$$

where dA is defined as in (2.14) and $\alpha > -1$ (the values $\pm\frac{1}{2}, 0$ of the parameter α are of special interest). The induced probability distribution of the (positive) eigenvalues is given by

$$\mathrm{const}_n \cdot \exp\Big(-\sum_{i=1}^n \lambda_i\Big) \cdot \prod_{i=1}^n \lambda_i^\alpha \cdot \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 \, d\lambda_1 \cdots d\lambda_n. \tag{2.26}$$

Employing the associated Laguerre polynomials

$$L_m^\alpha(x) \equiv \frac{1}{m!} e^x x^{-\alpha} \frac{d^m}{dx^m} \big(e^{-x} x^{m+\alpha}\big), \qquad m = 0, 1, \ldots,$$

one can rewrite (2.26) as

$$\frac{1}{n!} \det\big(K_n(x_i, x_j)\big)_{1 \le i,j \le n}, \tag{2.27}$$

where

$$K_n(x, y) = \sum_{\ell=0}^{n-1} \varphi_\ell^{(\alpha)}(x) \cdot \varphi_\ell^{(\alpha)}(y), \tag{2.28}$$

and

$$\left\{\varphi_\ell^{(\alpha)}(x) = \left(\Gamma(\alpha+1) \cdot \binom{n+\alpha}{n}\right)^{-\frac{1}{2}} L_\ell^\alpha(x)\right\}_{\ell=0}^\infty$$

is an orthonormal basis with respect to the weight $e^{-x} \cdot x^\alpha$ on the positive semi-axis. Applying Lemma 4 again, we can explicitly evaluate the k-point correlation functions and show that they are given by the determinants of $k \times k$ matrices with kernel (2.28).

f) Chain of correlated Hermitian matrices. Let $A_1, \ldots, A_p$ be complex Hermitian random $n \times n$ matrices with joint probability density

$$\mathrm{const}_n \cdot \exp\Big[-\operatorname{Tr}\Big(\frac{1}{2} V_1(A_1) + V_2(A_2) + \cdots + V_{p-1}(A_{p-1}) + \frac{1}{2} V_p(A_p) + c_1 A_1 A_2 + c_2 A_2 A_3 + \cdots + c_{p-1} A_{p-1} A_p\Big)\Big]. \tag{2.29}$$

We denote the real eigenvalues of A_j by $\widetilde{\lambda_j} = (\lambda_{j1}, \dots, \lambda_{jn})$, $j = 1, \dots, p$. The induced probability density of the eigenvalues is then given by

$$p_n(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_p) = \text{const}_n \cdot \left[\prod_{1 \le r < s \le n} (\lambda_{1r} - \lambda_{1s})(\lambda_{pr} - \lambda_{ps}) \right] \times \left[\prod_{k=1}^{p-1} \det \bigl[w_k(\lambda_{kr}, \lambda_{k+1,s}) \bigr]_{r,s=1,\dots,n} \right], \quad (2.30)$$

where

$$w_k(x, y) = \exp\biggl(-\frac{1}{2} V_k(x) - \frac{1}{2} V_{k+1}(y) + c_k xy \biggr). \quad (2.31)$$

Eynard and Mehta established [49] that the correlation functions of this model,

$$\rho_{k_1,\dots,k_p}(\lambda_{11}, \dots, \lambda_{1k_1}; \dots; \lambda_{p1}, \dots, \lambda_{pk_p}) = \prod_{j=1}^{p} \frac{n!}{(n-k_j)!} \int p_n(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_p) \cdot \prod_{j=1}^{p} \prod_{r_j = k_j + 1}^{n} d\lambda_{j r_j},$$

can be written as a $k \times k$ determinant with $k = k_1 + \dots + k_p$,

$$\det \bigl[K_{ij}(\lambda_{ir}, \lambda_{js}) \bigr]_{r=1,\dots,k_i;\, s=1,\dots,k_j;\, i,j=1,\dots,p}. \quad (2.32)$$

For the explicit formulae for the kernels $K_{ij}(x, y)$, we refer the reader to [49] (see also [50]). We note that (2.32) defines a determinantal random point field with one-particle space E which is the union of p copies of $\mathbb{R}^1$.

g) Universality in random matrix models. Airy, Bessel, and sine random point fields. We begin with a general class of kernels of the form

$$K(x, y) = \frac{\varphi(x)\psi(y) - \varphi(y)\psi(x)}{x - y}, \quad (2.33)$$

where

$$m(x)\varphi'(x) = A(x)\varphi(x) + B(x)\psi(x),$$
$$m(x)\psi'(x) = -C(x)\varphi(x) - A(x)\psi(x), \quad (2.34)$$

and $m(x)$, $A(x)$, $B(x)$, and $C(x)$ are polynomials. It was shown by Tracy and Widom [51] that the Fredholm determinants of integral operators with kernels (2.33) and (2.34) restricted to a finite union of intervals satisfy certain partial differential equations. Airy, Bessel, and sine kernels are special cases of (2.33) and (2.34). To define the sine kernel, we set

$$\varphi(x) \equiv \frac{1}{\pi} \sin(\pi x), \qquad \psi(x) \equiv \varphi'(x)$$
$$\bigl(m(x) \equiv 1, \quad A(x) \equiv 0, \quad B(x) \equiv 1, \quad C(x) \equiv \pi^2 \bigr);$$

for the Airy kernel we set

$$\varphi(x) \equiv A_i(x), \qquad \psi(x) \equiv \varphi'(x)$$
$$\big(m(x) \equiv 1, \quad A(x) \equiv 0, \quad B(x) \equiv 1, \quad C(x) \equiv -x\big);$$

finally, for the Bessel kernel we set

$$\varphi(x) \equiv J_\alpha(\sqrt{x}\,), \qquad \psi(x) \equiv x\varphi'(x)$$
$$\Big(m(x) \equiv x, \quad A(x) \equiv 0, \quad B(x) \equiv 1, \quad C(x) \equiv \frac{1}{4}(x-\alpha^2)\Big).$$

Here $\mathcal{A}_i(x)$ is the Airy function and $J_\alpha(x)$ is the Bessel function of order α (see [16]). For the corresponding kernels, the following exact representations hold [51]–[53]:

$$K_{\text{sine}}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}\,, \tag{2.35}$$

$$K_{\text{Airy}}(x,y) = \frac{\mathcal{A}_i(x)\cdot\mathcal{A}_i'(y) - \mathcal{A}_i(y)\cdot\mathcal{A}_i'(x)}{x-y} = \int_0^\infty \mathcal{A}_i(x+t)\cdot\mathcal{A}_i(y+t)\,dt, \tag{2.36}$$

$$K_{\text{Bessel}}(x,y) = \frac{J_\alpha(\sqrt{x}\,)\cdot\sqrt{y}\cdot J_\alpha'(\sqrt{y}\,) - \sqrt{x}\cdot J_\alpha'(\sqrt{x}\,)\cdot J_\alpha(\sqrt{y}\,)}{2\cdot(x-y)} = \frac{\sqrt{x}\cdot J_{\alpha+1}(\sqrt{x}\,)\cdot J_\alpha(\sqrt{y}\,) - J_\alpha(\sqrt{x}\,)\sqrt{y}\cdot J_{\alpha+1}(\sqrt{y}\,)}{2\cdot(x-y)}\,. \tag{2.37}$$

As mentioned above, the sine kernel arises as a scaling limit in the bulk of the spectrum in the G.U.E. ([15], Chapter 5). In turn, the Airy kernel arises as a scaling limit at the edge of the spectrum in the G.U.E. and at the (soft) right edge of the spectrum in the Laguerre ensemble, while the Bessel kernel arises as a scaling limit at the (hard) left edge in the Laguerre ensemble; see [54], [52], and [53]. The universality conjecture in random matrix theory asserts that such limits should be universal for a broad class of Hermitian random matrices. This conjecture was recently proved for unitary invariant ensembles (2.13) in the bulk of the spectrum ([55]–[57] and [35]) and for some classes of Wigner matrices in the bulk of the spectrum [58] and at the edge [59].

In the next subsection we completely characterize the determinantal random point fields on $\mathbb{R}^1(\mathbb{Z}^1)$ with independent identically distributed spacings.

2.4. Determinantal random point fields with independent identically distributed spacings. Renewal processes. We begin with some basic facts of the theory of renewal processes (see, for instance, [60] and [8]). Let $\{\tau_k\}_{k=1}^\infty$ be a sequence of independent identically distributed non-negative random variables and let τ_0 be a non-negative random variable independent of $\{\tau_k\}_{k=1}^\infty$ (in general, the distribution of τ_0 is different). We set

$$x_k = \sum_{j=0}^{k} \tau_j, \qquad k \geq 0. \tag{2.38}$$

This gives us a random configuration $\{x_k\}_{k=0}^{\infty}$ in $\mathbb{R}^1_+$. In probability theory, a random sequence $\{x_k\}_{k=0}^{\infty}$ is known as a delayed renewal process. We assume that the distribution of the random variables τ_k, $k \geq 1$, has a density $f(x)$, the so-called interval distribution density, and a finite expectation $\mathsf{E}\tau_1 = \int_0^{\infty} x f(x)\,dx$. The renewal density is then defined as follows:

$$u(x) = \sum_{k=1}^{\infty} f^{k*}(x) = f(x) + \int_0^x f(x-y)f(y)\,dy + \int_0^x \int_0^{x-y_2} f(x-y_1-y_2)f(y_1)f(y_2)\,dy_1\,dy_2 + \cdots . \tag{2.39}$$

One can express higher-order correlation functions of the renewal process via the corresponding one-point correlation function and the renewal density. Indeed (see [8], p. 136), the following formula holds for $t_1 \leq t_2 \leq \cdots \leq t_k$, where $k > 1$:

$$\rho_k(t_1, \ldots, t_k) = \rho_1(t_1) \cdot u(t_2 - t_1) \cdot u(t_3 - t_2) \cdot \ldots \cdot u(t_k - t_{k-1}). \tag{2.40}$$

It follows immediately from the above definitions that a random point field on $\mathbb{R}^1_+$ has independent identically distributed nearest spacings if and only if it is a renewal process (2.38). To make this process translation invariant, the probability density of τ_0 must be given as

$$\frac{1}{\mathsf{E}\tau_1} \int_x^{+\infty} f(t)\,dt \qquad \text{([8], p. 72 and [60], § XI.3).} \tag{2.41}$$

In this case, the one-point correlation function is identically constant, $\rho_1(x) \equiv \rho > 0$, and thus it follows from (2.40) that the distribution of the process is uniquely determined by the renewal density (in particular, one can recover ρ from the function $u(x)$ because $\rho = (\mathsf{E}\tau_1)^{-1}$, and the Laplace transforms of f and u admit a simple relationship). Macchi [7] treated a special class of translation-invariant renewal processes with interval distribution density given by the formula

$$f(x) = 2\rho(1-2\rho\alpha)^{-\frac{1}{2}} \cdot e^{-\frac{x}{\alpha}} \cdot \sinh\left((1-2\rho\alpha)^{\frac{1}{2}} \cdot \left(\frac{x}{\alpha}\right)\right), \tag{2.42}$$

where

$$2\rho\alpha \leq 1, \quad \rho > 0, \quad \alpha > 0, \tag{2.43}$$

and showed that it is a determinantal random point field with kernel

$$K(x,y) = \rho \cdot \exp(-|x-y|/\alpha) \tag{2.44}$$

(conditions (2.43) mean exactly that $0 < K \leq \mathrm{Id}$).

In the next theorem, we classify all delayed renewal processes that are also determinantal random point fields on $\mathbb{R}^1_+$.

