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DETERMINATION OF THE GLUON CONDENSATE AND THE FOUR QUARK CONDENSATE VIA FESR

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ABSTRACT

The dimension four gluon condensate and the dimension six four-quark condensate are estimated with the help of FESR by analyzing the ρ-meson channel. We find that the first is a factor 2-5 and the second is (in absolute value) a factor 5-8 larger than the corresponding "standard values". We have checked our results by the "heat evolution test" of Gauss transforms. They turn out to be consistent whereas the "standard values" are not.

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1. Introduction

Shifman, Vainshtein and Zakharov (SVZ hereafter) [1] have extended the notion of quark pair condensates to other vacuum condensates of QCD operators. According to their view these quantities represent confinement, they determine the binding between quark-antiquark pairs. With the help of some kind of dispersion relation - the so-called QCD sum rules - one can relate these condensates to experiment, to the resonances and continuum of a specific inclusive cross-section. In this way knowledge of the condensates provides information on physical quantities like masses and widths of resonances (and vice versa). For this reason rather accurate values for the condensates are of great importance. Unfortunately direct calculations of the condensates like the lattice calculations are still not precise enough (for a review see Ref. [2]) so that one has to rely on calculations from various types of sum rules. In fact this was the way how SVZ in their original work [1] determined the value of the gluon condensate by studying the charmonium channel. We want to refer to it now as the "standard value"

$$\left\langle \frac{\alpha_s}{\pi} GG \right\rangle = 0.012 \ \mathrm{GeV}^4 \quad . \tag{1.1}$$

For the four-quark condensate (vacuum expectation value of operators of mass dimension six)

$$C_6 \langle O_6 \rangle = -4\pi^3 \langle \alpha_s (\bar{u}\gamma_\mu\gamma_5 t^a u - \bar{d}\gamma_\mu\gamma_5 t^a d)^2 \rangle - \\ -\frac{8}{9}\pi^3 \left\langle \alpha_s (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} \bar{q}\gamma_\mu t^a q \right\rangle \tag{1.2}$$

SVZ propose the vacuum dominance hypothesis (VDH abbreviated)

$$\doteq -\frac{896}{81}\pi^3 \alpha_s \left| \langle \bar{q} q \rangle \right|^2 \tag{1.3}$$

$$= -0.06 \ \mathrm{GeV}^6 \quad . \tag{1.4}$$

In Eq. (1.3) the quark condensate has been estimated with the help of PCAC (see Refs. [1] and [3]).

However, there is still a large controversy about the values of the condensates.

i) Gluon condensate

Relying on non-relativistic approximations (potential theory) Bell and Bertlmann [4] conjectured that the moment procedure grossly underestimates the gluon condensate, Eq. (1.1), (about a factor of three).

British groups [5,6] working within two-dimensional QCD discovered quite similar features. Recently Marrow, Parker and Shaw [7] have re-examined the charmonium channel with ratios of exponential moments within QCD. When they take into account the higher states in the spirit of duality, and restrict the method to a compatibility region, they indeed obtain a gluon condensate value which is a factor 3-5 larger than the "standard" one, Eq. (1.1).

ii) Four-quark condensate

Although the VDH, Eq. (1.3), is widely used in the literature it has been noticed by several authors [8-11] some time ago that this approximation leads to inconsistencies. Also a recent work by the authors of Ref. [12] points clearly into this direction.

There exist many additional attempts in the literature [13] to determine the condensate values with one or the other variant of sum rules. But the outcome is that some authors agree with the "standard values" of Eqs. (1.1) and (1.4) and some others do not. In this rather unsatisfactory situation we try once more to estimate the condensates. We apply another type of sum rules, so-called finite energy sum rules (FESR abbreviated), and check the consistency of our results with the help of Gaussian distributions.

2. FESR

Finite energy sum rules have been derived in the past via Cauchy's theorem. On the other hand, Bertlmann, Launer and de Rafael [14] (BLR hereafter) have shown how they follow from Gaussian sum rules. Such a feature is certainly not peculiar to Gauss-Weierstrass transforms, the same results must also follow from Laplace (exponential) transforms or Hilbert (power law) transforms. In the Appendix we present a rigorous derivation of that (see also Ref. [15e]). Here we just quote the results for the ρ-channel we shall be concerned with

$$(-)^{n-1} C_{2n}\langle O_{2n}\rangle = 8\pi^2 \int_0^{s_0} ds\, s^{n-1} \frac{1}{\pi} \mathrm{Im}\,\Pi(s) - \frac{s_0^n}{n} F_{2n}(s_0) \tag{2.1}$$

with n = 1,2,...

The term $C_{2n}\langle O_{2n}\rangle$ represents the linear combination of renormalization group invariant vacuum expectation values of operators of a given mass dimension 2n; more specifically $C_4\langle O_4\rangle$ is defined by Eq. (4.1) and $C_6\langle O_6\rangle$ by Eq. (1.2). The function $F_{2n}(s_0)$ contains the perturbative corrections

$$F_{2n}(s_0) = 1 + \frac{\alpha_s(s_0)}{\pi} + \left(\frac{\alpha_s(s_0)}{\pi}\right)^2 \left[F_3 - \frac{\beta_2}{\beta_1} \log\log\frac{s_0}{\Lambda^2} - \frac{\beta_1}{2n} \right] \tag{2.2}$$

with

$$\frac{\alpha_s(s_0)}{\pi} = \frac{2}{-\beta_1 \log\frac{s_0}{\Lambda^2}} \tag{2.3}$$

and

$$\beta_1 = -\frac{11}{2} + \frac{n_f}{3}, \quad \beta_2 = -\frac{51}{4} + \frac{19}{12} n_f, \quad F_3 = 1.986 - 0.115\, n_f \,. \tag{2.4}$$

The parameter s_0 is the onset of the continuum and n_f is the number of quark flavours.

There is a Russian group of physicists [15] who advocate FESR already for quite some time. In a recent analysis some authors [15i] use FESR of the above type in order to predict resonance masses and widths. Other authors [16] work with somewhat modified FESR. Note that for n=1 Eq. (2.1) is just the "good old" global duality relation of Sakurai [17].

However, we want to use the relation (2.1) to calculate the condensates of a given mass dimension. For a specific example this was demonstrated already by BLR [14]. The problem is rather delicate because of the increasing power weight. At this stage we want to draw attention to some important features of FESR compared to the usual sum rules like Laplace - or Hilbert transforms.

i) As can be seen from Eq. (2.1) FESR weigh the high-energy region in contrast to the usual sum rules which do just the opposite. Therefore FESR require a very accurate parametrization of the hadronic spectral function. We will demonstrate this point in our discussion of the results.

ii) FESR have the advantage to project the condensates of one given mass dimension 2n - at least to the level of approximation at which they have been derived. Whereas in the usual sum rules condensates of different mass dimension are present and strongly correlated.

iii) The onset of the continuum s_0 is fixed within FESR; s_0 is a solution of the first FESR equation [n=1 in Eq. (2.1)]. This feature is also very different from the usual sum rule procedure where various authors choose their convenient s_0 just by hand.

iv) In addition we follow the principle of Pich and de Rafael [18] concerning the stability of FESR and accept only those solutions of Eq. (2.1) which are (rather) stable in s_0. Further successful applications [19] confirm this point of view. For comparison the criterion in the sum rules is usually given by some stability range in the short-distance expansion parameter.

From the above discussion it is clear that FESR and the usual sum rules are rather complementary tools for studying the features of QCD.

3. Gaussian sum rules

Gaussian sum rules have been introduced within the framework of QCD by BLR [14]. There the spectral function is convoluted with a Gaussian centered at an arbitrary point $\hat{s}$ with a width $\sqrt{2\tau}$

$$G(\hat{s},\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty ds \; e^{-\frac{(s-\hat{s})^2}{4\tau}} \frac{1}{\pi} \mathrm{Im}\,\Pi(s) \, . \tag{3.1}$$

Then BLR propose the following picture. At a certain "time" $\tau=0$ the spectral function $(1/\pi)\mathrm{Im}\pi(s)$ can be viewed as a "heat" (or "temperature") distribution in a semi-infinite rod $0 \leqslant \hat{s} \leqslant \infty$. Here the squared energy $\hat{s}$ plays the role of a "position" variable in the rod. The question is the "heat" distribution after some "time" $\tau > 0$ which is given by the "heat" equation

$$\left(\frac{\partial}{\partial \hat{s}}\right)^2 U(\hat{s},\tau) = \frac{\partial}{\partial \tau} U(\hat{s},\tau) \tag{3.2}$$

with the initial condition

$$U(\hat{s},\tau=0) = \frac{1}{\pi} \, Im\, \Pi(s) \quad . \tag{3.3}$$

For the two boundary conditions

i) $$U(\hat{s}=0,\tau) = 0 \longrightarrow U^{(-)} \tag{3.4a}$$

ii) $$\frac{\partial}{\partial \hat{s}} U(\hat{s},\tau)\Big|_{\hat{s}=0} = 0 \longrightarrow U^{(+)} \tag{3.4b}$$

there exist two independent solutions, the odd (3.4a) and the even (3.4b)

$$U^{(\mp)}(\hat{s},\tau) = \int_0^\infty ds \left[k(s-\hat{s},\tau) \mp k(s+\hat{s},\tau) \right] \frac{1}{\pi} \, Im\, \Pi(s) \quad . \tag{3.5$^{a}_{b}$}$$

The function $k(s-\hat{s},\tau)$ is the well-known source solution

$$k(s-\hat{s},\tau) = \frac{1}{\sqrt{4\pi\tau}} \, e^{-\frac{(s-\hat{s})^2}{4\tau}} \quad . \tag{3.6}$$

Clearly the Gaussian distribution (3.1) is just the average of both solutions (3.5a,b)

$$G(\hat{s},\tau) = \frac{1}{2}\left[U^{(-)}(\hat{s},\tau) + U^{(+)}(\hat{s},\tau) \right] . \qquad (3.7)$$

This is a powerful framework to formulate the concept of local duality. One certainly cannot confront QCD with the physical spectral function at $\tau=0$. But after some "time" when the "heat" distributions evolve the QCD distribution must match the experimental one. Furthermore the total "heat" is conserved - the areas of both distributions are equal at any "time". This BLR called "heat evolution test".

