Determination of Lamb mode eigenvalues

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An original method is presented to determine the complex Lamb wave spectrum by using a numerical spectral method applied to the elasticity equations. This method presents the advantage to directly determine complex wave numbers for a given frequency via a classical matricial eigenvalue problem, and allows the wave numbers to be determined at relatively high frequencies (i.e., corresponding to many propagating modes). It does not need initial guess values for the wave numbers, contrary to the usual method of root finding of the Rayleigh–Lamb frequency equations (dispersion relation) in the complex plane. Results are presented and the method is discussed. [DOI: 10.1121/1.1391248]

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I. INTRODUCTION

Lamb waves are involved in the nondestructive testing of plate structures because of their guided nature. The characterization of flaws through the scattering of elastic waves in such plates has received considerable attention in the past 20 years,[1] but Lamb wave propagation in an inhomogeneous medium has not been widely investigated.

A possible method for solving the problem of Lamb wave propagation in a medium with geometrical or material discontinuities is based on an eigenfunction expansion of the displacement and the stress, where the considered eigenfunctions are the Rayleigh–Lamb modes in an infinite plate. This expansion is combined with a mode-matching technique to treat the discontinuities. This method solves problems related to a semi-infinite plate[2,3] or to two dissimilar semi-infinite plates welded along their lateral boundaries.[4]

This method requires determining the complex wave number spectrum k_n associated with the eigenfunctions used in the expansion. In the context of Lamb waves, it is difficult to determine eigenfunctions and associated wave numbers, because of the spectrum complexity. It is well established that the Lamb wave spectrum in a free solid layer, composed of elastic material, consists of complex wave numbers k_n, real wave numbers corresponding to propagating Lamb waves, and complex wave numbers related to evanescent Lamb waves.[5–9] Usually, the dispersion relation $D(\omega,k)=0$ (the so-called Rayleigh–Lamb frequency equations) is numerically solved to determine the wave spectrum.

For propagating modes, the problem remains simple since the wave number is known to be real. On the other hand, when the mode-matching technique is used, evanescent modes have to be taken into account and a significant part of the entire complex spectrum has to be determined. The direct strategy of finding the roots of the dispersion relation $D(k,\omega)=0$, at a given frequency ω, is not well suited for systematic computation since the locations of the wave numbers are not known, *a priori*, in the complex plane. Consequently, initial guess values are not available for classical root finding routines. One technique to overcome this lack[4,2] consists in calculating the spectrum at zero frequency, where the dispersion relation is more tractable, and then to gradually increase the frequency to the desired value by using $dk=-(\partial_\omega D/\partial_k D)d\omega$ and/or $k(\omega)$ as an initial guess for $k(\omega+d\omega)$. Apart from the difficulties arising from the vanishing of the denominator, this ''step by step'' technique is time consuming because a series of spectra has to be computed for frequencies from zero to the desired frequency.

Here, we present an original method that directly projects the ordinary differential equation governing the Lamb modes on a spectral basis of orthogonal functions. Instead of solving the transcendental equation $D(k,\omega)=0$, one calculates the solutions of a classical eigenvalue problem in the form $(M-kI)\mathbf{X}=0$, where M is a matrix resulting from the projection of the differential equation. We obtain approximate eigenvalues that can be used as starting values for a more precise solution. This technique is classically used in the theory of fluid dynamics instability.[10,11] In this case, the coefficients of the differential equation governing the transverse modes are nonconstant; the method of finding the roots of the dispersion relation is then not natural since it is impossible to get an *analytical* dispersion relation.

The Lamb problem is posed in Sec. II. Then, spectral decomposition is performed (Sec. III), and a second-order polynomial system on k is derived (Sec. IV A). This system can be rewritten as a reduced eigenvalue problem for k_n^2 (Sec. IV B). Results are presented and discussed in Sec. V.

II. LAMB MODE PROBLEM

The Lamb mode problem (see Fig. 1) consists of searching for a solution of the elasticity equation in the waveguide defined by $-h\leqslant y\leqslant h$ with free boundaries, and for which

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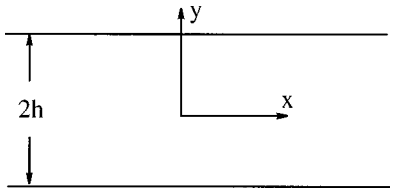

FIG. 1. Geometry of the Lamb wave problem.

displacements are in the (x, y) plane. The time dependence is $e^{-i\omega t}$ and will be omitted in the sequel. The equation of motion is

$$-\rho\omega^2\mathbf{w}=\mu\,\Delta\mathbf{w}+(\lambda+\mu)\boldsymbol{\nabla}(\mathrm{div}\,\mathbf{w}), \tag{2.1}$$

where ρ is the density, (λ, μ) are the Lamé's constants, and $\mathbf{w}=(\hat{u},\hat{v})$ is the vector of displacements, whose components are of the "modal" form:

$$\begin{pmatrix}\hat{u}(x,y)\\ \hat{v}(x,y)\end{pmatrix}=\begin{pmatrix}u(y)\\ v(y)\end{pmatrix}\exp(ikx). \tag{2.2}$$