Theorem 6. *A determinantal random point field on* $\mathbb{R}^1_+$ *with Hermitian kernel has independent identically distributed spacings if and only if the corresponding integral operator, which is locally of trace class, satisfies the following two conditions in addition to the condition* $0 \le K \le \mathrm{Id}$:

a)

$$K(x_1, x_2) \cdot K(x_2, x_3) = K(x_1, x_3) \cdot K(x_2, x_2) \tag{2.45}$$

for almost all $x_1 \le x_2 \le x_3$;

b) *the function*

$$K(x_2, x_2) - \frac{K(x_1, x_2) \cdot K(x_2, x_1)}{K(x_2, x_1)} \tag{2.46}$$

depends only on the difference $x_2 - x_1$ *for almost all* $x_1 \le x_2$. *Moreover, if the determinantal random point field is translation invariant, then it is given by the formulae* (2.42)–(2.44).

Remark 8. Of course, a translation-invariant determinantal random point field on $\mathbb{R}^1_+$ can be uniquely extended to a translation-invariant determinantal random point field on $\mathbb{R}^1$.

Proof of Theorem 6. Let us first prove the "only if" part of the theorem. Suppose that a determinantal random point field with kernel $K(x, y)$ is a delayed renewal process. It follows from (2.40) with $k = 2, 3$ that the renewal density satisfies the formula

$$u(y - x) = K(y, y) - \frac{K(x, y) \cdot K(y, x)}{K(x, x)}, \qquad y \ge x, \tag{2.47}$$

and the expression for $\rho_3(x_1, x_2, x_3)$, $x_1 \le x_2 \le x_3$, becomes

$$\begin{aligned}\rho_3(x_1, x_2, x_3) &= K(x_1, x_1) \cdot u(x_2 - x_1) \cdot u(x_3 - x_2) \\ &= K(x_1, x_1) \cdot \left(K(x_2, x_2) - \frac{K(x_1, x_2) \cdot K(x_2, x_1)}{K(x_1, x_1)} \right) \\ &\qquad \times \left(K(x_3, x_3) - \frac{K(x_2, x_3) \cdot K(x_3, x_2)}{K(x_2, x_2)} \right).\end{aligned} \tag{2.48}$$

Since the particles are in the set $A = \{x : K(x, x) > 0\}$ with probability 1, we can always consider the random point field restricted to A.

Comparing

$$\rho_3(x_1, x_2, x_3) = \det\bigl(K(x_i, x_j)\bigr)_{1 \le i,j \le 3} \tag{2.49}$$

with (2.48), we see that

$$\begin{aligned}&K(x_1, x_2) \cdot K(x_2, x_1) \cdot K(x_2, x_3) \cdot K(x_3, x_2) \cdot \frac{1}{K(x_2, x_2)} \\ &\quad = -K(x_1, x_3) \cdot K(x_3, x_1) \cdot K(x_2, x_2) + K(x_1, x_2) \cdot K(x_2, x_3) \cdot K(x_3, x_1) \\ &\qquad + K(x_1, x_3) \cdot K(x_3, x_2) \cdot K(x_2, x_1),\end{aligned}$$

which is equivalent to

$$\frac{1}{K(x_2,x_2)}\cdot\Big(K(x_1,x_2)\cdot K(x_2,x_3)-K(x_2,x_2)\cdot K(x_1,x_3)\Big)\\ \times\Big(K(x_3,x_2)\cdot K(x_2,x_1)-K(x_3,x_1)\cdot K(x_2,x_2)\Big)=0.$$

The third factor in the last formula is the complex conjugate of the second factor, and hence we obtain relation (2.45). Condition b) of the theorem has already been established in (2.47). For a translation-invariant determinantal random point field, the kernel $K(x,y)$ depends only on the difference $x-y$, and therefore $K(x,y)=\rho\cdot e^{-|x-y|/\alpha}\cdot e^{i\beta(x-y)}$, and the unitary equivalent kernel $e^{-i\beta x}K(x,y)\cdot e^{i\beta y}$ coincides with (2.44). Let us now prove the "if" part of the theorem. If a kernel satisfies conditions (2.45) and (2.46), then the renewal density must satisfy

$$u(x_2-x_1)=K(x_2,x_2)-\frac{K(x_1,x_2)\cdot K(x_2,x_1)}{K(x_1,x_1)}$$

for almost all $x_1\le x_2$. Let $x_1\le x_2\le\cdots\le x_k$. Our goal is to derive the algebraic identity

$$\det\big(K(x_i,x_i)\big)_{1\le i,j\le k}\\ =K(x_1,x_1)\cdot\prod_{i=1}^{k-1}\left(K(x_{i+1},x_{i+1})-\frac{K(x_i,x_{i+1})\cdot K(x_{i+1},x_i)}{K(x_i,x_i)}\right)\quad(2.50)$$

from the basic identities for the commuting variables $K(x_i,x_j)$ and $\overline{K(x_i,x_j)}$, which satisfy the following equations:

$$K(x_i,x_j)\cdot K(x_j,x_\ell)=K(x_i,x_\ell)\cdot K(x_j,x_j),\qquad 1\le i\le j\le\ell\le k,\\ K(x_i,x_j)=\overline{K(x_j,x_i)}.$$

Let us introduce the functions $a(x)=K(x,x)\cdot K(0,x)^{-1}$ and $b(x)=K(0,x)^{-1}$. Then for $i\le j$

$$K(x_i,x_j)=a(x_i)\cdot b(x_j)^{-1},\\ K(x_j,x_i)=\overline{a(x_i)}\cdot\overline{b(x_j)}^{-1}.$$

This enables us to rewrite the determinant in the form

$$\begin{vmatrix} a(x_1)\cdot b(x_1)^{-1}, & a(x_1)\cdot b(x_2)^{-1}, & \dots & ,a(x_1)\cdot b(x_n)^{-1}\\ \overline{a(x_1)}\cdot\overline{b(x_2)}^{-1}, & a(x_2)\cdot b(x_2)^{-1}, & \dots & ,a(x_2)\cdot b(x_n)^{-1}\\ \cdots & \cdots & \cdots & \cdots\\ \overline{a(x_1)}\cdot\overline{b(x_n)}^{-1}, & \overline{a(x_2)}\cdot\overline{b(x_n)}^{-1}, & \dots & ,a(x_n)\cdot b(x_n)^{-1}\end{vmatrix}=a(x_1)\cdot b(x_1)^{-1}\\ \times\prod_{i=1}^{n-1}\Big(b(x_{i+1})^{-1}\cdot\overline{b(x_{i+1})}^{-1}\cdot\big(a(x_{i+1})\cdot\overline{b(x_{i+1})}-\overline{a(x_i)}\cdot b(x_i)\big)\Big),\quad(2.51)$$

which is exactly the right-hand side of (2.50). Since the relation

$$\rho_k(x_1,\dots,x_k)=\rho_1(x_1)\cdot\prod_{i=1}^{n-1}u(x_{i+1}-x_i),\qquad x_1\le x_2\le\cdots\le x_k,$$

is thus established, we see that the remaining part of the proof is quite easy. Let $p_k(x_1,\dots,x_k)$ be the Janossy density, that is, the probability density of there being particles at the points $x_1,\dots,x_k$ and none in between. We recall that

$$p_k(x_1,\dots,x_k)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\cdot\int\rho_{k+j}(x_1,\dots,x_k;y_{k+1},\dots,y_{k+j})\,dy_{k+1}\cdots dy_{k+j},$$

where the jth term is integrated over the domain $\underbrace{(x_1,x_k)\times\cdots\times(x_1,x_k)}_{\longleftarrow\ j\text{ times}\ \longrightarrow}$. We claim that

$$p_k(x_1,\dots,x_k)=\rho_1(x_1)\cdot\prod_{i=1}^{k-1}f(x_{i+1}-x_i),\tag{2.52}$$

where the interval distribution density f and the renewal distribution density u are related via the convolution equation

$$u=f+u*f.\tag{2.53}$$

This completes the proof of Theorem 6.

Remark 9. An analogue of Theorem 6 holds in the discrete case as well. The proof is the same. One has only to replace (2.42) by the solution of the discrete convolution equation (2.53) with $u(n)=1-\rho\cdot e^{-2\beta n}$ and $K(n_1,n_2)=\rho\cdot e^{-\beta|n_1-n_2|}$, where $0<\rho\le 1$ and $\beta>0$, so that

$$\widehat{f}(t)=\sum_{n=0}^{\infty}f(n)e^{int}=\frac{(1-\rho)-(e^{-2\beta}-\rho)\cdot e^{it}}{(2-\rho)-(2e^{-2\beta}-\rho+1)\cdot e^{it}+e^{-2\beta}e^{2it}}\,.\tag{2.54}$$

Remark 10. Theorem 6 admits a generalization to the case in which the multiplicative identity (2.45) still holds but the renewal density

$$u(x_1,x_2)=K(x_2,x_2)-\frac{K(x_1,x_2)\cdot K(x_2,x_1)}{K(x_1,x_1)}$$

is no longer a function depending only on the difference of x_1 and x_2. These processes have independent spacings that need not be identically distributed because the distribution $f(x,y)dy$ of the spacings depends on the coordinate x of the left particle. Thus,

$$\begin{aligned}u(x_1,x_2)=f(x_1,x_2)&+\int_{x_1}^{x_2}f(x_1,y_1)\cdot f(y_1,x_2)\,dy_1\\&+\int_{x_1}^{x_2}\int_{x_1}^{y_2}f(x_1,y_1)\cdot f(y_1,y_2)\cdot f(y_2,x_2)\,dy_1\,dy_2+\cdots,\end{aligned}\tag{2.55}$$

where $f(x,y)$ is a one-parameter family of probability densities such that

$$\operatorname{supp} f(x,\,\cdot\,) \subset [x,+\infty], \qquad f \geq 0, \qquad \int f(x,y)\,dy = 1.$$

Let us recall the inversion formula for equation (2.55),

$$f(x_1,x_2) = u(x_1,x_2) - \int_{x_1}^{x_2} u(x_1,y_1)u(y_1,x_2)\,dy_1 + \int_{x_1}^{x_2}\int_{x_1}^{y_2} u(x_1,y_1)\cdot u(y_1,y_2)\cdot u(y_2,x_2)\,dy_1\,dy_2 - \cdots. \tag{2.56}$$

Writing $K(x,y) = a(x)b(y)^{-1}$, $x \leq y$, where $a(x) = \frac{K(x,x)}{K(0,x)}$, $b(y) = \frac{1}{K(0,y)}$, and $u(x,y) = \frac{1}{|b(y)|^2}\cdot(a(y)\overline{b(y)} - \overline{a(x)}b(x))$, we see that in principle (2.56) describes the class of corresponding interval distribution densities $u(x,y)$.

