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DETERMINATION OF A CONTROLLABLE SET FOR A CONTROLLED DYNAMIC SYSTEM

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Abstract

The controllable set of a controlled ordinary differential dynamic system to a given set is defined. Under certain reasonable conditions, the controllable set is characterised by a level set of the unique viscosity solution to some Hamilton-Jacobi-Bellman equation. The result is used to determine the asymptotic stable set of nonlinear autonomous differential equations.

1. Introduction

In this paper, we consider the following controlled system

$$\dot{y} = g(y(t), u(t)), \qquad t \in (0, \infty), \tag{1.1}$$

where y is the state valued in $\mathbb{R}^n$, and $u(\cdot)$ is the control valued in some metric space U. Let $Q \subset \mathbb{R}^n$ be a closed set. We define $C(Q)$ to be the set in $\mathbb{R}^n$ with the property that $x \in C(Q)$ if and only if one can find a control $u(\cdot)$ so that the trajectory $y(\cdot)$ of (1.1) corresponding to the initial state x and control $u(\cdot)$ satisfies the following

$$y(r) \in Q \quad , \text{ for some } r \geq 0. \tag{1.2}$$

We refer to $C(Q)$ as the controllable set of system (1.1) (to the given set Q).

It is clear that in general

$$C(Q) \neq \mathbb{R}^n \tag{1.3}$$

The purpose of this paper is to present a method to determine the set $C(Q)$ under some reasonable conditions. The idea is the following: We introduce a

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time optimal control problem. The value function associated with it is only defined on $C(Q)\backslash Q$. Then, we extend the value function to $\mathbb{R}^n\backslash Q$ and show that it is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation (HJB for short). Thus the value function can be found by solving certain partial differential equations. The set $C(Q)$ is characterised by a level set of the value function. Hence we have determined the set $C(Q)$ quantitatively. By using this result, we also determine the asymptotic stable sets of given nonlinear autonomous systems. This problem seems very important in many engineering areas and we expect that our result will have a powerful applicability in these areas. Meanwhile, we see that some efficient algorithms for the HJB equations are needed to implement our result. Some relevant results concerning this aspect can be found in [2,5,6,10,15].

After the original manuscript of this paper had been submitted, the works of Bardi [1] and Evans and James [9] were drawn to our attention. Both of these works deal with the minimum time function of the time optimal control problem with the terminal set being $\{0\}$, from a partial differential point of view. These works are considered to be very closely related to this paper.

2. Preliminaries

In this section, we list some basic assumptions and introduce some necessary notions. Certain preliminary results will also be given.

(H1) There exists a constant $L \geq 0$, such that

$$|g(y_1, u) - g(y_2, u)| \leq L|y_1 - y_2|, \qquad \forall y_1, y_2 \in \mathbb{R}^n, \ u \in U, \tag{2.1}$$

$$|g(y, u)| \leq L(1 + |y|), \qquad \forall (y, u) \in \mathbb{R}^n \times U. \tag{2.2}$$

(H2) The set $Q \subset \mathbb{R}^n$ is closed.

Next, we let

$$\mathcal{U} \equiv \{u(\cdot): [0, \infty) \to U | u(\cdot) \text{ is measurable}\}$$

Sometimes we use $\mathcal{U}[a, b]$, $\mathcal{U}(a, b]$, $\mathcal{U}[a, b)$, etc. to indicate the domain of the controls being $[a, b]$, $(a, b]$, $[a, b)$, etc. It is clear that under (H1), for any $u(\cdot) \in \mathcal{U}$, there exists a unique solution $y(\cdot)$ of the following problem:

$$\begin{cases} \dot{y}(t) = g(y(t), u(t)), & t \in [0, \infty), \\ y(0) = x. \end{cases} \tag{2.3}$$

We denote such a solution by $y(\cdot\,; x, u(\cdot))$. The following result is obvious.

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Proposition 2.1. *Let* (H1) *hold. Then, for any* $x_1, x_2 \in \mathbb{R}^n$, $u(\cdot) \in U$,

$$|y(t; x_1, u(\cdot)) - y(t; x_2, u(\cdot))| \le e^{Lt}|x_1 - x_2|, \qquad \forall t \ge 0, \tag{2.4}$$
$$|y(t; x, u(\cdot))| \le e^{Lt}(1 + |x|), \qquad \forall t \ge 0, x \in \mathbb{R}^n. \tag{2.5}$$

Now, we define the following: for any $x \in \mathbb{R}^n$,

$$\mathscr{A}_x \equiv \{(r, u(\cdot)) \in [0, \infty) \times U | y(r; x, u(\cdot)) \in Q\}, \tag{2.6}$$
$$U_x \equiv \{u(\cdot) \in U| \text{ there exists } r \in [0, \infty), \text{ such that } (r, u(\cdot)) \in \mathscr{A}_x\}, \tag{2.7}$$
$$C(Q) \equiv \{x \in \mathbb{R}^n | \mathscr{A}_x \neq \varnothing\}. \tag{2.8}$$

For any $x \in C(Q)$ and $u(\cdot) \in U_x$, we set (supposing (H2) is true)

$$\begin{aligned} T(x; u(\cdot)) &= \min\{r \in [0, \infty) | (r, u(\cdot)) \in \mathscr{A}_x\}, \\ &= \min\{r \in [0, \infty) | y(r; x, u(\cdot)) \in Q\}. \end{aligned} \tag{2.9}$$

The existence of the minima in (2.9) is clear since Q is closed. Trivially, we have that

$$Q \subseteq C(Q), \tag{2.10}$$

and

$$T(x, u(\cdot)) = 0, \qquad \forall x \in Q, \qquad u(\cdot) \in U. \tag{2.11}$$

The following notion will be important.