The final theoretical distribution which we are going to use (for vector-like and light quark currents) is [14]

$$\begin{aligned} G(\hat{x},\tau) = \frac{1}{16\pi^2} \Big\{ & \mathrm{Erfc}(-\hat{x}) \Big[1 + \frac{\alpha_s(\sqrt{\tau})}{\pi} + \\ & + \left(\frac{\alpha_s(\sqrt{\tau})}{\pi}\right)^2 \left(F_3 - \frac{1}{4}\beta_1 \gamma_E - \frac{\beta_2}{\beta_1} \log\log\frac{\sqrt{\tau}}{\Lambda^2} \right) \Big] - \\ & - \beta_1 \left(\frac{\alpha_s(\sqrt{\tau})}{\pi}\right)^2 \left(\int_0^{\hat{x}} dz \, D\,\mathrm{Erfc}(-z) - \frac{1}{2}\log 2 \; \mathrm{erf}(\hat{x}) \right) + \\ & + \frac{2}{\sqrt{\pi}} \sum_n \frac{(-)^{n-1}}{(n-1)!} \frac{C_{2n}\langle O_{2n}\rangle}{(2\sqrt{\tau})^n} H_{n-1}(\hat{x})\, e^{-\hat{x}^2} \Big\} . \end{aligned} \qquad (3.8)$$

For the specific notations we refer to the original literature [14].

4. Analysis in the ρ-meson channel

We want to determine in this paper the condensates of dimension four and six with the help of FESR, Eq. (2.1). We investigate the ρ-meson channel where many data are available. SVZ [1] have calculated $C_4\langle O_4\rangle$ in terms of QCD operators

$$C_4\langle O_4\rangle = \frac{\pi^2}{3}\left\langle \frac{\alpha_s}{\pi} GG \right\rangle + 4\pi^2 \left(m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle \right) \tag{4.1}$$

which leads to the "standard value"

$$C_4\langle O_4\rangle = 0.034 \ GeV^4 \ . \tag{4.2}$$

Note that the gluon condensate $\langle \alpha_s/\pi \ GG \rangle$, Eq. (1.1), provides the dominant contribution.

The dimension six condensate leads to a "standard value" which we have already quoted in Eqs. (1.2)-(1.4). Recall that a term $\langle g^3 GGG \rangle$ is absent here [20].

Now we begin with FESR. For n=1 we get

$$R_2(s_0) = \frac{8\pi^2}{s_0} \int_0^{s_0} ds \ \frac{1}{\pi} \mathrm{Im}\,\Pi(s) = F_2(s_0) + \frac{C_2\langle O_2\rangle}{s_0} \ . \tag{4.3}$$

Here $C_2\langle O_2\rangle$ is nothing but a short-hand notation for the perturbative quark mass insertion

$$C_2\langle O_2\rangle = - \frac{3(\hat{m}_u^2 + \hat{m}_d^2)}{\left[\frac{1}{2}\log \frac{s_0}{\Lambda^2}\right]^{4/(-\beta_1)}} \ . \tag{4.4}$$

Its contribution can be neglected for practical purposes since the invariant quark masses are very small ($\hat{m}_u = 8.2 \pm 1.5$ MeV, $\hat{m}_d = 14.4 \pm 1.5$ MeV), see for instance Ref. [3d] for the most recent determination). Equation (4.3) determines the values of s_0 which will be used to calculate $C_4\langle O_4\rangle$ and $C_6\langle O_6\rangle$

$$R_4(s_0) = 8\pi^2 \frac{2}{s_0^2} \int_0^{s_0} ds\, s\, \frac{1}{\pi} \mathrm{Im}\,\Pi(s) = F_4(s_0) - 2\frac{C_4\langle O_4\rangle}{s_0^2} , \tag{4.5}$$

$$R_6(s_0) = 8\pi^2 \frac{3}{s_0^3} \int_0^{s_0} ds\, s^2 \frac{1}{\pi} \mathrm{Im}\,\Pi(s) = F_6(s_0) + 3\frac{C_6\langle O_6\rangle}{s_0^3} . \tag{4.6}$$

As discussed in the previous chapter FESR are very sensitive to a specific parametrization of the hadronic spectral function and one has to be very careful in choosing it. We demonstrate this feature quantitatively by the following models.

MODEL A

The spectral function is approximated by a δ-function for the ρ-resonance (zero width approximation) and pure step for the continuum starting at s_0 (radiative corrections neglected)

$$\frac{1}{\pi} \mathrm{Im}\,\Pi(s) = \frac{M_\rho^2}{4 f_\rho^2}\, \delta(s - M_\rho^2) + \frac{1}{8\pi^2}\, \theta(s - s_0) \tag{4.7}$$

with M_ρ = 770 MeV and f_ρ^2 = 6.22 being the experimental values [21].

MODEL B

Towards a more realistic ansatz we approximate the ρ-resonance by a Breit-Wigner function

$$\frac{1}{\pi} \mathrm{Im}\,\Pi(s) = \frac{1}{\pi} \frac{M_\rho^2}{4 f_\rho^2} \frac{M_\rho \Gamma_\rho}{(s - M_\rho^2)^2 + M_\rho^2 \Gamma_\rho^2} + \frac{1}{8\pi^2}\, \theta(s - s_0) \tag{4.8}$$

with Γ_ρ = 153 MeV [21].

MODEL C

Like model B but it includes the radiative corrections [function $F_{2n}(s_0)$ of Eq. (2.2) with Λ = 50 MeV].

DATA FIT

Finally we fit the isospin I=1 data with a Gounaris-Sakurai-like formula [22] which allows for a destructive interference between the ρ- and ρ'-contribution. This fit is plotted in Fig. 1; it reproduces the data very well especially the dip between the ρ- and ρ'-meson (χ^2 = 103 for 71 data points and six parameters).

The results of the above models and data are collected in the Table. We observe that in model A the vacuum condensates emerge roughly an order of magnitude larger (in absolute values) than the corresponding "standard values". Although model A is widely used in the literature it is obviously oversimplified. If we approximate the broad ρ-resonance by a sharp δ-function we cannot expect to reproduce the physical mass correctly. Indeed a Breit-Wigner ansatz (model B) already diminishes the values by about a factor of two and the radiative corrections (model C) reduce further the values by about 30%. Note that all solutions can be found analytically [14,16a].

Let us consider next the behaviour of Eq. (4.3) versus s_0, the onset of the continuum, which we have depicted in Fig. 2a. Models A, B, C provide rather steep curves leading to high instabilities of the condensates $C_4\langle O_4\rangle$ and $C_6\langle O_6\rangle$ with respect to s_0. Only the real data fit with its dip between the two resonances ρ and ρ' stabilizes Eq. (4.3) as well as the vacuum condensates (see Figs. 3 and 4). Following Pich and de Rafael [18] concerning the stability of FESR we obtain now a region where we can trust the values of the condensates. The mean values are

$$C_4\langle O_4\rangle = 0.14 \text{ GeV}^4, \tag{4.9}$$

$$C_6\langle O_6\rangle = -0.45 \text{ GeV}^6. \tag{4.10}$$

In this procedure we have fitted the QCD parameter Λ in the perturbative corrections, Eqs. (2.2)-(2.4), to the data. We find (see Figs. 1 and 2a) that for Λ = 50 MeV there occurs exactly "duality resonance by resonance" very much in the spirit of Sakurai [17,23] and Bell-Bertlmann [24]. QCD "duals" the ρ-resonance by averaging in the interval

$$s_{\text{threshold}} \leqslant s \leqslant 2.08 \text{ GeV}^2 \tag{4.11}$$

and also the ρ'-resonance in the interval

$$2.08 \leqslant s \leqslant 4.6 \text{ GeV}^2. \tag{4.12}$$

Alternatively in the "heat" picture of BLR the total "heat" of each resonance equals separately the corresponding QCD "heat". For the actual computations we work in the first duality interval; we analyze only the ρ-resonance data. But recall that the presence of the ρ'-resonance is necessary in order to fix the duality interval.

Error analysis

Now we have to estimate the error on the condensate values which is rather tricky since the FESR are amplifying them strongly. In our analysis we rely on two principles: duality and stability. We finally compute the error in two different ways. Note that there exist other types of sum rules - the methods of analytic continuation [25] - where the error appears already in the sum rule itself. In this respect they might be advantageous.

i) We perform a "MINUIT" analysis of the data by including all errors in the resonance masses and coupling constants and obtain in this way upper and lower limits for Eqs. (4.3), (4.5) and (4.6)

$$R_{2,4,6}(s_0) \longrightarrow R_{2,4,6}(s_0) \pm \Delta R_{2,4,6}(s_0) \ . \qquad (4.13a,b,c)$$

Then the errors $\Delta R_{2,4,6}$ are given by the error matrix of the "MINUIT" program. The results for $R_2(s_0) \pm \Delta R_2(s_0)$ are plotted in Fig. 2b. In order to achieve duality s_0 can vary between

$$1.78 \leqslant s_0 \leqslant 2.08 \ \mathrm{GeV}^2 . \qquad (4.14)$$

Next we have plotted in Fig. 3 the limits on the condensate $C_4\langle O_4\rangle$ by taking into account $R_4(s_0) \pm \Delta R_4(s_0)$. Since s_0 can vary in the interval (4.14) we get the following lower and upper bound

$$0.07 \leqslant C_4\langle O_4\rangle \leqslant 0.16 \ \mathrm{GeV}^4 . \qquad (4.15)$$

The analogous treatment for $C_6\langle O_6\rangle$ (see Fig. 4) provides the bounds

$$-0.49 \leqslant C_6\langle O_6\rangle \leqslant -0.33 \ \mathrm{GeV}^6 . \qquad (4.16)$$

ii) Alternatively we calculate the error in the following way. We again include all errors of the data but now we achieve duality by adapting the QCD parameter Λ (which has been fitted anyhow)

$$R_2(s_0) \pm \Delta R_2(s_0) = F_2(s_0, \Lambda\pm) + \frac{C_2\langle O_2\rangle}{s_0} \quad . \tag{4.17}$$

This fixes lower and upper bounds on Λ

$$(\Lambda_- = 20\,\mathrm{MeV}) \leq \Lambda \leq (\Lambda_+ = 80\,\mathrm{MeV}) \; , \tag{4.18}$$

whereas s_0 on the other hand remains stable. This method respects both the minimal and the maximal error in the data of Eq. (4.17).