2.5. Plancherel measure on partitions and its generalizations, namely, *z*-measures and Schur measures. By a partition of $n = 1,2,\dots$ we mean a set of non-negative integers $\lambda = (\lambda_1,\dots,\lambda_m)$ such that $\lambda_1 + \cdots + \lambda_m = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. By $\operatorname{Par}(n)$ we denote the set of all partitions of n. For the basic facts about partitions we refer the reader to [61]–[64]. In particular, we recall that any partition λ of n (we use the notation $\lambda \vdash n$) can be identified with a Young diagram with $|\lambda| = n$ boxes. Let λ' be the partition corresponding to the transposed diagram. Let d be the number of diagonal boxes in λ (that is, the number of diagonal boxes in the Young diagram corresponding to λ). We denote by $(p_1,\dots,p_d \mid q_1,\dots,q_d)$ the Frobenius coordinates of λ, where $p_j = \lambda_j - j$ and $q_j = \lambda'_j - j$ for $j = 1,\dots,d$. The important role of partitions in representation theory is easy to see from the fact that the elements of $\operatorname{Par}(n)$ can be put in a one-to-one correspondence with the irreducible representations of the symmetric group S_n (see, for instance, [64] and [62]). The Plancherel measure M_n on the set $\operatorname{Par}(n)$ of all partitions of n is given by the formula

$$M_n(\lambda) = \frac{(\dim\lambda)^2}{n!}, \tag{2.57}$$

where $\dim\lambda$ is the dimension of the corresponding representation of S_n. The dimension $\dim\lambda$ can be expressed in terms of the Frobenius coordinates via the determinantal formula

$$\frac{\dim\lambda}{n!} = \det\left[\frac{1}{(p_i+q_j+1)\cdot p_i!\cdot q_i!}\right]_{1\leq i,j\leq d}, \qquad |\lambda| = n \tag{2.58}$$

([65], Proposition 2.6, (2.7)). Let $\operatorname{Par} = \bigsqcup_{n=0}^{\infty}\operatorname{Par}(n)$ and consider the following measure M^θ on Par, which can be referred to as the grand canonical ensemble by analogy with statistical mechanics:

$$M^\theta(\lambda) = e^{-\theta}\cdot\frac{\theta^n}{n!}M_n(\lambda) \quad \text{for} \quad \lambda \in \operatorname{Par}(n), \quad n = 0,1,2,\dots, \quad 0 \leq \theta < \infty. \tag{2.59}$$

The measure M^θ is also said to be the poissonization of the measure M_n. It follows from (2.59) that $|\lambda|$ is distributed by the Poisson law with mean θ, and M_n is the conditional measure of M^θ under the condition $|\lambda| = n$. Using the Frobenius coordinates, we can represent the measures M^θ and M_n as random point fields on the lattice $\mathbb{Z}^1$. It was recently proved by Borodin, Okounkov, and Olshanski [66] and independently by Johansson [67] that M^θ is a determinantal random point field (we note to be exact that in [67] only the restriction of M^θ to the first half $\bigl(p_1, \ldots, p_{d(\lambda)}\bigr)$ of the Frobenius coordinates was studied and, as a result, only the part of (2.60) corresponding to $xy > 0$ was obtained). To formulate the results of [66] and [67], we define the modified Frobenius coordinates of the partition λ as follows:

$$\mathrm{Fr}(\lambda) := \left\{p_1 + \frac{1}{2}, \ldots, p_d + \frac{1}{2}, -q_1 - \frac{1}{2}, \ldots, -q_d - \frac{1}{2}\right\}.$$

Let $\rho_k^\theta(x_1, \ldots, x_k)$ be the k-point correlation function of M^θ in the modified Frobenius coordinates, where

$$\{x_1, \ldots, x_k\} \subset \mathbb{Z}^1 + \frac{1}{2}\,.$$

Then

$$\rho_k^\theta(x_1, \ldots, x_k) = \det\bigl[K(x_i, x_j)\bigr]_{1 \le i,j \le k},$$

where K is the so-called discrete Bessel kernel,

$$K(x,y) = \begin{cases} \sqrt{\theta} \cdot \dfrac{J_{|x|-\frac{1}{2}}(2\sqrt{\theta}\,) \cdot J_{|y|+\frac{1}{2}}(2\sqrt{\theta}\,) - J_{|x|+\frac{1}{2}}(2\sqrt{\theta}\,) \cdot J_{|y|-\frac{1}{2}}(2\sqrt{\theta}\,)}{|x| - |y|} \\ \qquad \text{for } x \cdot y > 0, \\ \sqrt{\theta} \cdot \dfrac{J_{|x|-\frac{1}{2}}(2\sqrt{\theta}\,) \cdot J_{|y|-\frac{1}{2}}(2\sqrt{\theta}\,) - J_{|x|+\frac{1}{2}}(2\sqrt{\theta}\,) \cdot J_{|y|+\frac{1}{2}}(2\sqrt{\theta}\,)}{x - y} \\ \qquad \text{for } x \cdot y < 0, \end{cases} \tag{2.60}$$

and $J_x(\,\cdot\,)$ is the Bessel function of order x. We note that the kernel $K(x,y)$ is not Hermitian symmetric; however, the restrictions of this kernel to the positive and negative semiaxes are Hermitian. The formula (2.60) can be regarded as a limit case of a more general theorem obtained by Borodin and Olshanski for the so-called z-measures (see Theorem 3.3 in [68], and also [69]–[71] and the references therein). Let z and z' be complex numbers such that either

$$z' = \bar{z} \in \mathbb{C} \setminus \mathbb{Z} \tag{2.61}$$

or

$$[z] < \min(z, z') \le \max(z, z') < [z] + 1,$$

where z and z' are real and $[z]$ stands for the integral part of z. Let $(x)_j = x \cdot (x+1) \cdot \ldots \cdot (x+j-1)$ and $(x)_0 = 1$. Below we introduce a 2-parameter family of probability measures $M_{z,z'}^{(n)}$ on $\mathrm{Par}(n)$. These measures arose in harmonic analysis on the infinite symmetric group; see [71] and [65]. By definition,

$$M_{z,z'}^{(n)}(\lambda) = \frac{(z \cdot z')^{d(\lambda)}}{(z \cdot z')_n} \cdot \prod_{i=1}^{d(\lambda)} (z+1)_{p_i} \cdot (z'+1)_{p_i} \times (-z+1)_{q_i} \cdot (-z'+1)_{q_i} \cdot \frac{\dim^2 \lambda}{|\lambda|!}\,. \qquad (2.62)$$

The above conditions on z and z' are equivalent to the assumption that the products $(z)_j \cdot (z')_j$ and $(-z)_j \cdot (-z')_j$ are positive for any $j = 1, 2, \dots$. We note that $M_{z,z'}^{(n)}$ is convergent to the Plancherel measure M_n as $z, z' \to \infty$. The measure $M_{z,z'}^{(n)}$ is referred to as the z-measure of level n. Let us consider now the negative binomial distribution on non-negative integers,

$$(1-\xi)^{z \cdot z'} \cdot \frac{(z \cdot z')_n}{n!} \xi^n, \qquad n = 0, 1, \dots,$$

where ξ is an additional parameter, $0 < \xi < 1$. The corresponding mixture of the z-measures of levels $n = 0, 1, 2, \dots$ defines a measure $M_{z,z',\xi}$ on Par. We note that $M_{z,z',\xi}$ degenerates into M^θ as $z, z' \to \infty$ and $\xi \to 0$ in such a way that $zz'\xi \to \theta$. As shown in [68], the measure $M_{z,z',\xi}$ is a determinantal random point field on $\mathbb{Z}^1 + \frac{1}{2}$ in the modified Frobenius coordinates. The corresponding kernel can be expressed in terms of the Gauss hypergeometric function; this is the so-called hypergeometric kernel. It turns out that many known kernels can be obtained as degenerations of the hypergeometric kernel: in particular, the Hermite kernel ((2.2), (2.3), (2.7)), the Laguerre kernel ((2.2), (2.28)), the Meixner kernel ((2.67) below), and the Charlier kernel. For the hierarchy of degenerations of the hypergeometric kernel, we refer the reader to [69], §9. Recently, Okounkov [72] showed that the measures $M_{z,z',\xi}$ are special cases of an infinite-parameter family of probability measures on Par, the so-called Schur measures, which are defined as follows:

$$M(\lambda) = \frac{1}{z} s_\lambda(x) \cdot s_\lambda(y), \qquad (2.63)$$

where s_λ are the Schur functions (for the definition of the Schur functions, see [61] or [63]), and $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are sets of parameters such that the value

$$Z = \sum_{\lambda \in \mathrm{Par}} s_\lambda(x) \cdot s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \qquad (2.64)$$

is finite, and $\{x_i\}_{i=1}^\infty = \overline{\{y_i\}_{i=1}^\infty}$. The measures $M_{z,z',\xi}$ formally correspond to the case in which $\sum_{i=1}^\infty x_i^m = \xi^{\frac{m}{2}} \cdot z$ and $\sum_{i=1}^\infty y_i^m = \xi^{\frac{m}{2}} \cdot z'$, $m = 1, 2, \dots$. To be precise, one must regard the Newton power sums as real parameters and express the Schur functions as polynomials in the power sums. The reader probably would not be too surprised to learn that the Schur measures can also be regarded as determinantal random point fields ([72], Theorems 1 and 2)!

2.6. Two-dimensional random growth model. For the last example we take the following two-dimensional random growth model [73]. Let $\{a_{ij}\}_{i,j\geq 1}$ be a family of independent identically distributed random variables with a geometric law

$$p(a_{ij}=k)=p\cdot q^k, \qquad k=0,1,2,\dots, \tag{2.65}$$

where $0<q<1$ and $p=1-q$. The distribution (2.65) can be regarded as the distribution of the instant of first success in a series of Bernoulli trials. We set

$$G(M,N)=\max_{\pi}\sum_{(i,j)\in\pi} a_{ij}, \tag{2.66}$$

where the maximum is taken over all 'up/right' paths π from (1,1) to (M,N), or, in other words, over the paths

$$\pi=\bigl\{(i_1,j_1)=(1,1),\ (i_2,j_2),(i_3,j_3),\dots,(i_{M+N-1},j_{M+N-1})=(M,N)\bigr\}$$

such that $(i_{k+1},j_{k+1})-(i_k,j_k)\in\bigl\{(0,1),(1,0)\bigr\}$. We mention in passing that the distribution of the random variables $\{G(M,N)\}$ can be interpreted in terms of randomly growing Young diagrams and totally asymmetric exclusion processes with discrete time (for details, see [73]). Without loss of generality we can assume that $M\geq N\geq 1$. To state the connection with determinantal random point fields more explicitly, we introduce the discrete weight $w_K^q(x)=\binom{x+K-1}{x}\cdot q^x$, $K=M-N+1$, on the set of non-negative integers $x=0,1,2\dots$. Normalized orthogonal polynomials $\{M_n(x)\}_{n\geq 0}$ with respect to the weight w_K^q are proportional to the classical Meixner polynomials [74]. The kernel

$$K_{M,N}(x,y)=\sum_{j=0}^{N-1} M_j(x)M_j(y)\bigl(w_K^q(x)w_K^q(y)\bigr)^{1/2} \tag{2.67}$$

satisfies the conditions of Lemma 4 with respect to the counting measure on the non-negative integers. Therefore, the function

$$p_N(x_1,\dots,x_N)=\frac{1}{N!}\det\bigl(K_{M,N}(x_i,x_j)\bigr)_{1\leq i,j\leq N} \tag{2.68}$$

defines a discrete determinantal random point field. It was shown by Johansson that the distribution of the random variable $G(M,N)$ coincides with the distribution of the rightmost particle in (2.68). After an appropriate rescaling, this distribution is convergent as $N\to\infty$, $M\to\infty$, and $\frac{M}{N}\to$ const to the distribution of the rightmost particle in the Airy random point field (2.36). Additional information on the topic of the last two subsections can be found in the recent papers/preprints [75]–[88].

3. Translation-invariant determinantal random point fields

As above, let $(X,\mathcal{B},\mathsf{P})$ be a random point field with one-particle space E (so that X is a space of locally finite configurations of particles in E), let $\mathcal{B}$ be the Borel σ-algebra of measurable subsets of X, and let P be a probability measure on $(X,\mathcal{B})$. Throughout the section, we always assume that $E=\mathbb{R}^d$ or $\mathbb{Z}^d$. We define a continuous action $\{T^t\}_{t\in E}$ of the additive group E on X in the following natural way:

$$T^t\colon X\to X, \quad (T^t\xi)_i=(\xi)_i+t. \tag{3.1}$$

Definition 5. A random point field $(X, \mathcal{B}, \mathsf{P})$ is said to be *translation invariant* if

$$\mathsf{P}(T^{-t}A) = \mathsf{P}(A)$$

for any $A \in \mathcal{B}$ and any $t \in E$.

Translation invariance of a random point field implies translation invariance of the k-point correlation functions,

$$\rho_k(x_1 + t, \dots, x_k + t) = \rho_k(x_1, \dots, x_k) \quad \text{a.e.}, \qquad k = 1, 2, \dots, \quad t \in E. \tag{3.2}$$

Conversely, if $\{\rho_k\}$ are invariant under $\{T^t\}$, then there is a corresponding random point field that is translation invariant [3]. In particular, if translation-invariant correlation functions determine the measure P uniquely, then the random point field is translation invariant. In the case of a determinantal random point field, this implies the following criterion: a determinantal random point field is translation invariant if and only if the kernel K is translation invariant, that is, $K(x, y) = K(x - y, 0) =: K(x - y)$. In this section we restrict our attention to translation-invariant determinantal random point fields. We are interested in the ergodic properties of the dynamical system $(X, \mathcal{B}, \mathsf{P}, \{T^t\})$. For convenience of the reader, we recall basic definitions of ergodic theory [89].