Definition 2.2. System (1.1) is said to be locally controllable to set Q if for any $x \in Q$, there exists a $\delta = \delta(x) > 0$, such that whenever $|\hat{x} - x| < \delta$, one can find a control $u(\cdot) \in U$, with the property that

$$y(r; \hat{x}, u(\cdot)) \in Q, \qquad \text{for some } r > 0. \tag{2.12}$$

If the above δ is uniform in $x \in Q$, then we say that (1.1) is uniformly controllable to Q.

From the definition of $C(Q)$, we have the following

Proposition 2.3. *System* (1.1) *is locally controllable to* Q iff

$$Q \subset \operatorname{Int} C(Q), \tag{2.13}$$

and (1.1) *is uniformly controllable to* Q iff *there exists a* $\delta > 0$, *such that*

$$\mathscr{N}_\delta(Q) \equiv \{x \in \mathbb{R}^n | d(x, Q) < \delta\} \subset C(Q). \tag{2.14}$$

It is clear that if Q is compact, then local controllability to Q and uniform controllability to Q are equivalent.

We note that if (1.1) is uniformly (or locally) controllable to Q, then for x near Q, we have $x \in C(Q)$. However, it might take a very long time to steer x to Q, and the corresponding trajectory may also wind far away from Q. Sometimes such a situation is undesirable. The following notion excludes such a situation.

DEFINITION 2.4. System (1.1) is said to be locally s-controllable to Q, if for any $x \in Q$ and any $\varepsilon, \sigma > 0$, there exists a $\delta = \delta(x) > 0$, such that, whenever $|\hat{x} - x| < \delta$, one can find $u(\cdot) \in U$ with the following properties

$$\begin{cases} y(r; \hat{x}, u(\cdot)) \in Q & \text{for some } r \in [0, \varepsilon], \\ d(y(t; \hat{x}, (\cdot)), Q) \leq \sigma, & \forall t \in [0, r]. \end{cases} \tag{2.15}$$

If in the above δ is independent of $x \in Q$, then we say that (1.1) is uniformly s-controllable to Q.

Again, we see that if Q is compact, local s-controllability and uniform s-controllability are the same. For the case $Q = \{0\}$, the notion of local (or uniform) s-controllability to Q is also referred to as small time local controllability (STLC). It is known that for time optimal control problems with the system linear in the control variable, STLC is equivalent to continuity of the value function. We refer readers to [16] (also see [1,9]) for relevant details.

The following result gives a sufficient condition for (1.1) to be locally (or uniformly) s-controllable to Q.

THEOREM 2.5. *Let* (H1) *hold and let* Q *be a nonempty convex and closed subset of* $\mathbb{R}^n$. *Let* ∂Q *be the boundary of* Q *and for any* $x \in \partial Q$, *let*

$$N(x) = \{\eta \in \mathbb{R}^n \mid |\eta| = 1, (y - x, \eta) \leq 0, \ \forall y \in Q\}$$

(i) *Suppose for any* $x \in \partial Q$,

$$\sup_{\nu \in N(x)} \left(\inf_{u \in U} (\eta, g(x, u)) \right) < 0. \tag{2.16}$$

Then, (1.1) *is locally s-controllable to* Q.

(ii) *Suppose there exists a* $\delta_0 > 0$ *such that*

$$\sup_{\eta \in N(x), x \in \partial Q} \left(\inf_{u \in U} (\eta, g(x, u)) \right) \leq -\delta_0. \tag{2.17}$$

Then, the system is uniformly s-controllable to Q *with*

$$\mathcal{N}_{\delta_0/L}(Q) \equiv \{x \in \mathbb{R}^n \mid d(x, Q) < \delta_0/L\} \subset C(Q), \tag{2.18}$$

and for any $x \in \mathscr{N}_{\delta_0/L}(Q)$,

$$\inf_{u(\cdot)\in\mathscr{U}_x} T(x, u(\cdot)) \le \frac{Ld(x, Q)}{\delta_0 - Ld(x, Q)}, \tag{2.19}$$

where L *is given in* (H1).

The proof is contained in the relevant results of [18], in which some more general situations are discussed. Some relevant results can also be found in [11]. We note that (2.18) actually gives a "lower bound" for the controllable set.