Now the condensates are bounded by the limits $\Lambda\pm$ and $R\pm\Delta R$

$$C_4\langle O_4\rangle_\pm = \frac{s_0^2}{2}\left[F_4(s_0,\Lambda_\pm) - \left(R_4(s_0) \mp \Delta R_4(s_0)\right)\right] \tag{4.19}$$

$$C_6\langle O_6\rangle_\pm = -\frac{s_0^3}{3}\left[F_6(s_0,\Lambda_\mp) - \left(R_6(s_0) \pm \Delta R_6(s_0)\right)\right] \tag{4.20}$$

where (+) denotes the upper bound and (-) the lower bound.

The numerical results are

$$(C_4\langle O_4\rangle_- = 0.08) \leq C_4\langle O_4\rangle \leq (C_4\langle O_4\rangle_+ = 0.19\,\mathrm{GeV}^4) \tag{4.21}$$

$$(C_6\langle O_6\rangle_- = -0.51) \leq C_6\langle O_6\rangle \leq (C_6\langle O_6\rangle_+ = -0.37\,\mathrm{GeV}^6) \tag{4.22}$$

which are rather close to what we obtained before in Eqs. (4.15) and (4.16).

Combining now the outcome of both methods we quote our final result

$$0.07 \leq C_4\langle O_4\rangle \leq 0.19\,\mathrm{GeV}^4 \tag{4.23}$$

$$-0.51 \le C_6 \langle O_6 \rangle \le -0.33 \ \mathrm{GeV}^6 . \qquad (4.24)$$

Note that the previous presentations of the authors [26] differ from the above procedure but the results are quite similar.

"Heat Evolution Test"

Once having a set of QCD parameters we can check their consistency with the help of Gaussian sum rules or "heat" distributions. For our condensate results (4.9), (4.10), (4.23) and (4.24), we compare the theoretical - Eq. (3.8), with the experimental "heat" distribution, Eq. (3.1), for several "times" τ. The "time" τ = 0.1 GeV^4 is too short for a comparison since the structures are too pronounced (see Fig. 5a). But already for τ = 0.5 GeV^4 (see Fig. 5b) both distributions approach each other. For "times" $\tau \geqslant 1.0$ GeV^4 (see Fig. 5c) the distributions practically coincide. This shows that QCD - theory passes the "heat evolution test" - all their parameters are consistent with those of experiment.

For comparison we also plot the distributions by inserting the "standard values" (1.4) and (4.2) together with s_0 = 4 GeV^2 and Λ = 100 MeV, the values used by SVZ [1]. Then we find that even at large "times" $\tau \geqslant 1$ GeV^4 (see Figs. 6a,b,c) QCD does not match experiment. We always observe some missing total "heat". In this case the parameters are not consistent.

Last but not least we also checked the ratio of moments

$$R_{mn} = \frac{\int_0^{s_0} ds\, s^m \frac{1}{\pi} \mathrm{Im}\,\Pi(s)}{\int_0^{s_0} ds\, s^n \frac{1}{\pi} \mathrm{Im}\,\Pi(s)} \qquad (4.25)$$

which we have plotted for R_{21}, R_{10} and R_{20} in Fig. 7. They also show the nice stability - duality - as we have observed before (for comparison see Fig. 2a).

5. Summary and Conclusion

We investigated FESR in the ρ-meson channel. In principle they provide a simple tool to compute the vacuum condensates of a given mass dimension directly from the averaged hadronic spectral function. But unfortunately the values are strongly model dependent (see the Table). In our final calculation with the ρ-channel data we achieve both duality and stability of the results. Our main intention was to find the probable error. For that we used a MINUIT computer program. This is important since the FESR amplify the error of the condensates. Our results are consistent with the data in the sense that they pass the "heat evolution test". In contrast, the "standard values", Eqs. (1.4) and (4.2) together with $s_0 = 4\ GeV^2$ do not pass the test.

We notice a substantial increase of our condensate values in comparison with the "standard" ones. The condensate $C_4\langle O_4\rangle$ is a factor 2-5 larger and $|C_6\langle O_6\rangle|$ a factor 5-8. As brief conclusion, FESR support the need for a larger gluon condensate and also cast doubt on the validity of the vacuum dominance hypothesis for the condensate $C_6\langle O_6\rangle$.

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APPENDIX

In this section we want to present a derivation of FESR from QCD sum rules which differs from BLR [14]. It is valid not just for sufficiently large s_0 but for any finite value of s_0. The hadronic spectral function for the continuum in the ρ-channel is constant at a first stage

$$\frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) = c_0 = \frac{1}{8\pi^2} \qquad \text{for} \quad s \geq s_0 \tag{A}$$

since we first prove the pure asymptotic freedom case and incorporate the perturbative logarithmic corrections later on.

A) Derivation of FESR from Gauss Transforms

The keypoint now is to split the integral of the functions $U^{(\pm)}(\hat{s},\tau)$ [see Eqs. (3.5)] at the value s_0. We insert Hermite polynomials into the lower interval (see Eqs. (3.13) of Ref. [14]) and with the notation

$$x = \frac{s}{2\sqrt{\tau}} \quad , \quad \hat{x} = \frac{\hat{s}}{2\sqrt{\tau}} \quad \text{and} \quad x_0 = \frac{s_0}{2\sqrt{\tau}} \tag{A.1}$$

we get

$$U^{(+)}(\hat{x},\tau) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(2n)!} H_{2n}(\hat{x})\, e^{-\hat{x}^2} \int_0^{x_0} dx\, x^{2n}\, \frac{1}{\pi}\,\mathrm{Im}\,\Pi(2\sqrt{\tau}x) + \frac{c_0}{2}\left[\mathrm{Erfc}(x_0-\hat{x}) + \mathrm{Erfc}(x_0+\hat{x})\right] \tag{A.2}$$

and

$$U^{(-)}(\hat{x},\tau) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} H_{2n+1}(\hat{x})\, e^{-\hat{x}^2} \int_0^{x_0} dx\, x^{2n+1}\, \frac{1}{\pi}\,\mathrm{Im}\,\Pi(2\sqrt{\tau}x) + \frac{c_0}{2}\left[\mathrm{Erfc}(x_0-\hat{x}) - \mathrm{Erfc}(x_0+\hat{x})\right] . \tag{A.3}$$

Like BLR we start from the "principle of total heat conservation"

$$\int_0^\infty d\hat{x}\, H_{\substack{2n\\2n+1}}(\hat{x})\, U^{(\pm)}_{theory}(\hat{x},\tau) = \int_0^\infty d\hat{x}\, H_{\substack{2n\\2n+1}}(\hat{x})\, U^{(\pm)}_{exact}(\hat{x},\tau) \,. \tag{A.4}$$

Inserting the theoretical - , Eqs. (A.2), (A.3), and the exact distributions into relation (A.4) we obtain the following set of equations

$$2^{2n} \frac{C_0 C_{4n+2} \langle O_{4n+2} \rangle}{(2\sqrt{\tau})^{2n+1}} = \int_0^{x_0} dx\, (2x)^{2n} \frac{1}{\pi} \mathrm{Im}\,\Pi(2\sqrt{\tau}x) - \frac{C_0}{2} I_1 \tag{A.5}$$

$$-2^{2n+1} \frac{C_0 C_{4n+4} \langle O_{4n+4} \rangle}{(2\sqrt{\tau})^{2n+2}} = \int_0^{x_0} dx\, (2x)^{2n+1} \frac{1}{\pi} \mathrm{Im}\,\Pi(2\sqrt{\tau}x) + \frac{C_0}{2} I_2 \tag{A.6}$$

where $I_{1,2}$ represent the integrals

$$I_1 = \int_0^\infty d\hat{x}\, H_{2n}(\hat{x}) \left[\mathrm{erf}(\hat{x}+x_0) - \mathrm{erf}(\hat{x}-x_0) \right] \tag{A.7}$$

$$I_2 = \int_0^\infty d\hat{x}\, H_{2n+1}(\hat{x}) \left[\mathrm{erf}(\hat{x}+x_0) + \mathrm{erf}(\hat{x}-x_0) - 2\,\mathrm{erf}(\hat{x}) \right] . \tag{A.8}$$

These integrals are finite with the explicit result

$$I_1 = \frac{(2x_0)^{2n+1}}{2n+1} \tag{A.9}$$

$$I_2 = -\frac{(2x_0)^{2n+2}}{2n+2} \quad . \tag{A.10}$$

Combining all this together leads to the FESR

$$(-)^{n-1} C_0 C_{2n} \langle O_{2n} \rangle = \int_0^{s_0} ds\, s^{n-1} \frac{1}{\pi} \operatorname{Im} \Pi(s) - C_0 \frac{s_0^n}{n} \quad . \tag{A.11}$$

In order to include finally the radiative corrections we follow BLR [14] and replace

$$C_0 \longrightarrow C_0 F_{2n}(s_0) \tag{A.12}$$

with $F_{2n}(s_0)$ given by Eq. (2.2).

B. Derivation of FESR from Laplace Transforms

Next we derive FESR by starting with the Laplace transform. We use the zeta-function formalism which serves as a regulator for certain divergent integrals.