The faces $y=\pm h$ are free of traction, corresponding to boundary conditions:

$$\begin{aligned}&\tau_{xy}(x,\pm h)=\mu(\partial_y\hat{u}+\partial_x\hat{v})=0,\\ &\tau_{yy}(x,\pm h)=\lambda\,\partial_x\hat{u}+(\lambda+2\mu)\partial_y\hat{v}=0.\end{aligned} \tag{2.3}$$

Defining $k_t=\sqrt{\rho/\mu}\omega$, $k_l=\sqrt{\rho/(\lambda+2\mu)}\omega$, and $\gamma=(\lambda+2\mu)/\mu$, the system (2.1) can be written for (u,v):

$$k^2u-ik\frac{\gamma-1}{\gamma}v'-\left(k_l^2u+\frac{u''}{\gamma}\right)=0, \tag{2.4a}$$

$$k^2v-ik(\gamma-1)u'-(k_t^2v+\gamma v'')=0, \tag{2.4b}$$

and the boundary conditions (2.3) become

$$u'(\pm h)=-ikv(\pm h) \tag{2.5a}$$

$$v'(\pm h)=-ik\frac{\gamma-2}{\gamma}u(\pm h), \tag{2.5b}$$

where the prime and double prime stand for d/dy and d^2/dy^2.

III. SPECTRAL DECOMPOSITION

The eigenproblem (2.4)–(2.5) is known to be separable into symmetric and antisymmetric solutions, where symmetric (*resp.*, antisymmetric) modes correspond to even (*resp.*, odd) u and odd (*resp.*, even) v.[5] In the following, superscripts s and a, respectively, refer to symmetric and antisymmetric modes. Basis functions ϕ_n for u^s, v^a, and ψ_n for v^s, u^a, with $n\geqslant 1$, are chosen such that

$$\begin{aligned}&\phi_n''+\alpha_n^2\phi_n=0,\quad \text{and}\quad \phi_n'(0)=\phi_n'(h)=0,\\ &\psi_n''+\beta_n^2\psi_n=0,\quad \text{and}\quad \psi_n(0)=\psi_n'(h)=0,\end{aligned} \tag{3.1}$$

that yields

$$\begin{aligned}&\phi_n=\sqrt{\frac{\epsilon_n}{h}}\cos(\alpha_n y),\quad \text{with}\quad \begin{cases}\epsilon_1=1,\ \epsilon_n=2,\ \text{for}\ n\geqslant 2,\\ \alpha_n=\dfrac{(n-1)\pi}{h},\end{cases}\\ &\psi_n=\sqrt{\frac{2}{h}}\sin(\beta_n y),\quad \text{with}\quad \beta_n=\frac{(n-1/2)\pi}{h};\end{aligned} \tag{3.2}$$

ϕ_n and ψ_n are such that $(\phi_n|\phi_m)=\delta_{nm}$ and $(\psi_n|\psi_m)=\delta_{nm}$, where the scalar product is defined by $(f|g)=\int_0^h f(y)g(y)dy$.

Functions ϕ_n (*resp.*, ψ_n) form a complete basis to describe any even (*resp.*, odd) function because they are eigenfunctions of a classical Sturm–Liouville problem.[12] Thus, symmetric and antisymmetric solutions can be decomposed in these bases as

$$\begin{aligned}&u^s(y)=\sum_{n\geqslant 1}U_n^s\phi_n(y),\quad \text{and}\quad \mathbf{U^s}=(U_n^s),\\ &v^s(y)=\sum_{n\geqslant 1}V_n^s\psi_n(y),\quad \text{and}\quad \mathbf{V^s}=(V_n^s),\\ &u^a(y)=\sum_{n\geqslant 1}U_n^a\psi_n(y),\quad \text{and}\quad \mathbf{U^a}=(U_n^a),\\ &v^a(y)=\sum_{n\geqslant 1}V_n^a\phi_n(y),\quad \text{and}\quad \mathbf{V^a}=(V_n^a).\end{aligned} \tag{3.3}$$

The next step is then to obtain the projection of Eqs. (2.4) on the basis functions. This is presented in the following paragraphs.

A. Symmetric modes

The scalar product of (2.4a) by ϕ_n and (2.4b) by ψ_n is performed. Then, the projection of the derivatives are (using the same procedure as in Ref. 13):

$$\begin{aligned}((v^s)'|\phi_n)&=[v^s\phi_n]_0^h-(v^s|\phi_n')\\ &=\sum_{m\geqslant 1}(\phi_n(h)\psi_m(h)-(\phi_n'|\psi_m))V_m^s,\\ ((u^s)''|\phi_n)&=[(u^s)'\phi_n-u^s\phi_n']_0^h+(u^s|\phi_n'')\\ &=-ik\sum_{m\geqslant 1}\phi_n(h)\psi_m(h)V_m^s-\alpha_n^2U_n^s,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}((u^s)'|\psi_n)&=\sum_{m\geqslant 1}(\psi_n(h)\phi_m(h)-(\psi_n'|\phi_m))U_m^s,\\ ((v^s)''|\psi_n)&=-ik\frac{\gamma-2}{\gamma}\sum_{m\geqslant 1}\psi_n(h)\phi_m(h)U_m^s-\beta_n^2V_n^s.\end{aligned} \tag{3.5}$$