- A dynamical system is said to be *ergodic* if the measure $\mathsf{P}(A)$ of any invariant set A is either 0 or 1.
- A dynamical system has the *mixing property of multiplicity* $r \geq 1$ if

$$\lim_{t_1, \dots, t_r \to \infty} \int_X f_0(\xi) f_1(T^{t_1}\xi) : \dots : f_r(T^{t_1 + \dots + t_r}\xi)\, dF = \prod_{i=0}^{r} \int_X f_i(\xi)\, d\mathsf{P} \tag{3.3}$$

  for any functions $f_0, f_1, \dots, f_r \in L^{r+1}(X, \mathcal{B}, \mathsf{P})$.
- A dynamical system has *absolutely continuous spectrum* if

$$\int_X f(\xi)\overline{f(T^t\xi)}\, d\mathsf{P} = \int e^{i(t\cdot\lambda)} h_f(\lambda)\, d\lambda \tag{3.4}$$

  for any $f \in L^2(X, \mathcal{B}, \mathsf{P})$ orthogonal to the constants, where the integration on the right-hand side of (3.4) is over $\mathbb{R}^d$ in the continuous case and over $[0, 2\pi]^d$ in the discrete case, and $h_f(\lambda)d\lambda$ is a finite measure absolutely continuous with respect to Lebesgue measure. One can interpret (3.4) as follows. Let us introduce a d-parameter group $\{U^t\}_{t \in E}$ of unitary operators on the space $L^2(X, \mathcal{B}, \mathsf{P})$ by the formula

$$(U^t f)(\xi) = f(T^t\xi).$$

  Usually, this family of unitary operators is said to be *associated* with the dynamical system. We can readily see that the operators in $\{U^t\}$ commute with each other. Since $L^2(X, \mathcal{B}, \mathsf{P})$ is separable and $(U^t\psi, \varphi)$ is a measurable function of t for any $\psi, \varphi \in L^2(X, \mathcal{B}, \mathsf{P})$, it follows by von Neumann's theorem ([12], vol. 1, Theorem VIII.9) that U^t is strongly continuous.

In the case $E = \mathbb{R}^d$ one has $h_f(\lambda)d\lambda = d(f, Q_\lambda f)$, where dQ_λ is a projection-valued measure, namely,

$$Q_\lambda = Q_{(-\infty,\lambda_1)\times\cdots\times(-\infty,\lambda_d)} = \prod_{j=1}^{d} \chi_{(-\infty,\lambda_j)}(A_j),$$

where $\{A_j\}_{j=1}^d$ are the infinitesimal generators of the one-parameter groups $U^{(0,\dots,t_j,0,\dots,0)}$, and $\chi_{(-\infty,t)}$ is the characteristic function of the set $(-\infty, t)$ ([12], vol. 1, Theorem VIII.12). In the discrete case $E = \mathbb{Z}^d$, and dQ_λ is a projection-valued measure on the d-dimensional torus,

$$Q_{[1,e^{i\lambda_1}]\times\cdots\times[1,e^{i\lambda_d}]} = \prod_{j=1}^{d} \chi_{[1,e^{i\lambda_j}]}(U_j), \qquad U_j = U^{(0,\dots,t_j=1,\dots,0)}.$$

Theorem 7. *Let $(X, \mathcal{B}, \mathsf{P})$ be a translation-invariant determinantal random point field. Then the dynamical system $(X, \mathcal{B}, \mathsf{P}, \{T^t\})$ is ergodic, it has the mixing property of any multiplicity, and its spectrum is absolutely continuous.*

Remark 11. Recall that absolute continuity of the spectrum implies the mixing property of multiplicity 1, and, in turn, ergodicity follows from the latter property [89].

Proof of Theorem 7. We note that the linear combinations of the functions

$$f(\xi) = \prod_{j=1}^{N} S_{g_j}(\xi), \quad N \geq 1, \ \ S_g(\xi) = \sum_i g(x_i), \ \ g_j \in C_0^\infty(\mathbb{R}^d), \ \ j = 1, \dots N, \tag{3.5}$$

are dense in $L^2(X, \mathcal{B}, \mathsf{P})$. Therefore, it suffices to establish (3.3) and (3.4) for the functions of this form. We begin with a lemma calculating the expectation of (3.5).

Lemma 5. a)

$$\mathsf{E}_P \prod_{j=1}^{N} S_{g_j}(\xi) = \sum_{m=1}^{N} \sum_{\substack{\text{over the partitions} \\ \bigsqcup_{\ell=1}^{m} C_\ell = \{1,\dots,N\}}} \prod_{\ell=1}^{m} \Bigg[\sum_{k_\ell=1}^{\#(C_\ell)} \sum_{\substack{\text{over the partitions} \\ \bigsqcup_{i=1}^{k_\ell} B_{\ell i} = C_\ell}} \left\{ \sum_{\sigma \in S^{k_\ell}} \frac{(-1)^\sigma}{k_\ell} \cdot \int \prod_{i=1}^{k_\ell} g_{B_{\ell i}}(x_{\sigma(i)}) \cdot K(x_{\sigma(i+1)} - x_{\sigma(i)})\, dx_1 \cdots dx_{k_\ell} \right\} \Bigg], \tag{3.6}$$

where $g_{B_{\ell i}}(x) = \prod_{j \in B_{\ell i}} g_j(x)$.

b) $\mathsf{E} \prod_{j=1}^{N_1+\cdots+N_{r+1}} S_{g_j}(\xi) - \prod_{s=1}^{r+1} \left(\mathsf{E} \prod_{N_1+\cdots+N_{s-1}+1}^{N_1+\cdots+N_s} S_{g_j}(\xi) \right)$ *is equal to an expression similar to* (3.6) *with the only difference that the partitions*

$$\bigsqcup_{\ell=1}^{m} C_\ell = \left\{ 1, 2, \dots, \sum_{s=1}^{r+1} N_s \right\} \tag{3.7}$$

satisfy the following condition (*):

(*) *There is at least one element C_ℓ of the partition such that the intersections of C_ℓ with at least two of the sets* $\{1,\dots N_1\},\dots,\{N_1+\dots+N_{s-1}+1,\dots, N_1+\dots+N_s\},\dots,\{N_1+\dots+N_r+1,\dots,N_1+\dots+N_{r+1}\}$ *are non-empty.*

Proof of Lemma 5. The proof of part a) is rather straightforward and quite similar to that given at the beginning of [42], § 2 (see (2.1)–(2.7) in [42]). The proof of part b) follows from a).

To derive the mixing property (3.3), we replace $g_j(\,\cdot\,)$ for $N_1+\dots+N_{s-1}+1\le j\le N_1+\dots+N_s$, $s=1,\dots,r+1$, in (3.7) by $g_j(\cdot+t_1+\dots+t_{s-1})$. Choose a partition $\bigsqcup_{\ell=1}^m C_\ell=\{1,2,\dots,N_1+\dots+N_{r+1}\}$. Since $\{g_j\}$ are bounded functions with compact support, it follows that each of the m factors on the right-hand side of (3.6) is bounded. We claim that the ℓth factor (corresponding to C_ℓ, where ℓ is the same index as in (*)) tends to zero. To show this, we fix an arbitrary partition $\bigsqcup_{i=1}^{k_\ell} B_{\ell i}=C_\ell$ of C_ℓ. By assumption, C_ℓ contains indices u and v, $1\le u<v\le N_1+\dots+N_{r+1}$, that belong to different subsets in the list $\{1,\dots,N_1\},\dots,\{N_1+\dots+N_{s-1}+1,\dots,N_1+\dots+N_s\},\dots,\{N_1+\dots+N_r+1,\dots,N_1+\dots+N_{r+1}\}$. We claim that

$$\int\prod_{i=1}^{k_\ell} g_{B_{\ell i}}(x_{\sigma(i)})\cdot K(x_{\sigma(i+1)}-x_{\sigma(i)})\,dx_1\cdots dx_{k_\ell} \tag{3.8}$$

tends to zero as $\min\{t_s, 1\le s\le r\}\to\infty$. Indeed, if $\min\{t_s,1\le s\le r\}$ is sufficiently large, then the indices u and v belong to different sets $B_{\ell i}$, or, in other words, the corresponding quantity $g_{B_{\ell i}}$ is zero (the supports of the factors in $g_{B_{\ell i}}$ are disjoint). However, if u and v belong to different sets $B_{\ell i}$, then the argument in $K(x_{\sigma(i+1)}-x_{\sigma(i)})$ is greater than $\min\{t_s,1\le s\le r\}$ for some i. Since the Fourier transform $\widehat{K}(t)=\frac{1}{2\pi}\int e^{ixt}K(x)\,dx$ of $K(x)$ is a non-negative integrable function (bounded above by 1), it follows that $K(x_{\sigma(i+1)}-x_{\sigma(i)})$ tends to zero by the Riemann–Lebesgue lemma. Since the other factors in (3.8) are bounded, and the integration is over a bounded set, it follows that (3.8) tends to zero, which implies the mixing property.

To establish the absolute continuity of the spectrum, we apply (3.7) to the case $r=2$, $N_1=N_2=N$, $g_{N+j}(x)=\overline{g_j(x+t)}$, $j=1,\dots,N$, $f(\xi)=\prod_{j=1}^N S_{g_j}(\xi)$, $\overline{f(T^t\xi)}=\prod_{j=1}^N S_{\overline{g_j}}(T^t\xi)=\prod_{j=N+1}^{2N} S_{\overline{g_j}}(\xi)$. We have

$$\mathsf{E}(f(\xi)-\mathsf{E}f)\cdot(\overline{f(T^t\xi)}-\overline{\mathsf{E}f})=\sum_{m=1}^{2N}\ \sideset{}{^*}\sum_{\substack{\text{over the partitions}\\ \bigsqcup_{\ell=1}^m C_\ell=\{1,\dots,2N\}}}\ \prod_{\ell=1}^m\Biggl[\sum_{k_\ell=1}^{\#(C_\ell)}\ \sum_{\substack{\text{over the partitions}\\ \bigsqcup_{i=1}^{k_\ell} B_{\ell i}=C_\ell}}$$

$$\Biggl\{\sum_{\sigma\in S^{k_\ell}}\frac{(-1)^\sigma}{k_\ell}\cdot\int\prod_{i=1}^{k_\ell} g_{B_{\ell\sigma(i)}}(x_i)\cdot K(x_{i+1}-x_i)\,dx_1\cdots dx_{k_\ell}\Biggr\}\Biggr], \tag{3.9}$$

where it is assumed that $x_{k_\ell+1}=x_1$ in the integral, and the sum in $\sum^*$ is taken over the partitions $\{C_1,\dots,C_m\}$ such that both $C_\ell\cap\{1,2,\dots,N\}$ and

$C_\ell \cap \{N+1, \ldots, 2N\}$ are non-empty for at least one element C_ℓ of the partition (that is, the property (*) holds). The factors in the product $\prod_{\ell=1}^{m}$ that correspond to the values of ℓ for which (*) fails are constants as functions of t. Let us now choose an index ℓ satisfying (*). We claim that

$$\int \prod_{i=1}^{k_\ell} g_{B_{\ell\sigma(i)}}(x_i) \cdot K(x_{i+1} - x_i)\, dx_1 \cdots dx_{k_\ell}$$
$$= \left(\frac{1}{2\pi}\right)^{k_\ell} \prod_{i=1}^{k_\ell} \widehat{g}_{B_{\ell\sigma(i)}}(y_{i+1} - y_i) \cdot \widehat{K}(y_{i+1})\, dy_1 \cdots dy_{k_\ell} \tag{3.10}$$

can be represented as $\int e^{i(t\cdot\lambda)} h(\lambda)\, d\lambda$, where $h(\lambda)$ is an integrable function. The verification is rather straightforward, and we leave the details to the reader. We conclude that (3.9) is a linear combination of products of Fourier transforms of integrable functions. Since a product of Fourier transforms is a Fourier transform of the corresponding convolution, it follows that the spectrum is absolutely continuous. This proves Theorem 7.