Guided by the non-relativistic analogy where the zeta-function associated to the Hamilton operator H can be written as

$$\zeta_H(p) = \frac{1}{\Gamma(p)} \int_0^\infty d\tau\, \tau^{p-1}\, \mathrm{tr}\, e^{-H\tau} \tag{B.1}$$

we define a zeta-function in field theory by the following integral representation

$$\zeta(p) = \frac{1}{\Gamma(p)} \int_0^\infty d\sigma\, \sigma^{p-1} \int_0^\infty ds\, e^{-\sigma s} \frac{1}{\pi} \operatorname{Im} \Pi(s) \tag{B.2}$$

$$= \frac{1}{\Gamma(p)} \int_0^\infty d\sigma\, \sigma^{p-1} \mathcal{M}(\sigma) \quad . \tag{B.3}$$

The second integral in (B.2) is just the Laplace transform

$$\mathcal{M}(\sigma) = \int_0^\infty ds\, e^{-\sigma s} \frac{1}{\pi} \operatorname{Im} \Pi(s) \quad . \tag{B.4}$$

The theoretical QCD expression contains the vacuum condensates

$$\mathcal{M}(\sigma) = \frac{C_0}{\sigma}\left[1 + \sum_n \frac{1}{(n-1)!} C_{2n}\langle O_{2n}\rangle \sigma^n\right] \quad . \tag{B.5}$$

Comparing now in the ζ-function formalism the exact expression with its QCD partner we obtain the FESR.

We split the integral of Eq. (B.2) at s_0

$$\zeta_{exact}(p) = \int_0^{s_0} ds\, s^{-p} \frac{1}{\pi} \mathrm{Im}\,\Pi(s) - C_0 \frac{s_0^{1-p}}{1-p} \tag{B.6}$$

and recall its QCD partner

$$\zeta_{QCD}(p) = \frac{1}{\Gamma(p)} \int_0^{\infty} d\sigma\, \sigma^{p-1} \cdot \tag{B.7}$$

$$\cdot\left[\frac{C_0}{\sigma} + C_0 C_2\langle O_2\rangle + C_0 C_4 \langle O_4\rangle \sigma + \cdots\right] \quad .$$

In taking the limit $p \to 0$ the term $C_0 C_2\langle O_2\rangle$ becomes the residue which is the only surviving term. The comparison

$$\zeta_{QCD}(p \to 0) \equiv \zeta_{exact}(p \to 0) \tag{B.8}$$

provides the first FESR

$$C_0 C_2 \langle O_2\rangle = \int_0^{s_0} ds\, \frac{1}{\pi} \mathrm{Im}\,\Pi(s) - C_0 s_0 \quad . \tag{B.9}$$

We proceed in this spirit and calculate the higher-dimensional condensates by working with the derivatives of the Laplace transforms. Therefore we define

$$\mathcal{G}^{(n-1)}(p) = \frac{1}{\Gamma(p)} \int_0^\infty d\sigma\, \sigma^{p-1} \int_0^\infty ds\, s^{n-1}\, e^{-\sigma s}\, \frac{1}{\pi} \mathrm{Im}\,\Pi(s) \tag{B.10}$$

$$= \frac{1}{\Gamma(p)} \int_0^\infty d\sigma\, \sigma^{p-1} \left(-\frac{d}{d\sigma}\right)^{n-1} \mathcal{M}(\sigma) \quad . \tag{B.11}$$

We again split the integration at s_0

$$\mathcal{G}^{(n-1)}_{exact}(p) = \int_0^{s_0} ds\, s^{n-1-p}\, \frac{1}{\pi} \mathrm{Im}\,\Pi(s) - C_0 \frac{s_0^{\,n-p}}{n-p} \quad , \tag{B.12}$$

and we compare it with

$$\mathcal{G}^{(n-1)}_{QCD}(p) = \frac{1}{\Gamma(p)} \int_0^\infty d\sigma\, \sigma^{p-1} \, . \tag{B.13}$$

$$\cdot \left[\frac{(n-1)!\, C_0}{\sigma^n} + \cdots + (-)^{n-1} C_0 C_{2n} \langle O_{2n} \rangle + \cdots \right] \quad .$$

In the limit $p \to 0$ the condensate $(-)^{n-1} C_0 C_{2n} \langle O_{2n} \rangle$ becomes the residue and is the only term which survives. This provides the FESR, Eq. (A.11).

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	$C_4\langle O_4\rangle$	$C_6\langle O_6\rangle$
"standard values"	0.034	-0.06
model A	0.65	-1.55
model B	0.39	-0.89
model C	0.30	-0.69
data fit	$0.14^{+0.05}_{-0.07}$	$-0.45^{+0.12}_{-0.06}$

Table

We have collected the values of the condensates $C_4\langle O_4\rangle$ and $C_6\langle O_6\rangle$ for "standard values", Eqs. (1.4) and (4.2); for model A, Eq. (4.7); for model B, Eq. (4.8); for model C; and finally for our ρ-meson data fit including the error given by the data.

FIGURE CAPTIONS

Figure 1: The data of the ρ-meson channel are plotted together with our best fit (as described in ch. 4).

Figure 2a: a) The QCD function $F_2(s_0)$, Eq. (2.2), is compared with $R_2(s_0)$, Eq. (4.3), for b) best fit curve of ρ-channel data; c) model B, Eq. (4.8); d) model A, Eq. (4.7).

Figure 2b: a) The function $F_2(s_0)$ is compared with b) $R_2(s_0)\pm\Delta R_2(s_0)$ for the data fit including experimental errors.

Figure 3: The condensate $C_4\langle O_4\rangle$ is plotted versus s_0 including the errors given by the data (as described in ch. 4).

Figure 4: The condensate $C_6\langle O_6\rangle$ is plotted versus s_0 including the errors given by the data (as described in ch. 4).

Figure 5a: "Heat evolution test": the theoretical Gauss transform (broken curve), Eq. (3.8), with FESR condensate values (4.9), (4.10) is compared with the Gauss transform of our data fit (solid curve), Eq. (3.1) at a "time" $\tau = 0.1$ GeV^4.

Figure 5b: "Heat evolution test" as in Fig. 5a, but for $\tau = 0.5$ GeV^4.

Figure 5c: "Heat evolution test" as in Fig. 5a, but for $\tau = 1.0$ GeV^4.

Figure 6a: "Heat evolution test": the theoretical Gauss transform (broken curve) with the "standard values" (1.4), (4.2) is compared with the Gauss transform of our data fit at a "time" $\tau = 0.1$ GeV^4.

Figure 6b: "Heat evolution test" as in Fig. 6a, but for $\tau = 0.5$ GeV^4.

Figure 6c: "Heat evolution test" as in Fig. 6a, but for $\tau = 1.0$ GeV^4.

Figure 7: The ratios of moments R_{21}/s_0, R_{10}/s_0, R_{20}/s_0^2, Eq. (4.25), are plotted versus s_0. The solid curves correspond to the ρ-channel data fit and the broken lines to the QCD expressions.