THEOREM 2.6. *Let* (H1) *hold and let* (1.1) *be locally controllable to* Q. *Then,* $C(Q)$ *is open.*

PROOF. Let $x \in C(Q)$. Then, there exists a pair $(r, u(\cdot)) \in \mathscr{A}_x$. From Proposition 2.1, we know that for any $\varepsilon > 0$, there exists a $\delta > 0$, such that, provided $|\hat{x} - x| < \delta$, one has

$$|y(r; \hat{x}, u(\cdot)) - y(r; x, u(\cdot))| < \varepsilon, \tag{2.20}$$

with $y(r; x, u(\cdot)) \in Q$. Thus, by Definition 2.2, we have $\hat{x} \in C(Q)$, provided δ is small enough. Thus, $C(Q)$ is open.

3. Time optimal control problem

In this section we introduce a time optimal control problem to which the Hamilton-Jacobi-Bellman equation theory will be applied.

The usual time optimal control problem is of the following form ([4]):

PROBLEM T. For given $x \in C(Q)$, minimise $T(x, u(\cdot))$ over all possible $u(\cdot) \in \mathscr{U}_x$.

Instead of studying Problem T, we would like to introduce the following problem: (Let $\lambda > 0$ be fixed).

PROBLEM T$'$. For given $x \in C(Q)$, minimise

$$J_x(r, u(\cdot)) \equiv \int_0^r e^{-\lambda\tau} d\tau = (1 - e^{-\lambda r})/\lambda, \tag{3.1}$$

over $(r, u(\cdot)) \in \mathscr{A}_x$.

We see that minimising (3.1) over $(r, u(\cdot)) \in \mathscr{A}_x$ is equivalent to minimising the functional

$$\widehat{J}_x(u(\cdot)) \equiv (1 - e^{-\lambda T(x, u(\cdot))})/\lambda, \tag{3.2}$$

over $u(\cdot) \in \mathscr{U}_x$. Then, since the function $r \to (1 - e^{-\lambda r})/\lambda$ is monotonically increasing, we see that Problem T and Problem T′are equivalent. Now, we define

$$V(x) = \inf_{(r, u(\cdot)) \in \mathscr{A}_z} J_x(r, u(\cdot)), \ \forall x \in C(Q). \tag{3.3}$$

We call $V(\cdot)$ the value function of Problem T′. We note that the value function $V(\cdot)$ is only defined on the set $C(Q)$.

The main result of this section is the following:

THEOREM 3.1. *Let* (H1) *and* (H2) *hold. Let* (1.1) *be uniformly s-controllable to* Q. *Then, the value function* $V(\cdot)$ *is continuous in the set* $C(Q)$ *and*

$$V(x) = 0, \qquad \forall x \in Q, \tag{3.4}$$

$$0 < V(x) < \frac{1}{\lambda}, \qquad \forall x \in C(Q)\backslash Q, \tag{3.5}$$

$$\lim_{x \to x', x \in C(Q)} V(x) = \frac{1}{\lambda}, \qquad \textit{uniformly in } x' \in \partial C(Q). \tag{3.6}$$

PROOF. First of all, for any $x \in C(Q)$, there exists a pair $(r, u(\cdot)) \in \mathscr{A}_x$. Thus,

$$0 \le V(x) \le J_x(r, u(\cdot)) = \frac{1 - e^{\lambda r}}{\lambda} < \frac{1}{\lambda}.$$

Hence, (3.4) and (3.5) hold. Now, let us prove the upper semi-continuity of the value function $V(\cdot)$. To this end, let $x \in C(Q)$ and $\varepsilon_0 > 0$, such that

$$V(x) + \varepsilon_0 < \frac{1}{\lambda}.$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$, let $(r_\varepsilon, u_\varepsilon(\cdot)) \in \mathscr{A}_x$, such that

$$V(x) \le J_x(r_\varepsilon, u_\varepsilon(\cdot)) \le V(x) + \varepsilon < \frac{1}{\lambda} \tag{3.7}$$

Thus, we see that there exists a constant $C = C(x, \varepsilon_0)$, such that

$$0 \le r_\varepsilon \le C, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{3.8}$$

By Proposition 2.1 and Definition 2.4, we see that there exists a $\delta = \delta(\varepsilon) > 0$, such that for any $\hat{x} \in C(Q)$ with

$$|\hat{x} - x| < \delta,$$

there exists a pair $(\tilde{r}_\varepsilon, w_\varepsilon(\cdot)) \in \mathscr{A}_{y(r_\varepsilon; \hat{x}, u_\varepsilon(\cdot))}$, such that

$$0 \le \tilde{r}_\varepsilon \le \varepsilon. \tag{3.9}$$

Then, we let

$$\hat{u}_\varepsilon(\cdot) = u_\varepsilon(\cdot)\chi_{[0, r_\varepsilon]}(\cdot) + w_\varepsilon(\cdot - r_\varepsilon)\chi_{(r_\varepsilon, \infty)}(\cdot).$$

It is clear that

$$(r_\varepsilon + \tilde{r}_\varepsilon, \hat{u}_\varepsilon(\cdot)) \in \mathscr{A}_{\hat{x}}.$$