We can readily calculate the spectral density of the centralized linear statistics

$$S_g(\xi) = \mathsf{E} S_g = \sum_i g(x_i) - \mathsf{E} \sum_i g(x_i).$$

Namely,

$$\mathsf{E}(S_g - \mathsf{E}S_g)(\overline{S_g(T^t\cdot)} - \mathsf{E}\overline{S_g}) = \int e^{i(t\lambda)} \cdot \big(K(0) - \widehat{|K|^2}(\lambda)\big) \frac{1}{2\pi} |\widehat{g}(\lambda)|^2\, d\lambda,$$
$$h_{S_g}(\lambda) = (K(0) - \widehat{|K|^2}(\lambda)) \cdot \frac{1}{2\pi} |\widehat{g}(\lambda)|^2. \tag{3.11}$$

We thus conclude that

$$\mu(d\lambda) = \big(K(0) - \widehat{|K|^2}(\lambda)\big)\, d\lambda \tag{3.12}$$

is the spectral measure of the restriction of $\{U^t\}$ to the subspace of centralized linear statistics. Since $0 \le \widehat{K}(\lambda) \le 1$ and $K(0) = \frac{1}{2\pi} \int \widehat{K}(\lambda)\, d\lambda$, it follows that

$$0 \le \frac{d\mu}{d\lambda} = K(0) - \widehat{|K|^2}(\lambda) = K(0) - \frac{1}{2\pi} \int \widehat{K}(y) \widehat{K}(y - \lambda)\, dy \le K(0).$$

We note that $\frac{d\mu}{d\lambda} > 0$ for $\lambda \ne 0$, and $\frac{d\mu}{d\lambda}(0) = 0$ if and only if $\widehat{K}(\lambda)$ is a characteristic function. In particular, the spectral measure μ is equivalent to Lebesgue measure.

Before formulating the next lemma, we recall that $\#_{[-L,L]^d}(\xi)$ denotes the number of particles in $[-L, L]^d$.

Lemma 6.

$$\mathsf{D}(\#_{[-L,L]^d}) = \mathrm{Vol}([-L, L]^d) \cdot \left(\frac{d\mu}{d\lambda}(0) + o(1)\right) \quad \textit{as} \quad L \to \infty. \tag{3.13}$$

Proof of Lemma 6. Probabilists are familiar with the following analogue of this result in the theory of random processes. Let $\{\eta_n\}$ be an L^2-stationary random sequence and let $h(\lambda)$ be its spectral density, that is,

$$\mathsf{E}\eta_n\overline{\eta_m} = b(n-m) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda\cdot(n-m)}h(\lambda)\,d\lambda;$$

then $\mathsf{D}(\eta_n+\cdots+\eta_n) = (h(0)+o(1))\cdot n$ ([90], § XVIII.2). To prove the lemma, we write

$$\begin{aligned}\mathsf{D}(\#_{[-L,L]^d}) &= \int_{[-L,L]^d}\int_{[-L,L]^d}\big(\rho_2(x,y)-\rho_1(x)\rho_1(y)\big)\,dx\,dy + \int_{[-L,L]^d}\rho_1(x)\,dx \\ &= -\int_{[-L,L]^d}\int_{[-L,L]^d}|K|^2(x-y)\,dx\,dy + K(0)\operatorname{Vol}([-L,L]^d) \\ &= \left(K(0) - \int_{\mathbb{R}^d}|K|^2(x)\,dx + o(1)\right)\cdot\operatorname{Vol}([-L,L]^d) \\ &= \big(K(0) - \widehat{|K|^2}(0) + o(1)\big)\cdot\operatorname{Vol}([-L,L]^d).\end{aligned}$$

The lower-order terms in (3.13) also depend on the behaviour of $\frac{d\mu}{d\lambda}$ near the origin. For example, let $\widehat{K}(\lambda)$ be a characteristic function, $\widehat{K}(\lambda) = \chi_B(\lambda)$, where $B\subset\mathbb{R}^d$. As proved above, this is equivalent to the condition $\frac{d\mu}{d\lambda}(0) = 0$. For simplicity we assume that $d=1$. If B is a union of m disjoint intervals, then

$$\begin{aligned}\frac{d\mu}{d\lambda}(\lambda) &= K(0) - \frac{1}{2\pi}\int\widehat{K}(y)\cdot\widehat{K}(y-\lambda)\,dy \\ &= \frac{1}{2\pi}\cdot\big[\operatorname{length}(B) - \operatorname{length}(B\cap(B+\lambda))\big] \\ &= \frac{m}{2\pi}\cdot|\lambda|\cdot\big(1+o(1)\big),\quad \lambda\to 0,\end{aligned}\tag{3.14}$$

and a more exact estimate of the asymptotics of the expression

$$\int_{-L}^{L}\int_{-L}^{L}|K|^2(x-y)\,dx\,dy = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\widehat{|K|^2}(\lambda)\cdot\left(\frac{2\sin(L\cdot\lambda)}{\lambda}\right)^2 d\lambda$$

shows that

$$\mathsf{D}(\#_{[-L,L]^d}) = \frac{m}{\pi^2}\log L\cdot\big(1+o(1)\big).\tag{3.15}$$

Choosing $m=1$ and $\widehat{K}(\lambda) = X_{[-\pi,\pi]}(\lambda)$, one obtains the sine kernel $K(x-y) = \frac{\sin\pi(x-y)}{\pi(x-y)}$. The special role played by the sine kernel can be demonstrated by the fact that the rate of growth of the expression $\mathsf{D}(\#_{[-L,L]})$, which is equal to $\frac{1}{\pi^2}\log L$, is minimal among the translation-invariant kernels. If $B = \bigsqcup_{n\geq 1}[n, n+\frac{1}{n^\gamma}]$ and $\gamma>1$, then $\frac{d\mu}{d\lambda}\sim|\lambda|^{1-\frac{1}{\gamma}}$ and $\mathsf{D}(\#_{[-L,L]})\sim L^{\frac{1}{\gamma}}$. More generally, if $\frac{d\mu}{d\lambda}\sim|\lambda|^\alpha$ for $0<\alpha<1$, then $\mathsf{D}(\#_{[-L,L]})\sim L^{1-\alpha}$.

4. Central limit theorem for the counting function and for the empirical distribution function of spacings

In [91], Costin and Lebowitz proved the central limit theorem for $\#_{[-L,L]}$ in the case of the sine kernel. The paper contains a remark on p. 71, due to Widom, that the result holds for a larger class of random matrix models. In the general form, the theorem was published in [41].

Theorem 8. *Let E be defined as in* (1.1), *let $\{0 < K_t \le 1\}$ be a family of locally trace-class operators on $L^2(E)$, let $\{(X, \mathcal{B}, \mathsf{P}_t)\}$ be a family of corresponding determinantal random point fields on E, and let $\{I_t\}$ a family of measurable subsets of E such that*

$$\mathsf{D}_t\#_{I_t} = \operatorname{Tr}(K_t \cdot \chi_{I_t} - (K_t \cdot \chi_{I_t})^2) \to \infty \quad as \quad t \to \infty. \tag{4.1}$$

Then the distribution of the normalized number of particles in I_t (with respect to P_t) is convergent to the normal law, that is,

$$\frac{\#_t - \mathsf{E}\#_{I_t}}{\sqrt{\mathsf{D}_t\#_t}} \xrightarrow{\text{w}} N(0,1).$$

Remark 12. As shown in [41], condition (4.1) in Theorem 8 (on the growth of the variance) is satisfied for the Airy kernel ($K_t \equiv K$ in (2.36), with expanding I_t), for the Bessel kernel ($K_t \equiv K$ in (2.37), with expanding I_t), and for the families $\{K_n\}$ of kernels corresponding to random matrices for the classical compact groups (see § 2.3b and § 2.3c)). In all these cases, the growth of $\mathsf{D}_t\#_{I_t}$ is logarithmic with respect to $\mathsf{E}_t\#_{I_t}$.

Remark 13. To construct an example of a kernel defining an operator K, $0 \le K \le \mathrm{Id}$, such that $E\#_{[-n,n]} = \operatorname{Tr} K \cdot \chi_{[-n,n]} \to \infty$ as $n \to \infty$ and the quantity $\mathsf{D}\#_{[-n,n]} = \operatorname{Tr}(K \cdot \chi_{[-n,n]} - (K \cdot \chi_{[-n,n]})^2)$ remains bounded, we consider a set $\{\varphi_n(x)\}_{n=-\infty}^{\infty}$ of functions satisfying the following conditions:

a) $\operatorname{supp} \varphi_n \in (n, n+1)$,

b) $\|\varphi_n\|_{L^2} = 1$.

Then

$$K(x,y) = \sum_{n=-\infty}^{\infty} \left(1 - \frac{1}{n^2+1}\right) \cdot \varphi_n(x) \cdot \overline{\varphi_n(y)}$$

is the desired kernel. Indeed,

$$\mathsf{E}\#_{[-n,n]} = \sum_{k=-n}^{n} \left(1 - \frac{1}{k^2+1}\right) \underset{n\to\infty}{\longrightarrow} \infty,$$

$$\mathsf{D}\#_{[-n,n]} = \sum_{k=-n}^{n} \left(1 - \frac{1}{k^2+1}\right) \cdot \frac{1}{k^2+1} \underset{n\to\infty}{\longrightarrow} \sum_{-\infty}^{\infty} \left(1 - \frac{1}{k^2+1}\right) \cdot \frac{1}{k^2+1} < \infty.$$

On the other hand, if $0 \le K \le \mathrm{Id}$ and K is compact and locally of trace class, and $\operatorname{Tr} K \cdot \chi_{[-n,n]} \to +\infty$, then $\operatorname{Tr} K \cdot \chi_{[-n,n]} - (K \cdot X_{[-n,n]})^2 \to +\infty$.

The result of Theorem 8 can be generalized to the case of finitely many intervals. Namely, if $I_t^{(1)}, \dots, I_t^{(m)}$ are disjoint subsets such that $\mathrm{Cov}_t(\#_{I_t^{(i)}}, \#_{I_t^{(j)}})/V_t \to b_{ij}$ as $t \to \infty$ for $1 \le i, j \le m$, where V_t is a function of t that is infinitely increasing, then the distribution of the vector $\big((\#_{I_t^{(k)}} - \mathsf{E}_t \#_{I_t^{(k)}}/V_t^{1/2})_{k=1,\dots,m}\big)$ is convergent to an m-dimensional centralized normal vector with covariance matrix $(b_{ij})_{1\le i,j\le m}$ [41].