It can be noticed that both boundary conditions have been taken into account in the course of projection [Eq. (2.5a) for (3.4) and Eq. (2.5b) for (3.5)]. Eventually, a system of equations on $\mathbf{U^s}$, $\mathbf{V^s}$ is obtained:

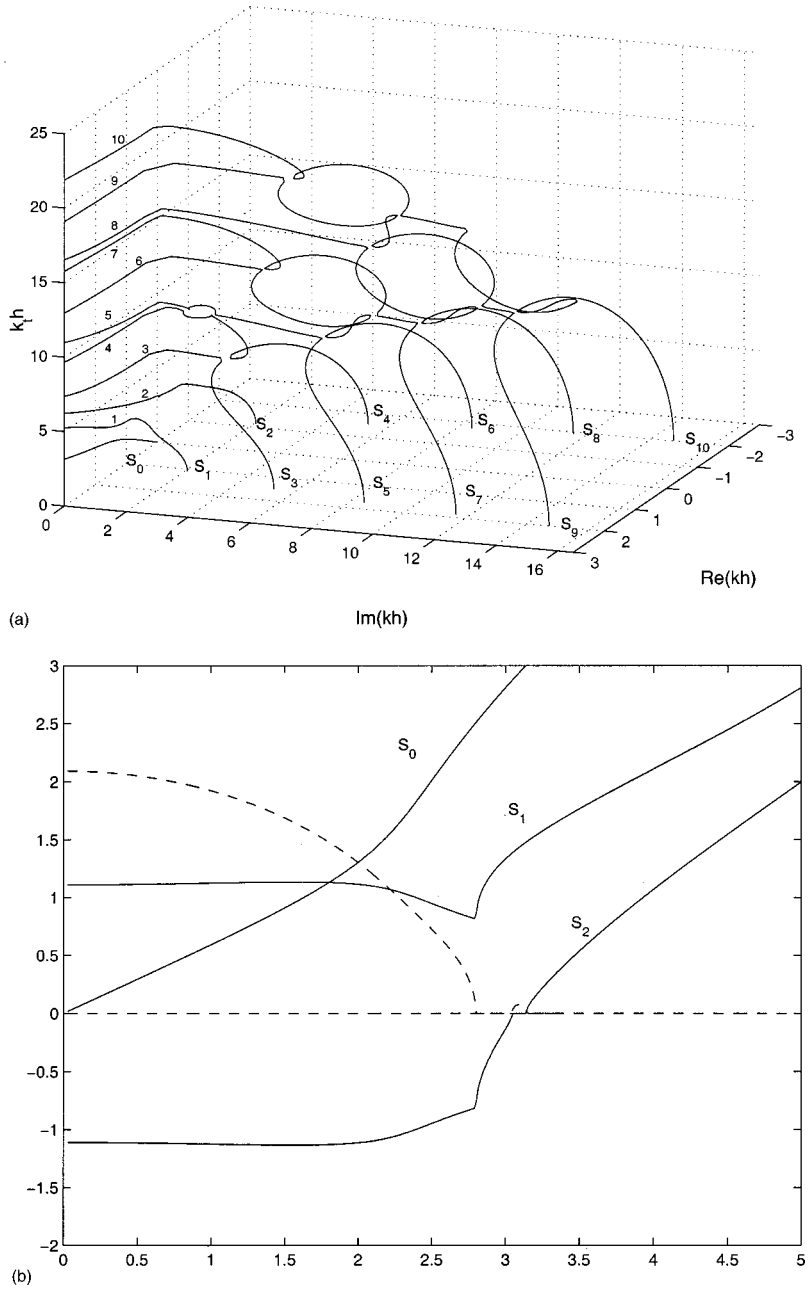

FIG. 2. (a) Tridimensional representation of the dimensionless complex wave numbers $k_n h$ for symmetric modes (S_0 to S_{10}) when the dimensionless frequency $k_t h$ varies, (b) real and imaginary parts of the dimensionless complex wave numbers $k_n h$ for symmetric modes (S_0 to S_3) as a function of the dimensionless frequency $k_t h$.

$$k^2\mathbf{U^s}+kA^s\mathbf{V^s}+B^s\mathbf{U^s}=0,$$
$$k^2\mathbf{V^s}+kC^s\mathbf{U^s}+D^s\mathbf{V^s}=0, \tag{3.6}$$

with matrices A^s, B^s, C^s, and D^s expressed by

$$A^s{}_{m,n}=i\left(\frac{\gamma-1}{\gamma}(\psi_m|\phi_n')+\frac{2-\gamma}{\gamma}\psi_m(h)\phi_n(h)\right)$$
$$=\begin{cases}\dfrac{\sqrt{2}i(2-\gamma)(-1)^m}{h\gamma}, & n=1,\\ \dfrac{2i(-1)^{m+n}(\alpha_n^2+(\gamma-2)\beta_m^2)}{h\gamma(\beta_m^2-\alpha_n^2)}, & n\geqslant 2,\end{cases}$$

$$B^s{}_{m,n}=\left(\frac{\alpha_m^2}{\gamma}-k_l^2\right)\delta_{mn}, \tag{3.7}$$

$$C^s{}_{m,n}=-\gamma A^s_{n,m}, \quad D^s{}_{m,n}=(\gamma\beta_m^2-k_t^2)\delta_{mn}.$$