Thus, one has

$$\begin{aligned} V(\hat{x}) &\le J_{\hat{x}}(r_\varepsilon + \tilde{r}_\varepsilon, \hat{u}_\varepsilon(\cdot)) \\ &= J_x(r_\varepsilon, u_\varepsilon(\cdot)) + e^{-\lambda r_\varepsilon}(1 - e^{-\lambda \tilde{r}_\varepsilon})/\lambda \\ &\le V(x) + \varepsilon + (1 - e^{-\lambda_\varepsilon})/\lambda, \quad \forall |\hat{x} - x| < \delta. \end{aligned} \tag{3.10}$$

Hence, we see that $V(\cdot)$ is upper semi-continuous in $C(Q)$. Now we prove the continuity of $V(\cdot)$. Again, let $x \in C(Q)$. Then, by the upper semi-continuity, we know that there exists a $\delta = \delta(x) > 0$, such that

$$V(\hat{x}) \le V(x) + \frac{\varepsilon_0}{2} < \frac{1}{\lambda} - \frac{\varepsilon_0}{2}, \qquad \forall |x - \hat{x}| < \delta. \tag{3.11}$$

Thus, for any $|\hat{x} - x| < \delta$ and any $\varepsilon \in (0, \varepsilon_0/2]$, if $(\hat{r}_\varepsilon, \hat{u}_\varepsilon(\cdot)) \in \mathscr{A}_{\hat{x}}$ has the property

$$V(\hat{x}) \le J_{\hat{x}}(\hat{r}_\varepsilon, \hat{u}_\varepsilon(\cdot)) \le V(\hat{x}) + \varepsilon, \tag{3.12}$$

then,

$$0 \le \hat{r}_\varepsilon \le C, \qquad \forall \varepsilon \in (0, \varepsilon_0/2], \qquad |\hat{x} - x| < \delta, \tag{3.13}$$

where $C = C(x, \varepsilon_0)$ is independent of $\varepsilon \in (0, \varepsilon_0/2]$ and $\hat{x}$ (with $|\hat{x} - x| < \delta$). Again by Proposition 3.1 and Definition 2.4, we know that if we shrink δ suitably, then, for $|\hat{x} - x| < \delta$, we have (uniformly in $\hat{x}$ and ε)

$$|y(\hat{r}_\varepsilon; x, \hat{u}_\varepsilon(\cdot)) - y(\hat{r}_\varepsilon; \hat{x}, \hat{u}_\varepsilon))| < \varepsilon,$$

and there exists a pair $(\bar{r}_\varepsilon, \bar{u}_\varepsilon(\cdot)) \in \mathscr{A}_{y(\hat{r}_\varepsilon; x, \hat{u}_\varepsilon(\cdot))}$, such that

$$0 \le \bar{r}_\varepsilon \le \varepsilon. \tag{3.14}$$

Thus, $(x, \hat{r}_\varepsilon + \bar{r}_\varepsilon) \in \mathscr{A}_x$ and

$$\begin{aligned} V(x) &\le \frac{1 - e^{-\lambda(\hat{r}_\varepsilon + \bar{r}_\varepsilon)}}{\lambda} \\ &\le V(\hat{x}) + \varepsilon + \frac{1 - e^{-\lambda \bar{r}_\varepsilon}}{\lambda} e^{-\lambda \hat{r}_\varepsilon} \\ &\le V(\hat{x}) + 2\varepsilon, \end{aligned} \tag{3.15}$$

for all $\hat{x}$ with $|\hat{x} - x| < \delta$. This means that $V(\cdot)$ is lower semi-continuous. Hence $V(\cdot)$ is continuous in $C(Q)$. Finally, we prove (3.6). Suppose it is not the case. Then, there exists a constant $\varepsilon_0 > 0$ and a sequence $\{x_k\} \subset C(Q)$, such that

$$d(x_k, \partial C(Q)) \le \frac{1}{k}, \qquad \forall k \ge 1, \tag{3.16}$$

$$V(x_k) < \frac{1}{\lambda} - \varepsilon_0, \qquad \forall k \ge 1. \tag{3.17}$$

From (3.17), we know that there exist $(t_k, u_k(\cdot)) \in \mathscr{A}_{x_k} (k \geq 1)$ and a constant $C_0 > 0$, such that

$$0 \leq t_k \leq C_0, \qquad \forall k \geq 1. \tag{3.18}$$

Thus, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$, such that whenever

$$|\hat{x} - x_k| < \delta,$$

one must have

$$y(t_k; \hat{x}, u_k(\cdot)) \in \mathscr{N}_\varepsilon(Q).$$

Hence, by uniform s-controllability of the system, we see

$$\hat{x} \in C(Q), \qquad \text{if } |\hat{x} - x_k| < \delta,$$

for some $\delta > 0$, independent of k. Thus $d(x_k, \partial C(Q)) \geq \delta$, $\forall k \geq 1$. This contradicts (3.16), completing the proof.

From the above theorem, we may extend the value function $V(\cdot)$ to $\mathbb{R}^n \backslash C(Q)$ as follows:

$$V(x) = \frac{1}{\lambda}, \qquad \forall x \in \mathbb{R}^n \backslash C(Q). \tag{3.19}$$

Then, we have the following:

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *hold. If the value function is extended as* (3.19), *then* $V(\cdot)$ *is continuous in* $\mathbb{R}^n$, *and the set* $C(Q)$ *is characterised by*

$$C(Q) = \{x \in \mathbb{R}^n | 0 \leq V(x) < \frac{1}{\lambda}\}. \tag{3.20}$$

Our next goal is to determine the value function $V(\cdot)$ without solving Problem T'. If we can do so, then (3.20) determines the controllable set $C(Q)$ for our system (1.1) and the given set Q.