Finally, we turn our attention to the problem of global distribution of spacings. Let $E = \mathbb{R}^d$ or $\mathbb{Z}^d$, let $\{B_j\}_{j=1}^k$ be bounded measurable subsets of E, and let $\{n_j\}_{j=1}^k$ be non-negative integers. We are interested in counting statistics of the following type:

$$\eta_L(B_1, \dots, B_k; n_1, \dots, n_k) := \#(x_i \in [-L, L]^d : \#_{x_i + B_j} = n_j,\ j = 1, \dots, k). \quad (4.2)$$

We can assume without loss of generality that the sets $\{B_j\}$ are disjoint and do not contain the origin. If $d = 1$, $k = 1$, and $B_1 = (0, s]$, then $\eta_L((0, s], 0)$ is the number of spacings in $[-L, L]$ with length greater than s, $\eta_L((0, s], 0) = \#\{x_i \in [-L, L] : x_{i+1} - x_i > s\}$, and $\eta_L((0, s], n)$ is the number of n-spacings with length greater than s, $\eta_L((0, s], n) = \#\{x_i \in [-L, L] : x_{i+n+1} - x_i > s\}$. In [40] we proved the convergence in law of the process $\frac{\eta_L((0,s],0) - \mathsf{E}\eta_L((0,s],0)}{L^{1/2}}$ to a limit Gaussian process in the case $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$. We recall that convergence in law (the functional central limit theorem) implies not only convergence of the finite-dimensional distributions but also convergence of all functionals that are continuous in an appropriate topology (for instance, in the locally uniform topology) on the space of sample paths. The proof of the central limit theorem for the finite-dimensional distributions of $\eta_L((0, s], 0)$ can be carried over more or less literally to the case of an arbitrary kernel $K(x, y)$ (not necessarily translation-invariant) and of dimension $d \ge 1$ under the assumption that conditions (4.33), (4.34), and (4.35) are satisfied. One can also replace the interval $(0, s]$ by an arbitrary measurable bounded set $B \subset E$. For convenience of the reader, we sketch the main ideas of the proof of the central limit theorem in the finite-dimensional case. Let us fix sets $B_1, \dots, B_k$ and indices $n_1, \dots, n_k$. We construct a new (the so-called modified) random point field such that $\eta_L(B_1, \dots, B_k; n_1, \dots, n_k)$ is equal to the number of all particles of the modified random point field on $[-L, L]^d$. Namely, we keep only the particles of the original random point field for which

$$\#_{x_i + B_j} = n_j, \qquad j = 1, \dots, k, \quad (4.3)$$

and we remove the particles for which condition (4.3) is violated. The modified random point field is no longer a determinantal random point field in general. However, it is of importance that, for this field, the correlation functions and cluster functions (see Definition 6 below) can be expressed in terms of the correlation functions of the original determinantal random point field. Let us denote by $\rho_\ell(x_1, \dots, x_\ell; B_1, \dots, B_k; n_1, \dots, n_k)$ the ℓ-point correlation function of the modified random point field. Suppose that

$$x_i \notin x_j + B_p, \qquad 1 \le i \ne j \le \ell, \quad 1 \le p \le k. \quad (4.4)$$

Then by the inclusion-exclusion principle,

$$\begin{aligned}
&\rho_\ell(x_1,\dots,x_\ell;B_1,\dots,B_k;n_1,\dots,n_k)\\
&\quad=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\underbrace{\int_{(x_1+B_1)^{n_1}\times\cdots\times(x_1+B_k)^{n_k}}\cdots\int_{(x_\ell+B_1)^{n_1}\times\cdots\times(x_\ell+B_k)^{n_k}}}_{\longleftarrow\ \ell\text{ times}\ \longrightarrow}\\
&\qquad\int_{((x_1+\bigsqcup_{j=1}^k B_j)\sqcup\cdots\sqcup(x_\ell+\bigsqcup_{j=1}^k B_j))^m}\rho_{\ell+\ell\cdot n+m}(x_1,\dots,x_\ell;\\
&\qquad\quad x_{11},\dots,x_{1n},x_{21},\dots,x_{2n},\dots,x_{\ell 1},\dots,x_{\ell n},y_1,\dots,y_m)\\
&\qquad\qquad\times dy_1\cdots dy_m\,dx_{\ell 1}\cdots dx_{\ell n}\cdots dx_{11}\cdots dx_{1n},\\
&\qquad\qquad\quad n=n_1+\cdots+n_k.
\end{aligned}\tag{4.5}$$

If (4.4) is violated, then the formula is quite similar, and the only difference is that the exponent n_j in $(x_i+B_j)^{n_j}=\underbrace{(x_i+B_j)\times\cdots\times(x_i+B_j)}_{\longleftarrow\ n_j\text{ times}\ \longrightarrow}$, $1\le i\le\ell$, $1\le j\le k$, must be replaced by the difference $n_j-\#(1\le r\ne i\le k: x_r\in x_i+B_j)$. Although the formulae (4.5) seem to be cumbersome and lengthy, they still turn out to be quite useful when calculating the asymptotics of the moments of the distribution $\eta_L(B_1,\dots,B_k;n_1,\dots,n_k)$. (Of course, the key role is played by the assumption that the correlation functions of the original random point field are determinants.) Let us recall the definition of cluster functions.

Definition 6. The *ℓ-point cluster functions* $r_\ell(x_1,\dots,x_\ell)$, $\ell=1,2,\dots$, of a random point field are defined by the formula

$$r_\ell(x_1,\dots,x_\ell)=\sum_G(-1)^{m-1}(m-1)!\cdot\prod_{j=1}^m\rho_{|G_j|}(\bar{x}(G_j)),\tag{4.6}$$

where the sum is taken over all partitions G of the set $[\ell]=\{1,2,\dots,\ell\}$ into subsets $G_1,\dots,G_m$, where $m=1,\dots,\ell$, $\bar{x}(G_j)=\{x_i: i\in G_j\}$, and $|G_j|=\#(G_j)$.

Cluster functions are also known in statistical mechanics as truncated correlated functions and Ursell functions. In the literature the right-hand sides of (4.6) are sometimes regarded as definitions of the functions $(-1)^{\ell-1}r_\ell$. The correlation functions can be obtained from cluster functions by the following inversion formula:

$$\rho_\ell(x_1,\dots,x_\ell)=\sum_G\prod_{j=1}^m r_{|G_j|}(\bar{x}(G_j)).\tag{4.7}$$

((4.6) is just the Möbius inversion formula applied to (4.7).) The integrals of the cluster functions $r_\ell(x_1,\dots,x_\ell)$ over the parallelepipeds of the form $\underbrace{[-L,L]^d\times\cdots\times[-L,L]^d}_{\longleftarrow\ \ell\text{ times}\ \longrightarrow}=[-L,L]^{\ell d}$ are closely related to the cumulants $C_j(L)$

of the number of particles in $[-L,L]^d$,

$$V_1(L) = \int_{[-L,L]^d} r_1(x_1)\,dx_1 = C_1(L) = \mathsf{E}\#_{[-L,L]^d},$$

$$\begin{aligned} V_2(L) :&= \int_{[-L,L]^d}\int_{[-L,L]^d} r_2(x_1,x_2)\,dx_1\,dx_2 \\ &= C_2(L) - C_1(L) = \mathsf{D}\#_{[-L,L]^d} - \mathsf{E}\#_{[-L,L]^d}, \end{aligned}$$

$$\begin{aligned} V_3(L) :&= \int_{[-L,L]^d}\int_{[-L,L]^d}\int_{[-L,L]^d} r_3(x_1,x_2,x_3)\,dx_1\,dx_2\,dx_3 \\ &= C_3(L) - 3C_2(L) + 2C_1(L). \end{aligned}$$

In general,

$$\sum_{n=1}^{\infty} \frac{C_n(L)}{n!} z^n = \sum_{n=1}^{\infty} \frac{V_n(L)}{n!}(e^z-1)^n \tag{4.8}$$

(see [91] and [40]). For determinantal random point fields,

$$r_\ell(x_1,\dots,x_\ell) = (-1)^{\ell-1} \sum_{\text{cyclic } \sigma\in S_\ell} K(x_1,x_2)\cdot K(x_2,x_3)\cdot\ldots\cdot K(x_\ell,x_1), \tag{4.9}$$

where the sum is taken over all cyclic permutations, and the expression under the symbol of the sum corresponds to $\sigma = (1\ 2\ 3\ldots\ell)$. One can also rewrite (4.9) as

$$\begin{aligned} r_\ell(x_1,\dots,x_\ell) &= (-1)^{\ell-1}\cdot\frac{1}{\ell} \\ &\times \sum_{\sigma\in S_\ell} K(x_{\sigma(1)},x_{\sigma(2)})\cdot K(x_{\sigma(2)},x_{\sigma(3)})\cdot\ldots\cdot K(x_{\sigma(\ell)},x_{\sigma(1)}). \end{aligned} \tag{4.10}$$

We note that the difference between the formula

$$\rho_\ell(x_1,\dots,x_\ell) = \sum_{\sigma\in S_\ell} (-1)^\sigma K(x_1,x_{\sigma(1)})\cdot K(x_2,x_{\sigma(2)})\cdot\ldots\cdot K(x_\ell,x_{\sigma(\ell)}) \tag{4.11}$$

for the ℓ-point correlation functions and the formula (4.9) is that the summation in (4.9) is taken over cyclic permutations only. It turns out that the relationship between $\rho_\ell(x_1,\dots,x_\ell;B_1,\dots,B_k;n_1,\dots,n_k)$ and $r_\ell(x_1,\dots,x_\ell;B_1,\dots,B_k;n_1,\dots,n_k)$ is of a similar nature.

Lemma 7. *Let condition* (4.4) *hold. Then*

$$\begin{aligned} & r_\ell(x_1,\dots,x_\ell;B_1,\dots,B_k;n_1,\dots,n_k) \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!} \underbrace{\int_{(x_1+B_1)^{n_1}\times\cdots\times(x_1+B_k)^{n_k}} \cdots \int_{(x_\ell+B_1)^{n_1}\times\cdots\times(x_\ell+B_k)^{n_k}}}_{\ell \text{ times}} \\ & \int_{((x_1+\bigsqcup_{j=1}^k B_j)\sqcup\cdots\sqcup(x_\ell+\bigsqcup_{j=1}^k B_j))^m} \rho_{\ell+\ell\cdot n+m,\ell}(x_1,\dots,x_\ell; \\ & \quad x_{11},\dots,x_{1n},x_{21},\dots,x_{2n},\dots,x_{\ell 1},\dots,x_{\ell n},y_1,\dots,y_m) \\ & \quad\times dy_1\cdots dy_m\,dx_{\ell 1}\cdots dx_{\ell n}\cdots dx_{11}\cdots dx_{1n}, \end{aligned} \tag{4.12}$$

where the function $\rho_{\ell+\ell\cdot n+m,\ell}$ *is defined below in* (4.13).

To define $\rho_{\ell+\ell\cdot n+m,\ell}$ we recall that

$$\rho_{\ell+\ell\cdot n+m}(x_1,\dots,y_m)=\sum_{\sigma\in S_{\ell+\ell\cdot n+m}}(-1)^{\sigma}K(x_1,\sigma(x_1))\cdot\ldots\cdot K(y_m,\sigma(y_m)),$$

where σ is a permutation on the set of variables $(x_1,\dots,x_\ell,x_{11},\dots,x_{\ell n},y_1,\dots,y_m)$. We write

$$\rho_{\ell+\ell\cdot n+m,\ell}(x_1,\dots,y_m)=\sum_{\sigma\in S_{\ell+\ell\cdot n+m}}^{*}(-1)^{\sigma}K(x_1,\sigma(x_1))\cdot\ldots\cdot K(y_m,\sigma(y_m)),\quad (4.13)$$

where the summation in $\sum^*$ is taken over all permutations σ satisfying the following condition.

Let τ be the multivalued map from $\{1,\dots,\ell\}$ to $\{1,\dots,\ell\}$ defined by

$$\tau(i)=\Bigg\{j:\sigma\Bigg(\{x_i,x_{i1},\dots,x_{in}\}\sqcup\Bigg(\{y_1,\dots,y_m\}\cap\Bigg(x_i+\bigsqcup_{p=1}^{k}B_p\Bigg)\Bigg)\Bigg)$$
$$\bigcap\Bigg(\{x_j,x_{j1},\dots,x_{jn}\}\sqcup\Bigg(\{y_1,\dots,y_m\}\cap\Bigg(x_j+\bigsqcup_{p=1}^{k}B_p\Bigg)\Bigg)\Bigg)\neq\varnothing\Bigg\}.\quad (4.14)$$

Then, for any $1\le i,j\le\ell$, there is an $N=N(i,j)$ such that

$$\tau^N(i)\ni j.\quad (4.15)$$

Remark 14. The proof of Lemma 7 in the case $d=1$, $K(x,y)=\frac{\sin\pi(x-y)}{\pi(x-y)}$, $B_1=(0,s]$, $n_1=0$ was given in § 3 of [40]. In the general case the argument is absolutely the same. As a consequence of Lemma 7, we obtain the following result.