B. Antisymmetric modes

Similar calculations are performed for antisymmetric modes. In this case, the scalar products of (2.4a) by ψ_n and (2.4b) by ϕ_n are performed and a system of equations on $\mathbf{U^a}$,$\mathbf{V^a}$ is obtained:

$$k^2\mathbf{U^a}+kA^a\mathbf{V^a}+B^a\mathbf{U^a}=0,$$
$$k^2\mathbf{V^a}+kC^a\mathbf{U^a}+D^a\mathbf{V^a}=0, \tag{3.8}$$

with matrices A^a, B^a, C^a, and D^a expressed by

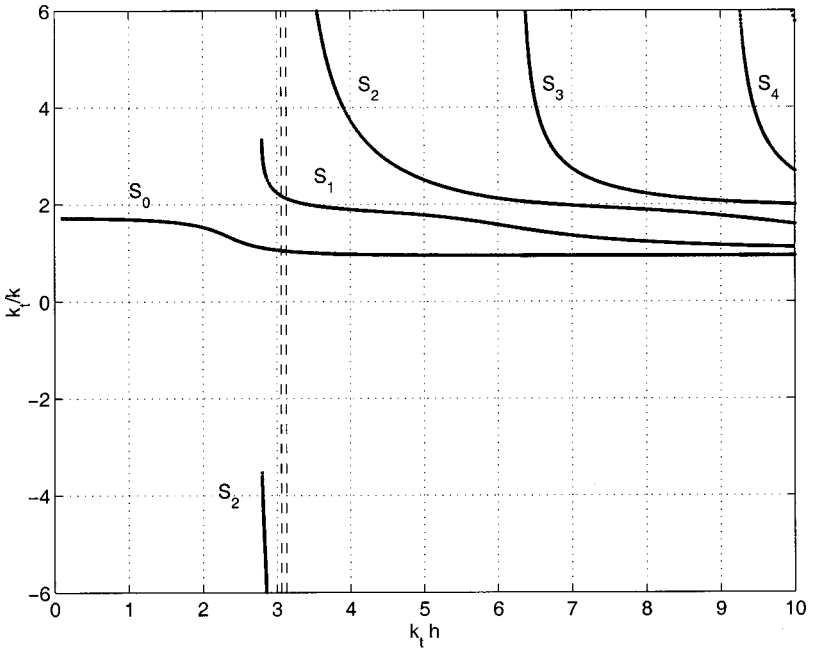

FIG. 3. Dimensionless phase velocities of symmetric Lamb waves k_t/k_n (for real k_n) as a function of the dimensionless frequency $k_t h$.

$$A^a{}_{m,n} = i\left(\frac{\gamma-1}{\gamma}(\phi_m|\psi_n') + \frac{2-\gamma}{\gamma}\phi_m(h)\psi_n(h)\right)$$

$$= \begin{cases} \dfrac{\sqrt{2}i(-1)^n}{h\gamma}, & m=1 \\ \dfrac{2i(-1)^{m+n}((\gamma-2)\alpha_m^2+\beta_n^2)}{h\gamma(\alpha_m^2-\beta_n^2)}, & m\geqslant 2 \end{cases}$$

$$B^a{}_{m,n} = \left(\frac{\beta_m^2}{\gamma} - k_l^2\right)\delta_{mn}, \tag{3.9}$$

$$C^a{}_{m,n} = -\gamma A^a_{n,m}, \quad D^a{}_{m,n} = (\gamma\alpha_m^2 - k_t^2)\delta_{mn}.$$

IV. RESOLUTION

The system (2.4) with boundary conditions (2.5) is an eigenvalue problem with differential operators. Owing to the spectral decomposition presented in Sec. III, it has become a discretized eigenvalue problem with matricial operators.

The discretized systems (3.6) and (3.8) are in the form of a nonlinear eigenvalue problem:

$$k^2\mathbf{U} + kA\mathbf{V} + B\mathbf{U} = 0, \tag{4.1a}$$

$$k^2\mathbf{V} + kC\mathbf{U} + D\mathbf{V} = 0, \tag{4.1b}$$

where matrices A, B, C, D result from projections of the original differential equations and also take into account the boundary conditions. System (4.1) can be easily expressed as a classical eigenvalue problem $(M-kI)\mathbf{X}=0$, as presented in the following section, Sec. IV A. In this case, for a given truncation corresponding to the first N basis functions, a $4N\times 4N$ system has to be solved to obtain $4N$ eigenvalues k. In Sec. IV B, it is shown that an alternative system can be derived, benefiting from the symmetry properties of the k spectrum; in this latter case, the system is only $2N\times 2N$, to also obtain $4N$ eigenvalues.