4. HJB equation viscosity solution

In this section, we will characterise the value function $V(\cdot)$ using the viscosity solution of a Hamilton-Jacobi-Bellman equation (HJB for short).

We should note that in our situation, the terminal state is constrained by

$$y(r; x, u(\cdot)) \in Q. \tag{4.1}$$

We have seen that this gives the admissible control set $\mathscr{U}_x$ depending on the initial state x, and in general, different x have different $\mathscr{U}_x$. We shall see that this leads to a dynamical programming principle a little different from the classical one ([3,13]).

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THEOREM 4.1. (*Dynamical Programming Principle*) *Let* (*H*1)-(*H*2) *hold. Let* (1.1) *be locally controllable to* Q. *Then, for any* $x \in C(Q)\backslash Q$, *there exists an* $s > 0$, *such that*

$$V(x) = \inf_{u(\cdot)\in\mathscr{U}} \left\{ \frac{1-e^{-\lambda t}}{\lambda} + e^{-\lambda t} V(y(t\,;x\,,u(\cdot))) \right\}, \qquad \forall t \in [0\,,s]. \tag{4.2}$$

PROOF. First of all, by definition, for given $x \in C(Q)$ we can find an $s_1 > 0$ such that

$$y(t\,;x\,,u(\cdot)) \in C(Q)\,, \qquad \forall u(\cdot) \in \mathscr{U}\,, \qquad t \in [0\,,s]. \tag{4.3}$$

Now, for any $u(\cdot) \in \mathscr{U}$ and $(r\,,w(\cdot)) \in \mathscr{A}_{y(t;x,u(\cdot))}$, we let $t \in [0\,,s_1]$ and

$$\hat{u}(\tau) = u(\tau)\chi_{[0,t]}(\tau) + w(\tau - t)\chi_{(t,\infty)}(\tau)\,, \qquad \tau \in [0\,,\infty).$$

Then, it is easy to check that $(t+r\,,\hat{u}(\cdot)) \in \mathscr{A}_x$. Hence,

$$\begin{aligned} V(x) \le J_x(t+r\,,\hat{u}(\cdot)) &= \int_0^{t+r} e^{-\lambda\tau}\,d\tau \\ &= \frac{1-e^{-\lambda t}}{\lambda} + \int_t^{t+r} e^{-\lambda\tau}\,d\tau \\ &= \frac{1-e^{-\lambda t}}{\lambda} + e^{-\lambda t}\int_0^{r} e^{-\lambda\tau}\,d\tau \\ &= \frac{1-e^{-\lambda t}}{\lambda} + e^{\lambda t} J_{y(t;x,u(\cdot))}(r\,,w(\cdot)). \end{aligned}$$

Since $(r\,,w(\cdot)) \in \mathscr{A}_{y(t;x,u(\cdot))}$ is arbitrary, we get

$$V(x) \le \frac{1-e^{-\lambda t}}{\lambda} + e^{-\lambda t} V(y(t\,;x\,,u(\cdot)))\,, \qquad \forall u(\cdot) \in \mathscr{U}. \tag{4.4}$$

Thus, we have

$$V(x) \le \inf_{u(\cdot)\in\mathscr{U}} \left\{ \frac{1-e^{\lambda t}}{\lambda} + e^{-\lambda t} V(y(t\,;x\,,u(\cdot)))\,, \right\} \equiv W(x\,,t). \tag{4.5}$$

Conversely, for any $\varepsilon > 0$, there exists a pair $(r_\varepsilon\,,u_\varepsilon(\cdot)) \in \mathscr{A}_x$, such that

$$V(x) \ge J_x(r_\varepsilon\,,u_\varepsilon(\cdot)) - \varepsilon. \tag{4.6}$$

Since $x \in C(Q)\backslash Q$, we can find an $s_0 > 0$, such that

$$r_\varepsilon \ge s_0\,, \qquad \forall \varepsilon > 0. \tag{4.7}$$

Then, for any $t \in [0\,,s_0]$, we let

$$\hat{u}_\varepsilon(\tau) = u_\varepsilon(t+\tau)\,, \qquad \tau \in [0\,,\infty).$$

We see that $(r_\varepsilon - t\,,\, \hat{u}_\varepsilon(\cdot)) \in \mathscr{A}_{y(t\,;\,x\,,\,u_\varepsilon(\cdot))}$. Hence, by (4.6), we have

$$\begin{aligned} V(x) + \varepsilon &\geq \frac{1 - e^{-\lambda r_\varepsilon}}{\lambda} \\ &= \frac{1 - e^{-\lambda t}}{\lambda} + e^{-\lambda t} J_{y(t\,;\,x\,,\,u_\varepsilon(\cdot))}(r_\varepsilon - t\,,\, \hat{u}_\varepsilon(\cdot)) \\ &\geq \frac{1 - e^{-\lambda t}}{\lambda} + e^{-\lambda t} V(y(t\,;\,x\,,\,u_\varepsilon(\cdot))) \geq W(x\,,\,t). \end{aligned}$$

Thus, (4.2) follows by taking $s = s_0 \wedge s_1$.