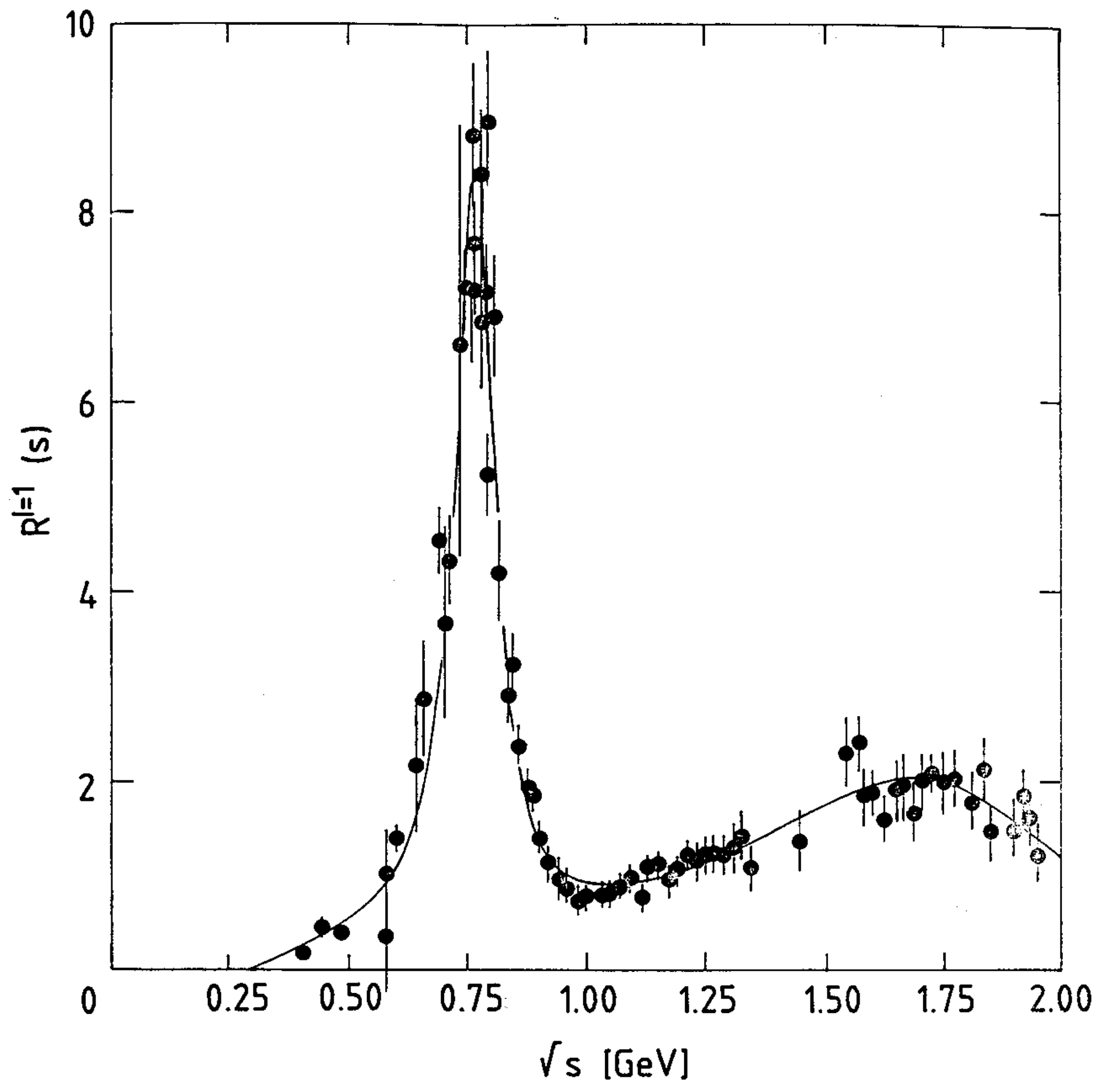


Fig. 1

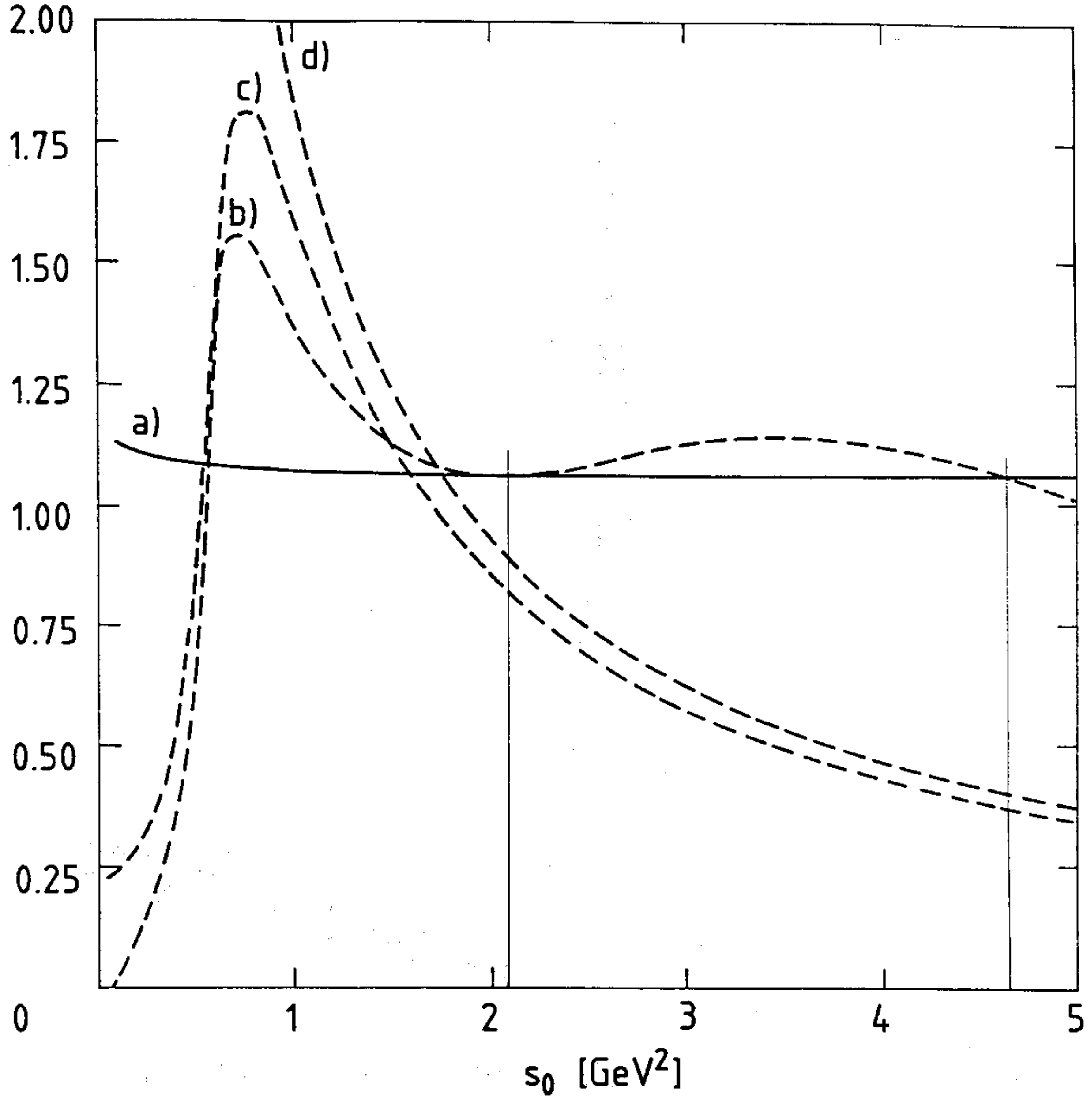


Fig. 2a

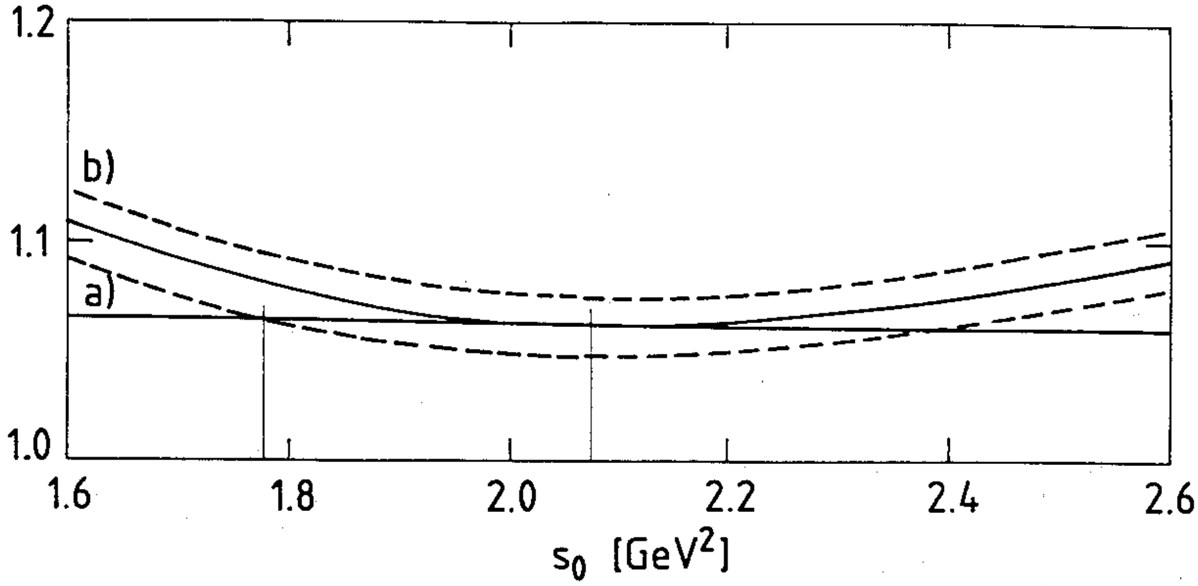
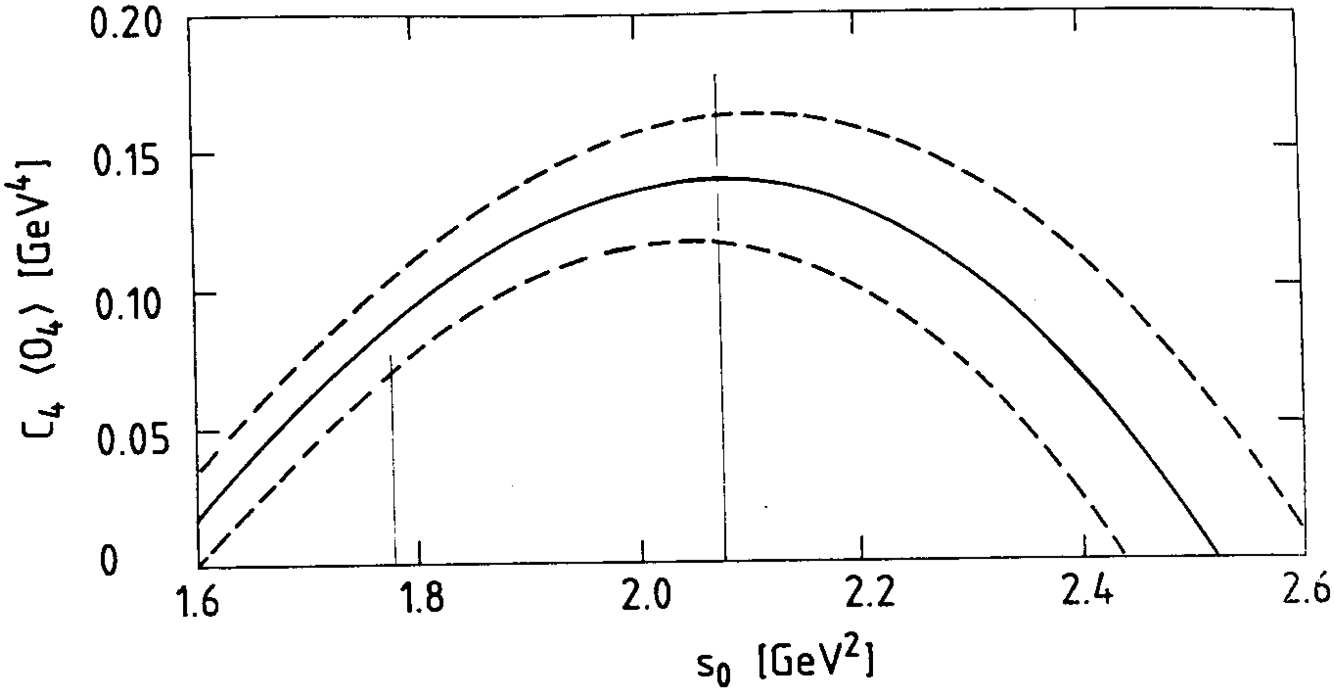


Fig. 2b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0.15
0.10
0.05
0
$C_4 \langle O_4 \rangle$ [GeV^4]
1.6
1.8
2.0
2.2
2.4
2.6
s_0 [GeV^2]