Lemma 8. *Let*

$$|K(x,y)|\le\psi(x-y),\quad (4.16)$$

where ψ is a bounded function, and let (4.4) *hold for an ℓ-tuple $(x_1,\dots,x_\ell)$. Then for any $\delta>0$ the following estimate holds*:

$$|r_\ell(x_1,\dots,x_\ell;B_1,\dots,B_k;n_1,\dots,n_k)|$$
$$\le\mathrm{const}(\ell,\delta)\sum_{\text{cyclic }\sigma\in S_\ell}\big(\psi(x_2-x_1)\cdot\psi(x_3-x_2)\cdot\ldots\cdot\psi(x_1-x_\ell)\big)^{1-\delta}.\quad (4.17)$$

For the proof of Lemma 8 we refer the reader to § 3 of [40]. The key point of the proof is to show that an expression of the form

$$\mathrm{const}_1(n,\ell)\cdot\frac{1}{m!}\,\mathrm{const}_2^m\cdot\min\Bigg\{\mathrm{const}_3(n,\ell);(\ell+\ell n+m)!$$
$$\times\sum_{\text{cyclic }\sigma\in S_\ell}\big(\psi(x_2-x_1)\cdot\psi(x_3-x_2)\cdot\ldots\cdot\psi(x_1-x_\ell)\big)\Bigg\}$$

is an upper bound for the absolute value of the mth term of the series (4.12). If $\psi^{1-\delta} \in L^2(E)$ for some δ with $0 < \delta < 1$, then

$$\int_{[-L,L]^d} \cdots \int_{[-L,L]^d} \psi(x_2 - x_1)^{1-\delta} \cdot \ldots \cdot \psi(x_1 - x_\ell)^{1-\delta}\, dx_1 \cdots dx_\ell \\ \leq \text{const}(\psi) \cdot \int_{[-L,L]^d} \psi(x-y)^{2-2\delta}\, dx\, dy = O(L^d),$$

and therefore it follows from Lemma 8 that

$$\int_{[-L,L]^{d\ell} \cap (4.4)} r_\ell(x_1, \ldots, x_\ell; B_1, \ldots, B_k; n_1, \ldots, n_k)\, dx_1 \cdots dx_\ell = O(L^d). \tag{4.18}$$

for $\ell = 1, 2, \ldots$. In particular,

$$\mathsf{E}\eta_L(B_1, \ldots, B_k; n_1, \ldots, n_k) \\ = V_1(L) = \int_{[-L,L]^d} r_1(x; B_1, \ldots, B_k; n_1, \ldots, n_k)\, dx = O(L^d). \tag{4.19}$$

Suppose that one could show that

$$\mathsf{D}\eta_L(B_1, \ldots, B_k; n_1, \ldots, n_k) = V_1(L) + V_2(L) \\ = \int_{[-L,L]^d} r_1(x; B_1, \ldots, B_k; n_1, \ldots, n_k)\, dx \\ + \int_{[-L,L]^d} \int_{[-L,L]^d} r_2(x_1, x_2; B_1, \ldots, B_k; n_1, \ldots, n_k)\, dx_1\, dx_2 \\ = \text{const} \cdot L^d\big(1 + o(1)\big), \tag{4.20}$$

$$\int_{[-L,L]^{\ell d} \setminus (4.4)} r_\ell(x_1, \ldots, x_\ell; B_1, \ldots, B_k; n_1, \ldots, n_k)\, dx_1 \cdots dx_\ell \\ = o\big(L^{\frac{\ell d}{2}}\big), \quad \ell > 2. \tag{4.21}$$

The ℓth cumulant of η_L is a linear combination of $V_i(L)$, $i = 1, 2, \ldots \ell$ (see (4.8)). Hence, the estimates (4.18)–(4.21) would imply that the ℓth cumulant of η_L behaves like const $\cdot L \cdot (1 + \bar{o}(1))$ for $\ell = 2$ and increases slower than $L^{\frac{\ell d}{2}}$ for $\ell > 2$. In turn, this would imply that the second cumulant of $\frac{\eta_L - \mathsf{E}\eta_L}{\sqrt{\mathsf{D}\eta_L}}$ is equal to 1 and the other cumulants of $\frac{\eta_L - \mathsf{E}\eta_L}{\sqrt{\mathsf{D}\eta_L}}$ tend to zero as $L \to +\infty$, which is equivalent to the assertion that the moments of $\frac{\eta_L - \mathsf{E}\eta_L}{\sqrt{\mathsf{D}\eta_L}}$ converge to the corresponding moments of the normal distribution, and, in particular,

$$\frac{\eta_L - \mathsf{E}\eta_L}{\sqrt{\mathsf{D}\eta_L}} \xrightarrow{\text{w}} N(0,1).$$

However, a more detailed consideration faces complications. Apparently, there are no convenient expressions for the formulae (4.12) and (4.13) in the case when (4.4) fails.

Below we show how to overcome these difficulties in the case of $\eta_L(B;0)$ (that is, for $k=1$ and $n_1=0$). We introduce the centralized ℓ-point correlation functions by the formula

$$\rho_\ell^{(c)}(x_1,\dots,x_\ell)=\sum_G{}^{**}\prod_{j=1}^m r_{|G_j|}(\overline{x}(G_j)),\tag{4.22}$$

where the sum $\sum^{**}$ is taken over all partitions $G=\{G_1,\dots,G_m\}$, $m=1,2,\dots$, of the set $\{1,\dots,\ell\}$ into at least two-element subsets (that is, $|G_j|>1$ for $j=1,\dots,m$). It follows from (4.7) and (4.22) that

$$\begin{aligned}\rho_\ell^{(c)}(x_1,\dots,x_\ell)&=\rho_\ell(x_1,\dots,x_\ell)\\&\quad+\sum_{p=1}^{\ell}(-1)^p\sum_{1\le i_1<\dots<i_p\le\ell}\prod_{s=1}^p\rho_1(x_{i_s})\cdot\rho_{\ell-p}\big((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})\big)\\&=\rho_\ell(x_1,\dots,x_\ell)\\&\quad-\sum_{p=1}^{\ell}\sum_{1\le i_1<\dots<i_p\le\ell}\prod_{s=1}^p\rho_1(x_{i_s})\rho_{\ell-p}^{(c)}\big((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})\big).\end{aligned}\tag{4.23}$$

Let us denote by $M_{(\ell)}^{(c)}(L)$ the integral of the centralized ℓ-point correlation function of the modified random point field over $[-L,L]^{\ell d}$,

$$M_{(\ell)}^{(c)}(L)=\int_{[-L,L]^d}\cdots\int_{[-L,L]^d}\rho_\ell^{(c)}(x_1,\dots,x_\ell;B_1;0)\,dx_1\cdots dx_\ell.\tag{4.24}$$

We have

$$\begin{aligned}&\sum_{\ell=0}^\infty\frac{t^\ell}{\ell!}\mathsf{E}(\eta_L-\mathsf{E}\eta_L)^\ell\\&\qquad=e^{-t\mathsf{E}\eta_L}\cdot\sum_{\ell=0}^\infty\frac{t^\ell}{\ell!}\mathsf{E}\eta_L^\ell\\&\qquad=e^{-t\mathsf{E}\eta_L}\cdot\sum_{\ell=0}^\infty\frac{(e^t-1)^\ell}{\ell!}\mathsf{E}\eta_L\cdot(\eta_L-1)\cdot\ldots\cdot(\eta_L-\ell+1)\\&\qquad=e^{-t\mathsf{E}\eta_L}\cdot e^{(e^t-1)\mathsf{E}\eta_L}\cdot\sum_{\ell=0}^\infty\frac{(e^t-1)^\ell}{\ell!}M_{(\ell)}^{(c)}(L).\end{aligned}\tag{4.25}$$

If we were able to show that

$$M_\ell^{(c)}(L)=\begin{cases}(2n-1)!!\ \mathrm{const}_1^n\cdot L^{nd}\cdot\big(1+o(1)\big)&\text{for }\ell=2n,\\ o\big(L^{\frac{\ell d}{2}}\big)&\text{for }\ell=2n+1,\end{cases}\tag{4.26}$$

and

$$\mathsf{E}\eta_L=\mathrm{const}_2\cdot L^d\cdot\big(1+o(1)\big),\tag{4.27}$$

then (4.25) would imply that

$$\mathsf{E}(\eta_L - \mathsf{E}\eta_L)^\ell = \begin{cases} (2n-1)!!\,(\mathrm{const}_1 + \mathrm{const}_2)^n L^{nd} \cdot \big(1+o(1)\big) \\ \qquad\qquad \text{for } \ell = 2n, \\ o\big(L^{\frac{\ell d}{2}}\big) \qquad \text{for } \ell = 2n+1, \end{cases} \tag{4.28}$$

and

$$\frac{\eta_L - \mathsf{E}\eta_L}{L^{d/2}} \xrightarrow{\text{w}} N(0, \mathrm{const}_1 + \mathrm{const}_2).$$

In principle, one can calculate $M_\ell^{(c)}(L)$ by using (4.12) and (4.13). Indeed, if

$$x_i - x_j \notin B \tag{4.29}$$

(we note that the condition (4.29) is exactly the condition (4.4) in the case $k=1$, $n_1 = 0$), then the expression for $\rho_\ell^{(c)}(x_1,\dots,x_\ell;B;0)$ can be obtained from (4.22), (4.12), and (4.13). Otherwise $\rho_\ell(x_1,\dots,x_\ell;B;0) = 0$, and it follows from (4.23) that

$$\rho_\ell^{(c)}(x_1,\dots,x_\ell;B;0) = \sum_{p=1}^{\ell}(-1)^p \cdot \sum_{1\le i_1<\dots<i_p\le\ell} \prod_{s=1}^{p} r_1(x_{i_s};B;0) \\ \times \rho_{\ell-p}\big((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})\big). \tag{4.30}$$

If (4.29) fails for an $(\ell-p)$-tuple $(x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})$, then the corresponding factor $\rho_{\ell-p}((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p}))$ in (4.30) vanishes. If (4.29) holds for $(x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})$, then we iterate (4.23) again,

$$\rho_{\ell-p}\big((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})\big) = \rho_{\ell-p}^{(c)}\big((x_1,\dots,x_\ell)\setminus(x_{i_1},\dots,x_{i_p})\big) + \sum \cdots .$$

We thus obtain the following assertion.

Lemma 9. *Let* (4.29) *fail for an ℓ-tuple* $(x_1,\dots,x_\ell)$. *Then*

$$\rho_\ell^{(c)}(x_1,\dots,x_\ell;B;0) = \sum_{\varnothing\subseteq D\subset\{1,\dots,\ell\}} C_D \cdot \prod_{i\notin D} r_1(x_i;B;0)\cdot \rho_{|D|}^{(c)}(\bar{x}(D)), \tag{4.31}$$

where

$$C_D = \sum_{\substack{A\supseteq D \\ (4.29)\text{ holds for } \bar{x}(A)}} (-1)^{|A|}. \tag{4.32}$$

In particular, $C_D = 0$ *if* (4.29) *fails for* $\bar{x}(D)$, *and also if there is an i such that* $1 \le i \le \ell$, $i \notin D$, *and* $x_i - x_j \notin B \cup (-B)$ *for any* $1 \le j \le \ell$.

The proof readily follows from the above considerations.