REMARK 4.2. It is very important that in (4.2), the infimum is taken over the whole of $\mathscr{U}$ instead of just over $\mathscr{U}_x$. Due to this, the classical relevant argument concerning the HJB equation is applicable to our case. The difference between Theorem 4.1 and the classical dynamical programming principle is that x is not arbitrary in $\mathbb{R}^n$ and $s > 0$ is small (depending on x).

PROPOSITION 4.3. *Let the assumptions of Theorem* 4.1 *hold. Suppose that the value function* $V(\cdot)$ *is in* $C^1(\mathbb{R}^n)$. *Then,* $V(\cdot)$ *satisfies the following HJB equation*

$$\begin{cases} \lambda V(x) - H(x\,,\, V_x(x)) = 0\,, & x \in \mathbb{R}^n \backslash Q \\ V|_{\partial Q} = 0\,, \end{cases} \tag{4.8}$$

where

$$H(x\,,\,p) \equiv 1 + \inf_{u \in U} (p\,,\, g(x\,,\,u))\,, \qquad \forall (x\,,\,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.9}$$

PROOF. By Theorem 4.1, a classical argument applies ([13]).

We know that the value function $V(\cdot)$ is not $C^1(\mathbb{R}^n)$ in general. Thus, the above proposition is formal. To make it rigorous, let us adopt the notion of the viscosity solution introduced by Crandall and Lions [8] (see also [7]). We consider the following

$$\begin{cases} \lambda V(x) - H(x\,,\, V_x(x)) = 0\,, \ x \in \mathbb{R}^n \backslash Q \\ V|_{\partial Q} = 0. \end{cases} \tag{4.10}$$

DEFINITION 4.4. A function $V(\cdot) \in C_b(\overline{\mathbb{R}^n \backslash Q}) \equiv \{$bounded, continuous functions on $\overline{\mathbb{R}^n \backslash Q}\}$ is called a viscosity solution of (4.10), if $V(x) = 0$, on ∂Q, and for any $\varphi(\cdot) \in C^1(\mathbb{R}^n \backslash Q)$, whenever $V(\cdot) - \varphi(\cdot)$ attains a local maximum (minimum) at $x_0 \in \mathbb{R}^n \backslash Q$, one always has

$$\lambda V(x_0) - H(x_0\,,\, \varphi_x(x_0)) \leq 0\,, \qquad (\geq 0). \tag{4.11}$$

One of the main results of this section is the following:

THEOREM 4.5. *Let* $(H1)$–$(H2)$ *hold. Let* (1.1) *be uniformly s-controllable to* Q. *Let the value function* $V(\cdot)$ *be extended as* (3.19). *Then,* $V(\cdot)$ *is the unique viscosity solution of* (4.10)).

PROOF. By (3.19) and (4.2), we see that $V(\cdot)$ satisfies (4.10) in the viscosity sense in $\mathbb{R}^n\backslash\overline{C(Q)}$ and $C(Q)\backslash Q$. Also the boundary condition is satisfied and uniqueness follows [8] (see also [7,13,14]). Thus, it remains to prove that $V(\cdot)$ satisfies (4.10) in the viscosity sense on $\partial C(Q)$. To this end, let $\varphi(\cdot)\in C^1(\mathbb{R}^n\backslash Q)$ and let $V(\cdot)-\varphi(\cdot)$ attain a local maximum at $x^0\in\partial C(Q)$. By the definitions of $C(Q)$ and uniform s-controllability, we know that for any $u\in U$,

$$y(t;x_0,u)\notin C(Q),\qquad \forall t\geq 0. \tag{4.12}$$

Thus, by the assumption

$$V(x_0)-\varphi(x_0)\geq V(x)-\varphi(x),\qquad \text{for } x \text{ near } x_0,$$

we have (note(3.19))

$$\varphi(x_0)\leq\varphi(y(t;x_0,u)),\qquad \forall t \text{ small.}$$

Thus,

$$0\leq(\varphi_x(x_0),g(x_0,u)),\qquad \forall u\in U.$$

Hence,

$$\begin{aligned}\lambda V(x_0)-H(x_0,\varphi_x(x_0)) &= 1-1-\inf_{u\in U}(\varphi_x(x_0),g(x_0,u))\\ &= -\inf_{u\in U}(\varphi_x(x_0),g(x_0,u))\leq 0.\end{aligned}$$

Similarly, if $V(\cdot)-\varphi(\cdot)$ attains a local minimum at $x_0\in\partial C(Q)$, then

$$\varphi(x_0)\geq\varphi(y(t;x_0,u)),\qquad \forall t \text{ small.}$$

Thus,

$$0\geq(\varphi_x(x_0),g(x_0,u)),\qquad \forall u\in U.$$

Hence,

$$\lambda;V(x_0)-H(x_0,\varphi_x(x_0))=-\inf_{u\in U}(\varphi_x(x_0),g(x_0,u))\geq 0.$$

We have seen that in the above proof, (4.12) is crucial.