Fig. 3

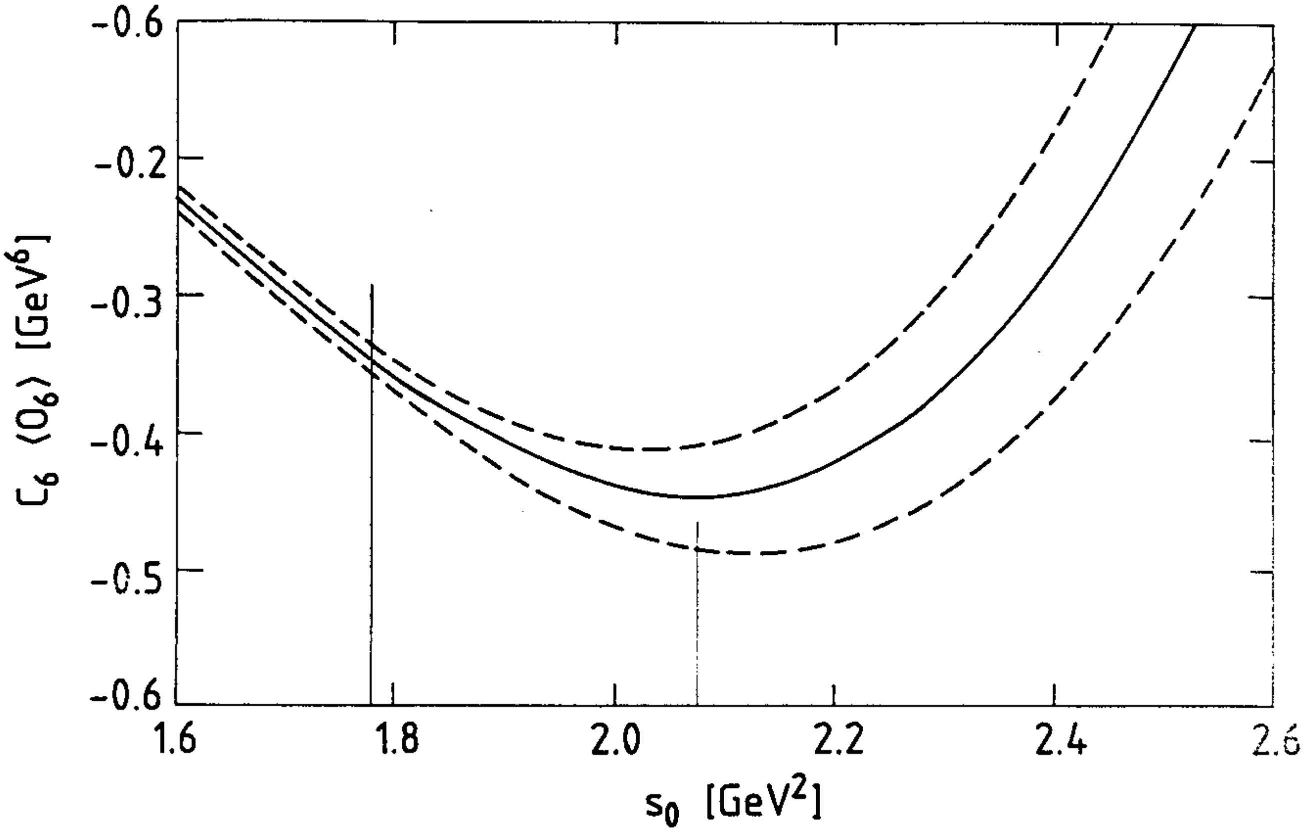
-0.6
-0.2
-0.3
-0.4
-0.5
-0.6
$C_6\ \langle O_6 \rangle$ [GeV^6]
1.6
1.8
2.0
2.2
2.4
2.6
s_0 [GeV^2]