Theorem 9. *Let* $(X, \mathcal{B}, \mathsf{P})$ *be a determinantal random point field with kernel*

$$|K(x,y)| \le \psi(x-y), \tag{4.33}$$

where ψ *is a bounded non-negative function such that* $\psi \cdot (\log(\frac{\psi+1}{\psi}))^n \in L^2(E)$ *for any* $n > 0$. *Let the estimate*

$$\mathsf{D}\eta_L(B;0) = \sigma^2 \cdot L^d \cdot \big(1+o(1)\big) \tag{4.34}$$

be valid for $\eta_L(B;0) = \#(x_i \in [-L,L]^d : \#(x_i + B) = 0)$. *Then the central limit theorem holds,*

$$\frac{\eta_L(B,0) - \mathsf{E}\eta_L(B;0)}{L^{d/2}} \xrightarrow{\text{w}} N(0,\sigma^2).$$

Remark 15. If $\mathrm{Cov}(\eta_L(B_i;0), \eta_L(B_j;0)) = b_{ij} \cdot L^d \cdot (1+o(1)),\ \ 1 \le i,j \le p$, then

$$\left(\frac{\eta_L(B_i;0) - \mathsf{E}\eta_L(B_i;0)}{L^{d/2}}\right)_{1\le i\le p} \xrightarrow{\text{w}} N(0,(b_{ij})_{1\le i,j\le p}). \tag{4.35}$$

We recall that

$$\begin{aligned}
&\mathrm{Cov}\big(\eta_L(B_i;0); \eta_L(B_j;0)\big) = \mathsf{E}\big(\eta_L(B_i;0) - \mathsf{E}\eta_L(B_i,0)\big)\cdot\big(\eta_L(B_j;0) - \mathsf{E}\eta_L(B_j;0)\big) \\
&\quad = \int_{[-L,L]^{2d}\cap\{x_1-x_2\notin B_i\cup(-B_j)\}} \Bigg(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \\
&\qquad\qquad \times \int_{((x_1+B_i)\sqcup(x_2+B_j))^m} \rho_{2+m,2}(x_1,x_2;y_1,\dots,y_m)\,dy_1\cdots dy_m\Bigg)\,dx_1\,dx_2 \\
&\qquad - \int_{[-L,L]^d} r_1(x_1;B_i;0)\cdot \int_{(x_1+B_i)\cup(x_1-B_j)} r_1(x_2;B_j;0)\,dx_2\,dx_1 \\
&\qquad + \int_{[-L,L]^d} r_1(x;B_1\cup B_2,0)\,dx.
\end{aligned}$$

Remark 16. Lemma 8 assumes a slightly more restrictive condition on ψ, namely, $\psi^{1-\delta} \in L^2(E)$ for some $0 < \delta < 1$. However, looking at the proof of Lemma 6, we can readily see that $\psi^{1-\delta}$ in (4.17) can be replaced by $\psi \cdot (\log(\frac{\psi+1}{\psi}))^n$ with $n > 3\ell$.

Proof of Theorem 9. From (4.25)–(4.28), it suffices to prove that

$$\begin{aligned}
&\int_{[-L,L]^{2nd}} \rho_{2n}^{(c)}(x_1,\dots,x_{2n};B;0)\,dx_1\cdots dx_{2n} \\
&\quad = (2n-1)!!\Bigg(\int_{[-L,L]^{2d}\cap\{x-y\notin B\cap(-B)\}} r_2(x,y;B;0)\,dx\,dy \\
&\qquad - \int_{[-L,L]^d} r_1(x;B;0)\int_{(x+B)\cup(x-B)} r_1(y;B;0)\,dy\,dx\Bigg)^n + o(L^{nd}), \qquad (4.36)\\
&\qquad\qquad n = 1,2,\dots,
\end{aligned}$$

$$\begin{aligned}
&\int_{[-L,L]^{(2n+1)d}} \rho_{2n+1}^{(c)}(x_1,\dots,x_{2n+1};B;0)\,dx_1\cdots dx_{2n+1} = o\big(L^{\frac{2n+1}{2}d}\big), \qquad (4.37)\\
&\qquad\qquad n = 1,2,\dots.
\end{aligned}$$

Lemma 10.

$$\int_{[-L,L]^{2nd}\cap(4.29)} \rho_{2n}^{(c)}(x_1,\dots,x_{2n};B;0)\,dx_1\cdots dx_{2n}$$
$$= (2n-1)!!\left(\int_{[-L,L]^{2d}\cap\{x-y\notin B\cup(-B)\}} r_2(x,y;B;0)\,dx\,dy\right)^n + o(L^{nd}), \quad (4.38)$$

$$\int_{[-L,L]^{(2n+1)d}\cap(4.29)} \rho_{2n+1}^{(c)}(x_1,\dots,x_{2n+1};B;0)\,dx_1\cdots dx_{2n+1} = o(L^{\frac{2n+1}{2}d}). \quad (4.39)$$

We recall that all functions $r_\ell(x_1,\dots,x_\ell;B;0)$ are bounded (see (4.17)). Let us rewrite relation (4.22) as

$$\rho_\ell^{(c)}(x_1,\dots,x_\ell) = \sum_G{}' \prod_{j=1}^m r_{|G_j|}(\bar{x}(G_j)) + \sum_G{}'' \prod_{j=1}^m r_{|G_j|}(\bar{x}(G_j)),$$

where $\sum'$ is the sum over all partitions of $\{1,\dots,\ell\}$ into pairs, and the sum in $\sum''$ is taken over all other partitions (into sets with at least two elements). Let ℓ be even, $\ell = 2n$. Integrating the sum $\sum_G'$ over $[-L,L]^{2nd}\cap(4.29)$, we obtain exactly the right-hand side of (4.38) (there are precisely $(2n-1)!!$ partitions of $\{1,\dots,2n\}$ into two-element sets). It follows from (4.17) and the estimates after Lemma 8 that

$$\int_{[-L,L]^{\ell d}} |r_\ell(x_1,\dots,x_\ell;B;0)|\,dx_1\cdots dx_\ell = O(L^d).$$

Therefore, the integral of $\sum_G''$ over $[-L,L]^{2nd}\cap(4.29)$ is of order $o(L^{nd})$. The formula (4.39) can be proved in a similar way.

To estimate the integral

$$\int_{[-L,L]^{2nd}\setminus(4.29)} \rho_{2n}^{(c)}(x_1,\dots,x_{2n};B;0)\,dx_1\cdots dx_{2n}, \quad (4.40)$$

we introduce an equivalence relation on $\{x_1,\dots,x_{2n}\}$; namely, we say that x_i and x_j are 'neighbours' if there is a sequence of indices $1\le i_0,i_1,\dots,i_u\le 2n$, $1\le u\le 2n$, such that $i_0=i$, $i_u=j$, and $x_{i_{s+1}}-x_{i_s}\in B\cup(-B)$ for $s=0,\dots,u-1$. We claim that contributions of order $O(L^{nd})$ can occur in (4.40) only from sets of points $(x_1,\dots x_{2n})$ for which any equivalence class of 'neighbours' contains either one or two indices. For example, let us consider the case of k two-element equivalence classes $\{x_1,x_2\},\dots,\{x_{2k-1},x_{2k}\}$ and $2n-2k$ one-element equivalence classes $\{x_{2k+1}\},\dots,\{x_{2n}\}$. By calculations similar to those in [40], pp. 596–597, we can see that the integral of $\rho_{2n}^{(c)}(x_1,\dots,x_{2n};B;0)$ over the subset of $[-L,L]^{2nd}$ corresponding to the above partition is equal to

$$(2n-2k-1)!!\left(-\int_{[-L,L]^d} r_1(x;B;0)\int_{(x+B)\cup(x-B)} r_1(y;B;0)\,dy\,dx\right)^k$$
$$\times\left(\int_{[-L,L]^{2n}\cap\{x-y\notin B\cap(-B)\}} r_2(x,y;B;0)\,dx\,dy\right)^{n-k} + o(L^{nd}). \quad (4.41)$$

After summation over all partitions into one- and two-element equivalence classes of 'neighbours' (we note that (4.38) corresponds to the partition into singletons), we obtain exactly the asymptotic relation (4.36). It follows from Lemma 7 and (4.17) that all other partitions into equivalence classes give negligible contributions. The estimate (4.37) can be proved in a similar way.

The conditions of Theorem 9 are quite non-restrictive in the case of translation-invariant kernels. In this case, the covariance function of the limit Gaussian process w-lim $\frac{\eta_L((0,\bar{s}],0)-\mathsf{E}\eta_L((0,\bar{s}];0)}{L^{d/2}}$ is given by d-dimensional analogues of the formulae (37), (38), and (26) in [40] (of course, one has to replace the kernel $\frac{\sin\pi(x-y)}{\pi(x-y)}$ by $K(x-y)$). Here and henceforth, we denote by $(0,\bar{s}]$ the parallelepiped $(0,s_1]\times\cdots\times(0,s_d]$, where $\bar{s}=(s_1,\dots,s_d)$. In particular, if $K(x)$ is continuously differentiable, then the limit Gaussian process is Hölder continuous with any exponent less than $\frac{1}{2}$. Among other characteristics of a modified random point field (with respect to $B=(0,\bar{s}]$ and $n=0$), the spectral measure of the restriction of the group $\{U^t\}$ to the subspace of centralized linear statistics is of special interest. We denote this spectral measure by $\mu^{(s)}(d\lambda)$. Let us recall that the spectral measure $\mu^{(0)}(d\lambda)=\mu(d\lambda)$ of the original determinantal random point field is given by the formula (3.12). In particular, for the sine kernel we obtain

$$\frac{d\mu}{d\lambda}=\begin{cases}\dfrac{|\lambda|}{2\pi}, & |\lambda|\le 2\pi,\\ 1, & |\lambda|>2\pi.\end{cases}$$

For the sine kernel we can see after cumbersome but rather straightforward calculations that

$$\frac{d\mu^{(s)}}{d\lambda}=\frac{\pi^2 s^3}{9}+\frac{|\lambda|}{2\pi}\cdot\left(1-\frac{4}{3}\pi^2 s^3\right)+O(s^4)+O(|\lambda|\cdot s^4)+O(|\lambda^2|\cdot s^2). \tag{4.42}$$

We note that $\frac{d\mu^{(s)}}{d\lambda}(0)\neq 0$ for small $s\neq 0$, which is consistent with the estimate

$$\mathsf{D}\eta_L((0,s];0)\sim L.$$

For the proof of the functional central limit theorem, we refer the reader to [40], p. 577. Suppose that the functions

$$L^{-d}\frac{\partial}{\partial s}\eta_L\big((0,\bar{s}];0\big),\quad L^{-d}\frac{\partial}{\partial s}\operatorname{Cov}\big(\eta_L\big((0,\bar{s}];0\big);\eta_L\big((0,\bar{t}];0\big)\big) \tag{4.43}$$

are uniformly bounded with respect to L, $\bar{s}$, and $\bar{t}$, where $\bar{s}$ and $\bar{t}$ belong to compact subsets of $\mathbb{R}^d_+$ (of $\mathbb{Z}^d_+$). By smoothing the δ-function by a C^∞ approximating function, one can construct a continuous approximation $\tilde{\eta}_L((0,\bar{s}];0)$ of $\eta_L((0,\bar{s}];0)$ such that

$$|\tilde{\eta}_L((0,\bar{s}];0)-\eta_L((0,\bar{s}];0)|\le 1.$$

As a result,

$$\frac{\tilde{\eta}_L((0,\bar{s}];0)-\mathsf{E}\tilde{\eta}_L((0,\bar{s}];0)}{L^{d/2}}$$

is a random continuous function of $\bar{s}$, and

$$\left| \frac{\tilde{\eta}_L((0,\bar{s}];0) - \mathsf{E}\tilde{\eta}_L((0,\bar{s}];0)}{L^{d/2}} - \frac{\eta_L((0,\bar{s}];0) - \mathsf{E}\eta_L((0,\bar{s}];0)}{L^{d/2}} \right| \le \frac{2}{L^{d/2}}. \qquad (4.44)$$

The distribution of the random process $\frac{\tilde{\eta}_L((0,\bar{s}];0)-\mathsf{E}\tilde{\eta}_L((0,\bar{s}];0)}{L^{d/2}}$ defines a probability measure on $C([0,\infty)^d)$. By convergence in law of random processes we mean weak convergence of the induced probability measures on $C([0,\infty)^d)$ (see [92]; in the general case, one can consider other spaces of sample paths, for instance, the space of functions with jump discontinuities).

Theorem 10. *Let* (4.33), (4.34), (4.35), *and* (4.43) *be satisfied. Then the random process*

$$\frac{\tilde{\eta}_L((0,\bar{s}];0) - \mathsf{E}\tilde{\eta}_L((0,\bar{s}];0)}{L^{1/2}}$$

is convergent in law to a Gaussian process.

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Received 11/APR/00

Typeset by AMS-TEX

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