A. Eigenvalue problem

System (4.1) can be rewritten as

$$k^2\mathbf{X_1} + kF_1\mathbf{X_1} + G_1\mathbf{X_1} = 0, \tag{4.2}$$

with

$$\mathbf{X_1} = \begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}, \quad F_1 = \begin{pmatrix}0 & A\\ C & 0\end{pmatrix}, \quad \text{and} \quad G_1 = \begin{pmatrix}B & 0\\ 0 & D\end{pmatrix}. \tag{4.3}$$

Then, following Ref. 11, with $\mathbf{Y_1} = k\mathbf{X_1}$ and

$$\mathbf{Z_1} = \begin{pmatrix}\mathbf{X_1}\\ \mathbf{Y_1}\end{pmatrix}, \quad M_1 = \begin{pmatrix}0 & I_{2N}\\ -G_1 & -F_1\end{pmatrix}, \tag{4.4}$$

where I_{2N} is the $2N\times 2N$ identity matrix; the system (4.2) is rewritten as

$$M_1\mathbf{Z_1} - k\mathbf{Z_1} = 0. \tag{4.5}$$

It corresponds to a classical eigenvalue problem for the $4N\times 4N$ matrix M_1, in which the eigenvalue k appears linearly.

B. Reduction of the matrix dimension

By inspection, it can be noticed that system (4.1) possesses (fortunately!) the usual symmetries of the Lamb modes $k\rightarrow -k$ and $k\rightarrow k^*$. In order to reduce the dimension of the involved matrices, and, consequently, to increase numerical efficiency, it is possible to take advantage of the symmetry $k\rightarrow -k$. This can be done by casting (4.1) in the form of a nonlinear eigenvalue problem, where only the even powers of the eigenvalue appear.

Expressing $\mathbf{V}$ as a function of $\mathbf{U}$ in (4.1b), (4.1a) can be written as

$$(k^2 I_N - k^2 A(k^2+D)^{-1}C + B)\mathbf{U} = 0. \tag{4.6}$$

In (4.6), $\mathbf{U}\in \mathrm{Ker}(k^2 I_N - k^2 A(k^2+D)^{-1}C + B)$ and a solution $\mathbf{U}$ corresponds to k such that $\det(k^2 I_N - k^2 A(k^2+D)^{-1}C$

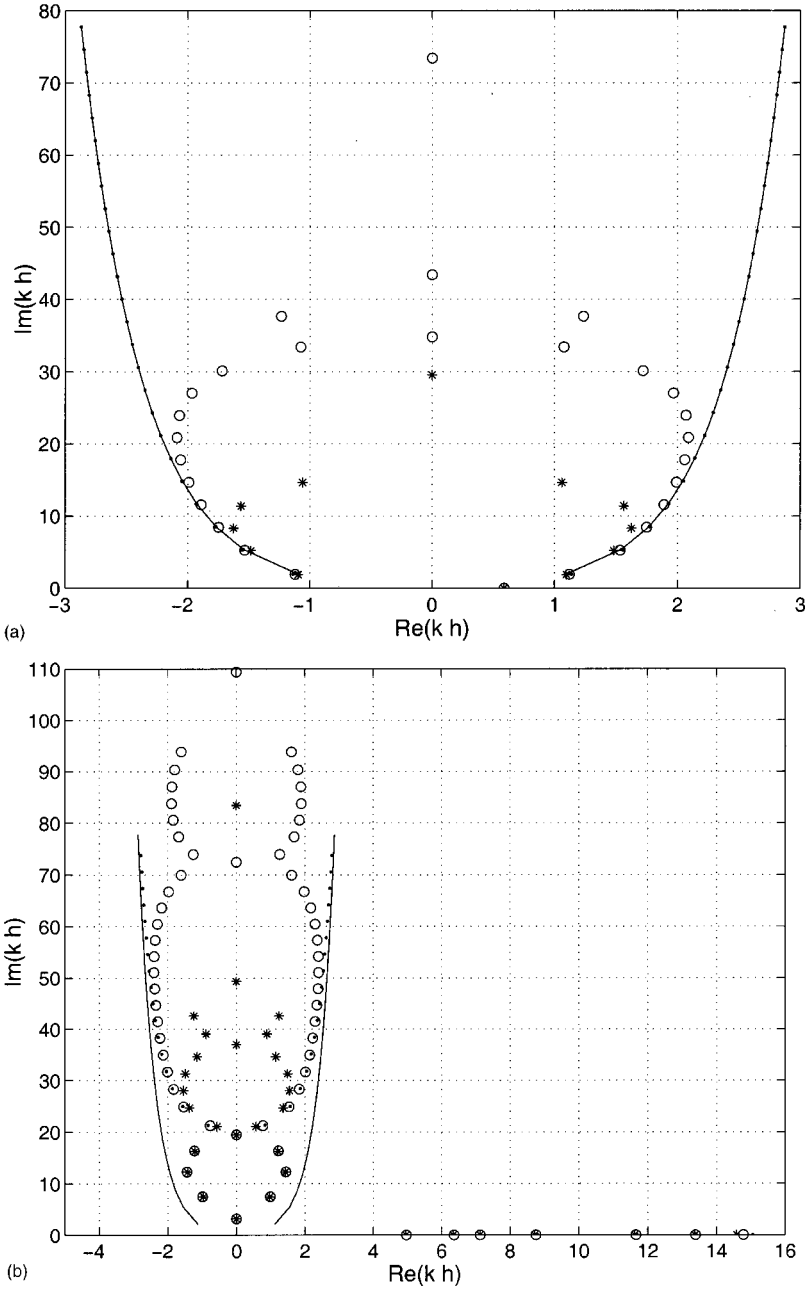

FIG. 4. Dimensionless complex Lamb wave spectrum at (a) $k_t h=1$: (*): $N=6$, (○): $N=14$, (·): exact values k^e and (—) asymptotic values k^a, (b) $k_t h=14$: (*): $N=16$, (○): $N=32$, (·): exact values k^e and (—) asymptotic values k^a, and (c) $k_t h=28$: (*): $N=24$, (○): $N=44$, (·): exact values k^e and (—) asymptotic values k^a.