Fig. 4

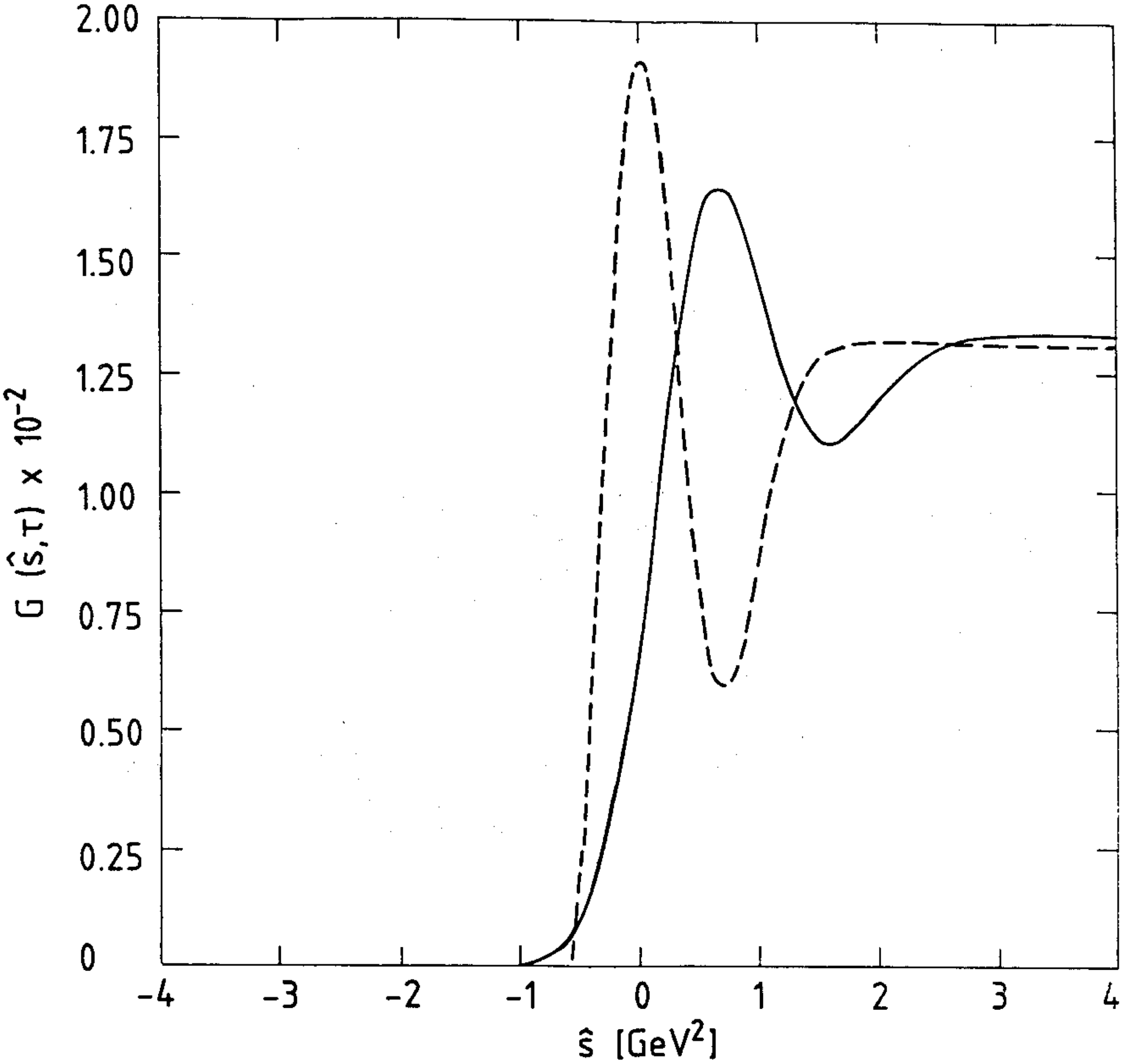

Fig. 5a

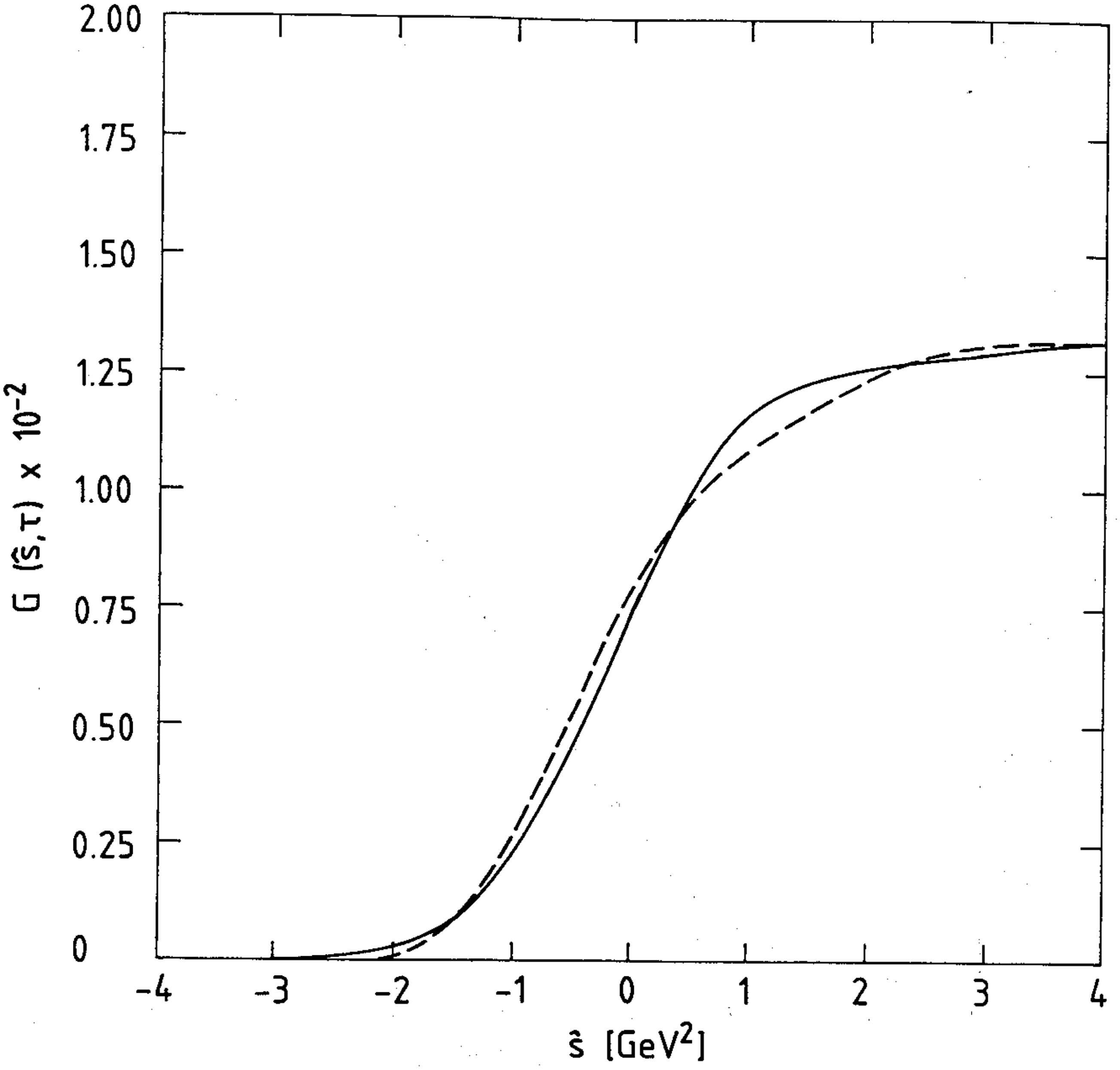

Fig. 5b

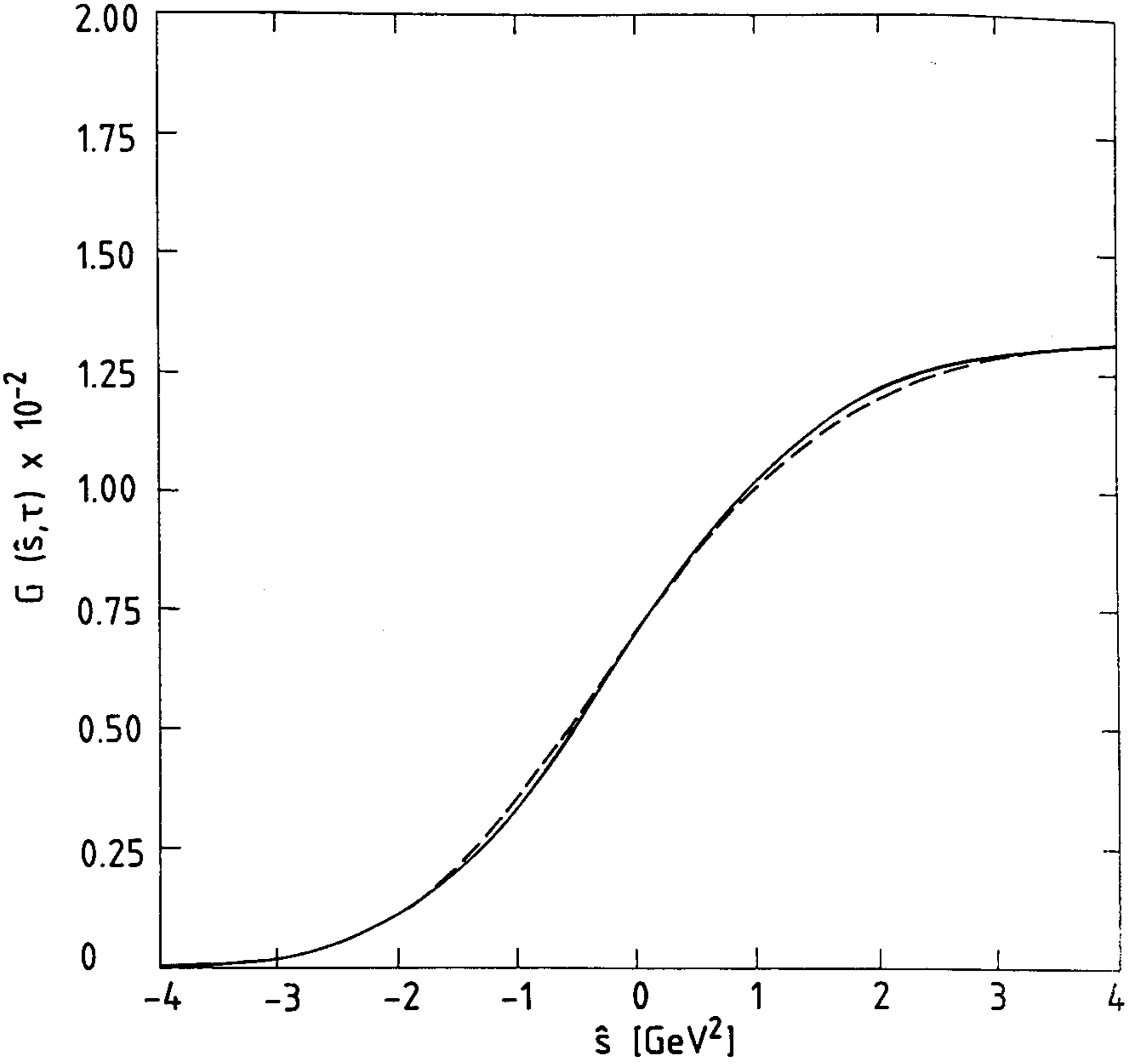

Fig. 5c