From the above, we know that one can solve (4.10) in the viscosity sense. Namely, solve some elliptic boundary value problem and then take the limit to get $V(\cdot)$ (see [2,8,13,14]). Then, by (3.20), we obtain $C(Q)$.

5. An application

In this section, we use the result obtained in the previous section to determine the asymptotic region of a nonlinear autonomous equation. More precisely, we consider the following

$$\dot{y}(t) = f(y(t)), \qquad t \geq 0. \tag{5.1}$$

We assume the following:

$$f(0) = 0, \qquad |f(x) - f(\hat{x})| \leq K|x - \hat{x}|, \qquad \forall x, \hat{x} \in \mathbb{R}^n, \tag{5.2}$$

or

$$\begin{aligned} f(0) = 0, \qquad |f(x) - f(\hat{x})| \leq K_R|x - \hat{x}|, \qquad \forall x, \hat{x} \in \mathbb{R}^n, |x|, |\hat{x}| \leq R, \\ \langle f(x), x\rangle \leq K(1 + |x|^2), \qquad \forall x \in \mathbb{R}^n, \end{aligned} \tag{5.2$'$}$$

and $y \equiv 0$ is an asymptotic stable solution of (5.1) in the sense that there exists a $\delta > 0$, such that for any

$$x \in B_\delta(0) \equiv \{x \in \mathbb{R}^n | \, \|x\| < \delta\},$$

the unique solution of (5.1) on $[0, \infty)$ with

$$y(0) = x, \tag{5.3}$$

satisfies

$$\lim_{t\to\infty} y(t) = 0. \tag{5.4}$$

In many real problems, one can find a $\delta > 0$ which satisfies the above (if it exists).

Definition 5.1. We call an $x \in \mathbb{R}^n$ an asymptotic stable initial state if solution $y(\cdot)$ of (5.1) with (5.3) satisfies (5.4). We let

$$S \equiv \{x \in \mathbb{R}^n | x \text{ is an asymptotic stable initial state }\}.$$

The set S is called the asymptotic stable set of the system. It is clear that $y(\cdot) \equiv 0$ is an asymptotic solution of (5.1) if

$$B_\delta(0) \subset S, \qquad \text{for some } \delta > 0. \tag{5.5}$$

Some relevant notions and results can be found in [18].

Now, our question is whether one can determine the asymptotic stable set S. The rest of this section gives a solution to this problem using our obtained results.

First of all, as we remarked, we can assume that $\delta > 0$ is known and that it satisfies (5.5). Next, we let $h(\cdot) \in C_0^\infty(\mathbb{R}^n)$ with the following properties:

$$\sup h \subset B_\delta(0), \qquad 0 \leq h \leq 1, \tag{5.6}$$

$$h(x) = 1, \qquad x \in B_{\delta/2}(0). \tag{5.7}$$

Then, we consider the following controlled system:

$$\dot{y}(t) = f(y(t)) + h(y(t))u(t), \qquad t \geq 0, \tag{5.8}$$

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with control $u(\cdot)$ valued in $B_1(0) \subset \mathbb{R}^n$. We see that if we take $Q = \{0\}$, then $(H1)$–$(H2)$ hold. On the other hand, it is easy to see that

$$C(\{0\}) = S. \tag{5.9}$$

Thus we obtain the following

THEOREM 5.2. *The asymptotic stable set S is given by*

$$S = \{x \in \mathbb{R}^n | V(x) < 1\}, \tag{5.10}$$

where $V(\cdot)$ is the unique viscosity solution of the following problem:

$$\begin{cases} V(x) - 1 - \langle f(x), V_x(x)\rangle + h(x)|V_x(x)| = 0, & x \in \mathbb{R}^n \backslash \{0\}, \\ V(0) = 0. \end{cases} \tag{5.11}$$

6. Illustrative examples

In this section, we present two examples. For simplicity, we only consider one-dimensional cases.

EXAMPLE 6.1. The control system we are interested in is the following:

$$\dot{x} = g_0(x) + g_1(x)u + g_2(x)u^2, \tag{6.1}$$

where g_0, g_1, $g_2 \colon \mathbb{R} \to \mathbb{R}$ are uniformly Lipschitz continuous. We take $U = [-1, 1]$ and $Q = \{0\}$. Thus, we see that (H1) and (H2) hold. We also assume

$$\begin{cases} \inf_{|u|\le 1}(g_0(0) + g_1(0)u + g_2(0)u^2) < 0, \\ \sup_{|u|\le 1}(g_0(0) + g_1(0)u + g_2(0)u^2) > 0. \end{cases} \tag{6.2}$$

By Theorem 2.5 and the compactness of Q, we know that (5.1) is uniformly s-controllable to Q. Thus, all the assumptions of our theory hold. Hence, to determine the controllable set, we only need to calculate the (viscosity) solution $V(\cdot)$ of the corresponding HJB equation. To this end, we first calculate the Hamiltonian. We let

$$\begin{cases} h_1(x) = \inf_{|u|\le 1}\{g_0(x) + g_1(x)u + g_2(x)u^2\}, \\ h_2(x) = \sup_{|u|\le 1}\{g_0(x) + g_1(x)u + g_2(x)u^2\}. \end{cases} \tag{6.3}$$