$+B)=0$. If this equation is multiplied by $\det((k^2+D)A^{-1})$ on the left-hand side and $\det(A)$ on the right-hand side, we obtain $\det(K^2+(D-CA+A^{-1}BA)K+DA^{-1}BA)=0$, with $K=k^2$. The reduced eigenvalue problem on $K=k^2$ is

$$(K^2+F_2K+G_2)\mathbf{X_2}=0, \tag{4.7}$$

with $F_2=(D-CA+A^{-1}BA)$, $G_2=DA^{-1}BA$.

In the course of the derivation of the reduced system, it has been assumed that $\det(A)\neq 0$ and $\det(k^2+D)\neq 0$. The former assumption has been numerically verified. The latter assumption is verified as long as $k^2\neq k_t^2-\gamma\beta_n^2$.

As previously, we now introduce

$$\mathbf{Y_2}=K\mathbf{X_2}, \quad \mathbf{Z_2}=\begin{pmatrix}\mathbf{X_2}\\ \mathbf{Y_2}\end{pmatrix},$$

and

$$M_2=\begin{pmatrix}0 & I_N\\ -G_2 & -F_2\end{pmatrix},$$

where I_N is the $N\times N$ identity matrix. An eigenproblem for $K=k^2$ with the $2N\times 2N$ matrix M_2 is obtained:

$$M_2\mathbf{Z_2}-K\mathbf{Z_2}=0. \tag{4.9}$$

V. RESULTS

In order to check the validity and the efficiency of our technique, results obtained using the spectral method are presented. Without loss of generality, we will focus on the symmetric Lamb modes, but similar results can be obtained for antisymmetric modes. The material properties are those of

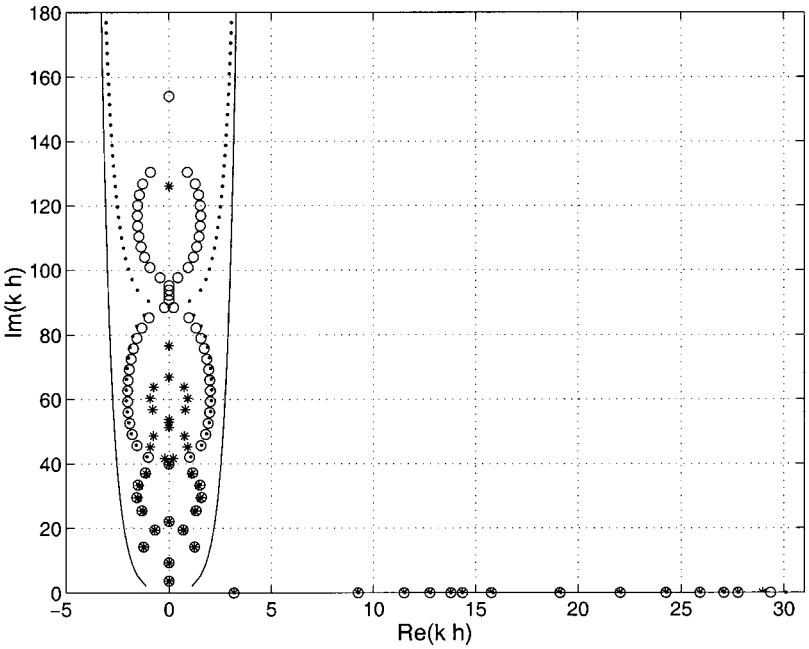

FIG. 4. (*Continued.*)

copper: $v_s=\sqrt{\mu/\rho}=2150\,\mathrm{m\,s^{-1}}$ and $v_l=\sqrt{(\lambda+2\mu)/\rho}=4170\,\mathrm{m\,s^{-1}}$ ($\gamma=(v_l/v_s)^2$) and the plate thickness is taken to be $2h=0.02\,\mathrm{m}$.

For u and v projected on N spectral basis functions, $4N$ eigenvalues are calculated from the $2N\times 2N$ system (4.9). For clarity, only the $2N$ rightgoing modes will be presented; the other $2N$ modes, which are leftgoing, are simply obtained by the symmetry $k\rightarrow -k$. On the other hand, as presented in the forthcoming section Sec. V C, a part of the calculated spectrum corresponds to spurious eigenvalues. For this reason, a qualitative criterium is used and only a subset of the determined spectrum is selected.

In the following, in order to assess the obtained values, we refer to "exact" values k^e, obtained from a Newton convergence method with a tolerance of 10^{-10}. Here, the wave numbers obtained from the spectral method are used as initial guess values and it has been verified that these "exact" values correspond to actual zeros of the dispersion relation.