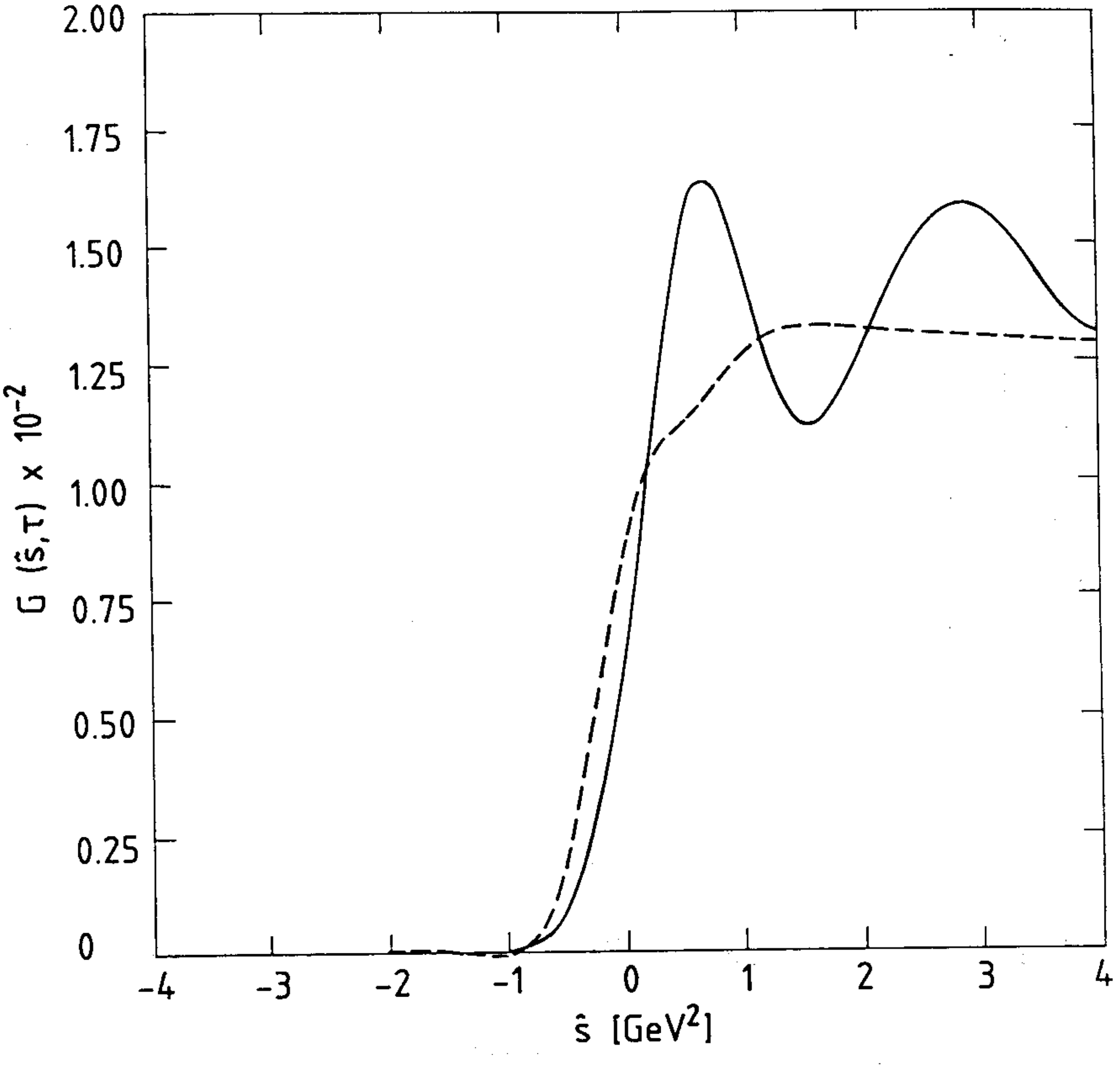

Fig. 6a

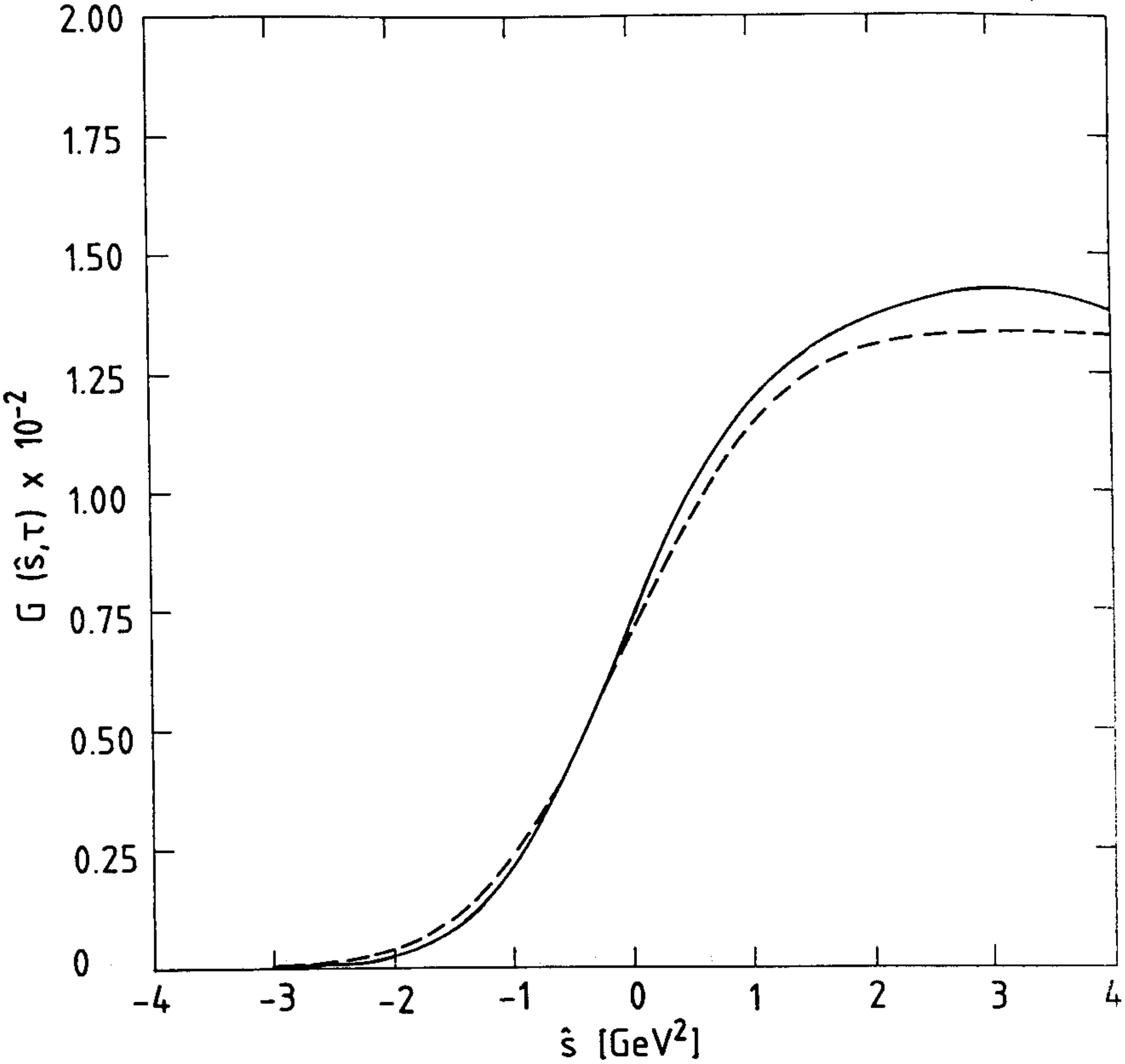

Fig. 6b

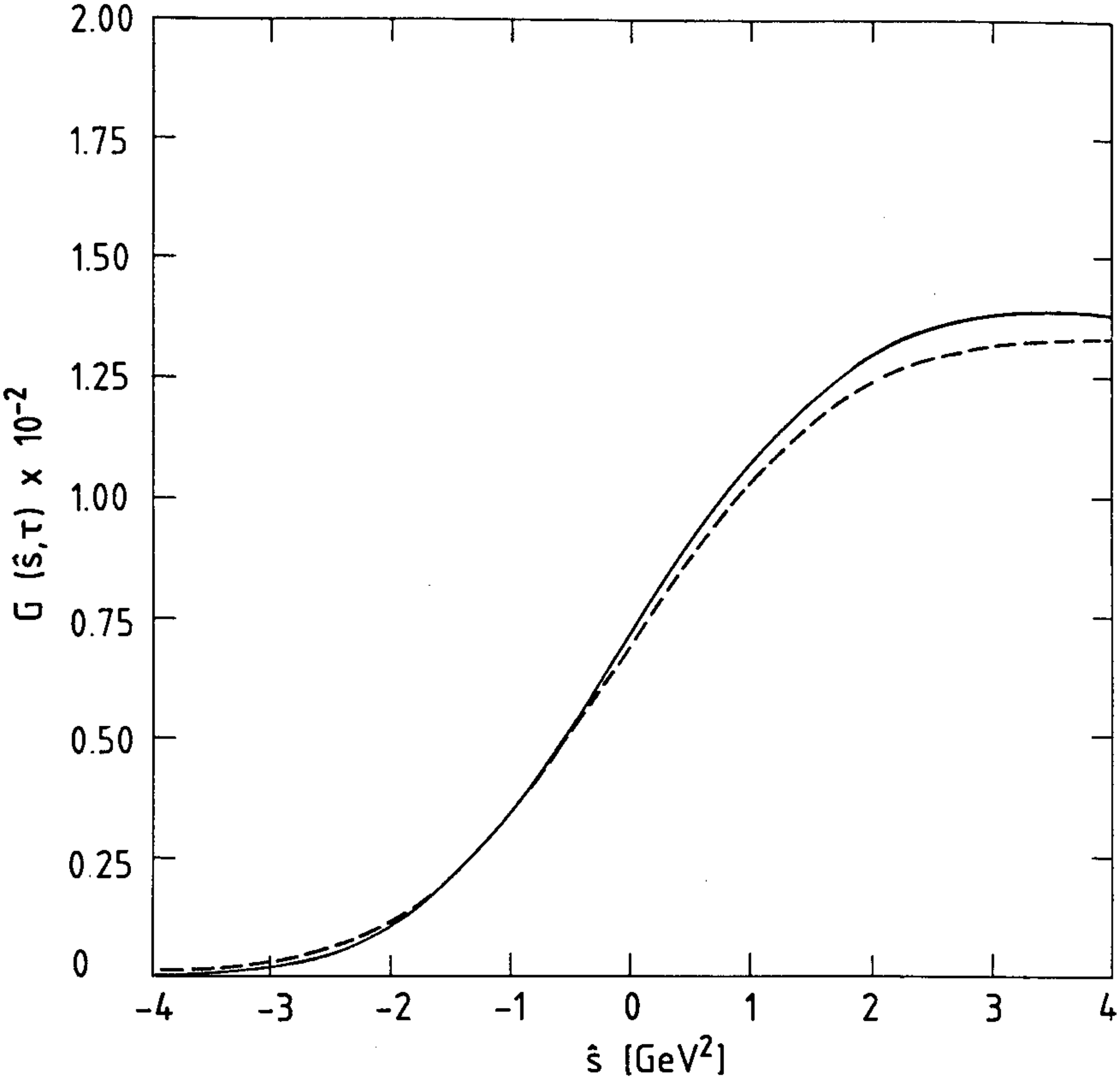

Fig. 6c

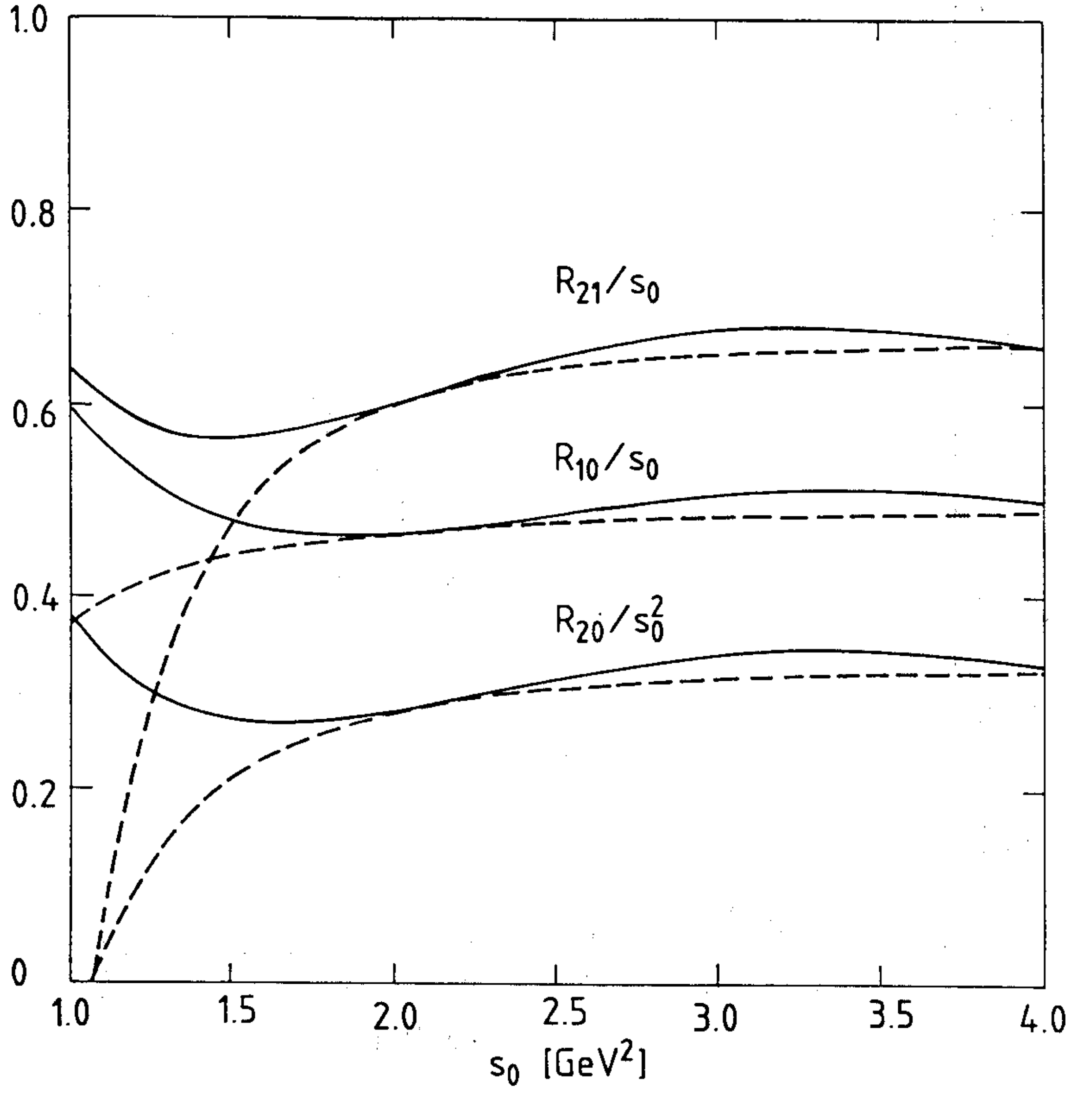

Fig. 7