Then, we see that

$$H(x, p) = 1 + h_1(x)p^+ - h_2(x)p^-, \qquad \forall (x, p) \in \mathbb{R}^2, \tag{6.4}$$

where $p^+ = \max\{p, 0\}$ and $p^- = \max\{-p, 0\}$. Taking $\lambda = 1$, we obtain the corresponding Hamilton-Jacobi-Bellman equation of the following type:

$$\begin{cases} V(x) - 1 - h_1(x)V_x(x)^+ + h_2(x)V_x(x)^- = 0, & x \in \mathbb{R}\backslash\{0\} \\ V(0) = 0. \end{cases} \tag{6.5}$$

We know that there exists a unique viscosity solution $V(\cdot)$ of (6.5) and the controllable set is given by

$$\{x \in (-\infty, \infty) | V(x) < 1\}.$$

On the other hand, if we set

$$h(x) = h_1(x)\chi_{(0,\infty)}(x) + h_2(x)\chi_{(-\infty,0)}(x), \qquad x \in \mathbb{R}, \tag{6.6}$$

then it is clear that there exists a maximal interval $(a, b) \subseteq \mathbb{R}$, such that

$$\int_0^x \frac{ds}{h(s)} \text{ is convergent }, \qquad \forall x \in (a, b). \tag{6.7}$$

It is easy to see that the (viscosity) solution of (6.5) is given by

$$V(x) = \begin{cases} 1 - e^{-\int_0^x ds/h(x)}, & x \in (a, b), \\ 1, & x \notin (a, b), \end{cases} \tag{6.8}$$

Hence, the controllable set of our system is given by

$$\left\{x \in \mathbb{R} \,\middle|\, \int_0^x ds/h(s) \text{ is convergent } \right\}.$$

Example 6.2. Consider the following system:

$$\dot{y} = -\sin y \equiv f(y), \tag{6.9}$$

with $Q = \{0\}$. We would like to determine the asymptotic stable set of the system. To this end, we let the control domain be $[-1, 1]$ and let $\delta > 0$ be small enough so that (5.5) holds with S being the asymptotic stable set of the system. Then, we take a nonnegative continuous map h satisfying (5.6)–(5.7) and $h(-x) = h(x)$. Now, we consider the modified system

$$\begin{aligned} &\dot{y}(t) = f(y(t)) + h(y(t))u(t), \qquad t \geq 0, \\ &y(0) = x. \end{aligned} \tag{6.10}$$

We let $V(x)$ be the corresponding value function. Then it is the unique viscosity solution of (5.11) with f and h being given as above. On the other hand, we see that

$$\begin{cases} V'(x) \geq 0, & x > 0, \\ V'(x) \leq 0, & x < 0. \end{cases} \tag{6.11}$$

Thus (5.11) can be written as

$$\begin{cases} V(x) - 1 - V'(x)[f(x) - h(x)] = 0, & x > 0, \\ V(0) = 0. \end{cases} \tag{6.12}$$

and

$$\begin{cases} V(x) - 1 - V'(x)[f(x) + h(x)] = 0, & x < 0, \\ V(0) = 0. \end{cases} \tag{6.13}$$

Let us look at (6.12). By direct computation, we see that

$$V(x) = 1 - e^{-\int_0^x [h(s)-f(s)]^{-1} ds}, \qquad x \geq 0. \tag{6.14}$$

It is clear that (note δ is small enough)

$$\begin{aligned} &[h(s) - f(s)]^{-1} > 0, \ \forall s \in [0, \pi), \\ &\lim_{x \uparrow \pi} \int_0^x [h(s) - f(s)]^{-1} ds = +\infty. \end{aligned} \tag{6.15}$$

Thus,

$$\lim_{x \uparrow \pi} V(x) = 1. \tag{6.16}$$

Similarly, we can show that

$$V(x) = 1 - e^{-\int_0^x [h(s)+f(s)]^{-1} ds}, \qquad x \leq 0, \tag{6.17}$$

and

$$\lim_{x \downarrow -\pi} V(x) = 1. \tag{6.18}$$

Now, we set

$$V(x) = 1, \qquad \forall x \in (-\infty, -\pi] \cup [\pi, +\infty). \tag{6.19}$$

We can check that $V(\cdot)$ is the unique viscosity solution of (5.11) with f and h being given as above. Thus, the asymptotic stable set S of our system is given by

$$S = \{x \in \mathbb{R} \mid V(x) < 1\} = (-\pi, \pi). \tag{6.20}$$

It is possible to cook up some higher dimensional examples in the same manner, but they would involve much more computation. For the purpose of illustrating our ideas, we prefer not to go into complex details. As we said in the introduction, some efficient numerical algorithms for solving HJB equations are definitely necessary for implementing our theory. We refer the readers to [2,5,6,10,15] for these. Some further investigations will also be carried out in our future publications.

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