A. Eigenvalues in the complex plane

Figure 2 shows the k spectrum obtained in the complex plane varying ω. In the computation, $N=16$ leads to 32 rightgoing eigenvalues k; as discussed in Sec. V C, only about N values are identified as correct values (11 eigenvalues are shown in the figure). The usual behavior of Lamb modes is recovered: for low frequency, only S_0 is propagating and, to increase the frequency leads to more and more propagating modes. We recover also the particular behavior of S_2: it becomes propagating with negative phase velocity and recovers a positive phase velocity at higher frequency.

B. Phase velocities of Lamb modes

To recover the usual representation of propagating modes,[5,14] we have plotted in Fig. 3 the dimensionless phase velocities of symmetric Lamb waves, k_t/k_n, for real k_n, as a function of the dimensionless pulsation k_th. This is obtained in Fig. 2 at constant $\mathrm{Im}(k)=0$. The expected form is obtained but we underline the fact that the branch $k<0$ of the S_2 mode corresponds to a rightgoing mode with negative phase velocity but positive group velocity. Incidently, it can be noticed that, in some papers, this negative phase velocity branch is erroneously identified as a part of an S_1 mode[5] or is not represented.[14] The S_2 mode turns purely imaginary in a frequency band corresponding to k_th between around 3.06 and 3.13, in agreement with previous studies (see, for instance, Rokhlin *et al.*[15]).

C. Representation of the spectrum for a given ω

Figures 4 show the evolution of the rightgoing complex wave spectrum derived from the spectral method when N increases. These $2N$ values are compared with the exact values k^e and the asymptotic values k^a derived by Merkulov *et al.*[6] for large k:

$$k_n^a h=\frac{1}{2}\ln\left[2\pi\left(n+\frac{1}{2}\right)\right]-\frac{i}{2}\left[\pi\left(n+\frac{1}{2}\right)-\frac{\ln[2\pi(n+1/2)]}{\pi(n+1/2)}\right]. \tag{5.1}$$

It can be seen from these figures that the k spectrum found with the spectral method coincides with the exact one (k^e) for the N or so first values, and this, independently of the complexity of the spectral structure when the frequency increases. A qualitative criterion to select the useful part of the calculated spectrum can be to restrict the complex wave spectrum to the first N values.

D. Convergence

The evolution of the relative error is shown in Fig. 5 as a function of N at two different frequencies. The relative error is defined as $|\Delta(k)/k|$, where $\Delta(k)=k-k^e$. In both cases, it appears that the method converges as $1/N$ for large N.

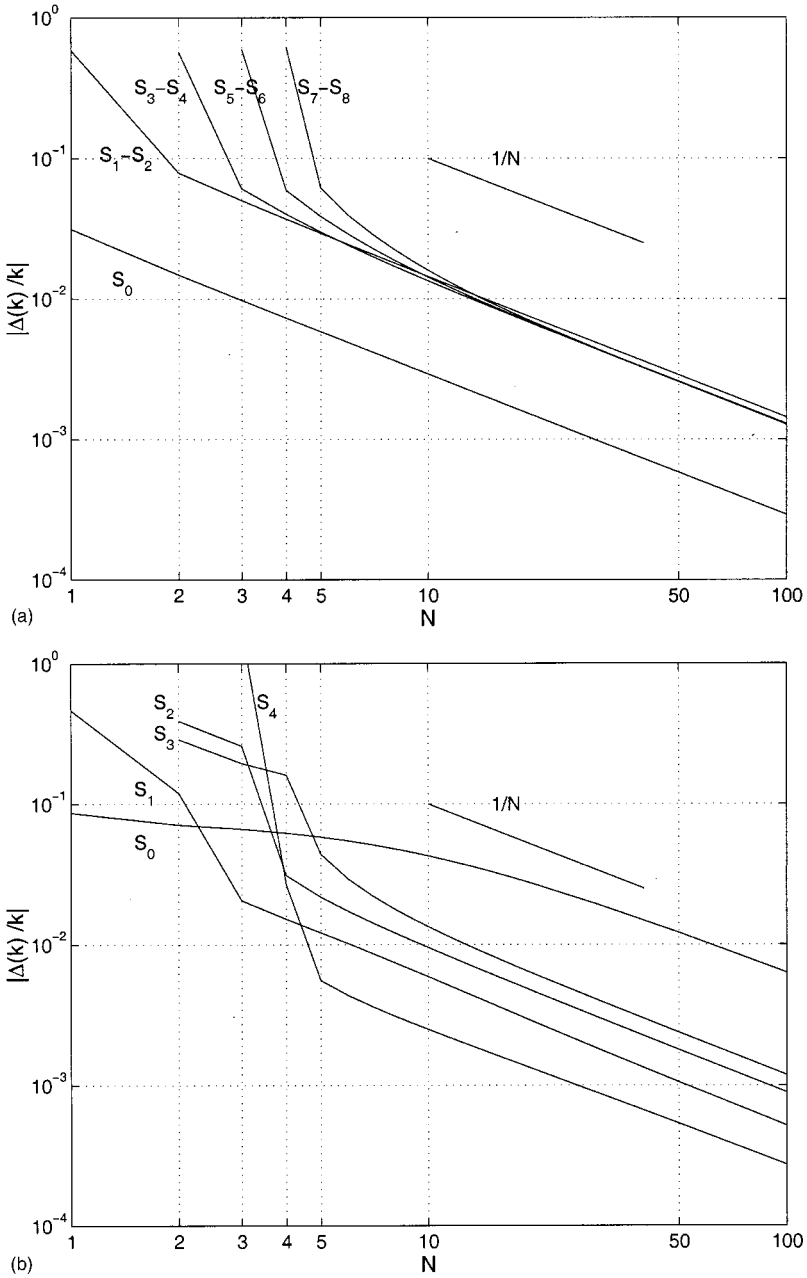

FIG. 5. Convergence for the first modes as a function of the order N of the truncation for a given frequency, (a) $k_t h=1$ (modes S_0 to S_8) and (b) $k_t h=14$ (modes S_0 to S_4).

For $k_t h=1$, only the mode S_0 is propagating; the symmetry properties of the spectrum for evanescent modes implies $k_{2n}=-k^*_{2n-1}$ for $n\geqslant 1$, implying the same convergence for the pairs $(2n,2n-1)$.

For $k_t h=14$, 14 modes are propagating; we give the convergence for the first five modes. It can be noticed that the mode S_0 reaches a type $1/N$ convergence law only for $N>10$, but is given with a reasonable accuracy (10%) as soon as $N=1$.

VI. CONCLUSION

A new method for the determination of the Lamb wave spectrum has been presented. This method is based on a spectral projection of the equation of the elasticity, leading to a classical eigenvalue problem.

This method is an alternative to the usual method of root finding of the Rayleigh–Lamb dispersion relation. The following points make the method attractive.

(i) It is easy to implement. For a given frequency, it is very simple to take the expression of the M_2 matrix from (3.9), (4.10), and (4.11), to put in it the material (γ, k_l, and k_t) and geometrical (h) properties and then, to use any eigenvalue solver package to obtain the wave numbers.

(ii) In a step by step method, the series of spectra calculated from zero to a given frequency can cross a critical frequency, for which two wave numbers collapse, leading to $\partial_k D=0$. A particular treatment then has to be applied to go through this critical frequency since $\partial k/\partial\omega=\partial_\omega D/\partial_k D$ diverges. With our method, these critical frequency cases have bearing on the determination of the spectrum because its calculation does not depend on the history of the spectrum.

(iii) If the goal is to obtain the wave numbers with a prescribed precision, the results of the spectral method can

be used as very good initial guess values in a Newton–Raphson method. That is to say that our method can be viewed as a ''super initial guess value provider.''

(iv) The method can be easily generalized to other wave guided modes. In fact, it can be implemented as long as the transverse problem can be written in the form of a differential equation with an integer power of the eigenvalue in the coefficients. Then, this technique offers the possibility of much greater certainty in finding all the families of transverse modes. For instance, it may be applied to fluid-loaded plates, plates with damping, or transversally layered plates.

[1] D. Ensminger, *Ultrasonics, Fundamentals, Technology, Applications* (Marcel Dekker, New York, 1988).

[2] R. D. Gregory and I. Gladwell, ''The reflection of a symmetric Rayleigh–Lamb wave at a fixed or free edge of a plate,'' J. Elast. **13**, 185–206 (1983).

[3] R. D. Gregory and I. Gladwell, ''The generation of waves in a semi-infinite plate by a smooth oscillating piston,'' ASME J. Appl. Mech. **51**, 787–791 (1984).

[4] C. Scahdrett and N. Vasudevan, ''The propagation of time harmonic Rayleigh–Lamb waves in a bimaterial plate,'' J. Acoust. Soc. Am. **89**, 1606–1614 (1991).

[5] I. A. Viktorov, *Rayleigh and Lamb Waves: Physical Theory and Applications* (Plenum, New York, 1967), Chap. 2.

[6] L. G. Merkulov, S. I. Rokhlin, and O. P. Zobnin, ''Calculation of the spectrum of wave numbers for Lamb waves in a plate,'' Sov. J. Nondestruct. Test. **6**, 369–373 (1970).

[7] J. Miklowitz, *The Theory of Elastic Waves and Waveguides* (North-Holland, New York, 1978), Chap. 4.

[8] J. D. Achenbach, *Wave Propagation in Elastic Solids* (Elsevier, Amsterdam, 1984).

[9] R. D. Mindlin, ''Waves and vibrations in isotropic elastic plates,'' *Structural Mechanics*, edited by J. N. Goodier and N. J. Hoff (Pergamon, New York, 1960).

[10] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1982).

[11] T. J. Bridges and P. J. Morris, ''Differential eigenvalue problems in which parameter appears non-linearly,'' J. Comput. Phys. **55**, 437–460 (1984).

[12] P. M. Morse and H. Feshback, *Mathematical Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[13] V. Pagneux, N. Amir, and J. Kergomard, ''A study of wave propagation in varying cross-section waveguides by modal decomposition. Part I. Theory and validation,'' J. Acoust. Soc. Am. **100**, 2034–2048 (1996).

[14] Y. Cho and J. L. Rose, ''A boundary element solution for a mode conversion study on the edge reflection of Lamb waves,'' J. Acoust. Soc. Am. **99**, 2097–2109 (1996).

[15] S. I. Rokhlin, D. E. Chimenti, and A. H. Nayfeh, ''On the topology of the complex wave spectrum in a fluid-coupled elastic layer,'' J. Acoust. Soc. Am. **85**, 1074–1080 (1989).