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Determination of the Pion-Nucleon Coupling Constant by means of the Nucleon-Nucleon Dispersion Relation

Sadahiko MATSUYAMA

Department of Physics, University of Tokyo, Tokyo

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Making use of the dispersion relation for nucleon-nucleon scattering, the coupling constant between pion and nucleon is determined from the neutron-proton scattering length in triplet state and the binding energy of the deuteron. The coupling constant $g^2/4\pi$ thus determined coincides with those deduced by other theories.

§ 1. Introduction

Attempts to determine the pion-nucleon coupling constant have been made by various authors. Otsuki and his colleagues[1] were perhaps the first group that has succeeded to bring out this programme in a reliable way, and obtained $g^2/4\pi=0.065\sim0.090$. However, their arguments, based mainly on the electric quadrupole moment of the deuteron, depends on the nature of nuclear forces at short distances, and could not avoid some ambiguities in the course of calculations.* Concerning the pion-nucleon scattering and pion photo-production, calculations were made by Chew and Low[2] by using their own formulation and by Bernardini[3] et al. with the aid of the Kroll-Ruderman theorem.

Recently, the dispersion relations for pion-nucleon scattering of Goldberger and others[4] have enabled us to calculate the quantity in a way independent of the detailed character of meson theory, and many investigations have been made along this line.[5] However, the pion-nucleon scattering data at present contain large experimental errors, and do not permit a very accurate determination, leading to apparently inconsistent conclusions in some cases.[6] At any rate, the striking agreement between these values of the coupling constant determined in various ways gives a strong support to the pion theory of nuclear forces.

The dispersion relations for nucleon-nucleon scattering derived by Miyazawa and the present author[7] provides another approach for this problem. These formulae present a method to calculate the quantity in question *directly* (without solving the Schrödinger equation) from the scattering amplitudes (or phase shifts) of the nucleon-nucleon scattering.

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* Ohmura has shown that the restrictions imposed by them on the nature of nuclear forces at short distances are too stringent and that the lower limit of $g^2/4\pi$ becomes 0.05 when these points are remedied.

Since the experimental data for nucleon-nucleon scattering are more accurate than those for pion-nucleon scattering, especially in the low energy region, and since the binding energy of the deuteron is very well known, it seems likely at first glance that our formula provides a highly accurate means for this purpose. Unfortunately, however, we have no experimental data for this case *separately* for singlet and triplet states except for extremely low energies, and there appears a continuous spectrum in our formula in the unphysical region in contrast with the pion-nucleon case. These two situations give rise to some complications in our treatment, and obscures a part of the merits of our method.

In this connection, the value of $g^2/4\pi$ must be determined in such a way that the low energy scattering amplitudes give the main contribution to our result as will be illustrated in sections 2 and 3, where the effective range approximation plays an essential rôle.

Errors caused by the approximating methods in our calculations are estimated in detail in section 4. Suggestions as to how to increase the accuracy are made in the last section.

§ 2. Formulation

Our fundamental formula is the dispersion relation for nucleon-nucleon scattering in the laboratory system and in the forward direction with one subtraction, i. e.,

$$\frac{D^{(s)}(\omega_1)-D^{(s)}(m)}{\omega_1-m}=8\pi\frac{R^{-1}}{1-R^{-1}r}\frac{1}{m^2(\omega_1-\omega_D)(m-\omega_D)}-\frac{U^{(s)}(\omega_1)-U^{(s)}(m)}{\omega_1-m}$$
$$+\frac{1}{\pi}\mathrm{P}\int_m^\infty\frac{A^{(s)}(\omega)}{(\omega-\omega_1)(\omega-m)}d\omega. \qquad (2\cdot1)$$

All the notations are the same as those in I. The first term on the right-hand side originates in the fact that the two nucleons can form the deuteron, and should therefore be retained only in the case of neutron-proton scattering in triplet state. The second term is the integral of the same function as in the third term with the region of integration $\omega<m$. $U^{(s)}(\omega)$ reduces as was proved in I, to the Fourier transform of the nuclear force potential in the static limit. Therefore the second term can be calculated at least in principle in a field theoretical way and can be represented as a power series of g, the pion-nucleon coupling constant.

If we choose the energy ω_1 sufficiently small, the main contribution comes from the potential of the second order in g, fourth and higher order potential being small corrections to the second order one. We shall take in this paper only, the second and fourth order potentials neglecting all higher order effects.

Thus eq. $(2\cdot1)$ becomes an quadratic equation of $g^2/4\pi$, and we can determine the value of $g^2/4\pi$ from this equation if all the quantities in the other terms, D,

A, R and r, are known. Assuming the charge independence of coupling between pion and nucleon, we can write down the second order nuclear potential as

$$V^{(2)}(\omega)=-\frac{g^2}{\mu^2}\frac{k^2}{k^2+\mu^2}(\sigma^{(1)}\cdot e)(\sigma^{(2)}\cdot e)(\tau^{(1)}\cdot\tau^{(2)}), \tag{2·2}$$

where k is the momentum of the incident nucleon, $m^2+k^2=\omega^2$, and e the unit vector in the direction of $\boldsymbol{k}$. Pseudoscalar coupling and pseudovector coupling give the same result by the so-called equivalence theorem[8] as far as the second order potential is concerned.

As mentioned in section 1, we do not know at present any experiment that deals with the nucleon scattering in singlet and triplet states separately, except for extremely low energies, and therefore we must insert the averaged value of $A^{(s)}(\omega)$ and $D^{(s)}(\omega)$ into (2·1) if the high energy quantities give large contribution to this equation. On the other hand, if the average of the second order potential (2·2) is taken over the singlet and triplet states, it vanishes and the fourth order potential now becomes the main part of $U(\omega)$.

The Fourier transform of the fourth order potential, which arises from a part of the continuous spectrum of $A(\omega)$ in the unphysical region, cannot be calculated in any reliable way from our present stage of knowledge of quantum field theory, and therefore, our formula (2·1) gives no information about the value of the coupling constant if we choose ω_1 so that the high energy data contribute largely to (2·1).

At low energies (" thermal " energies) the neutron-proton scattering cross section is very well known for spin singlet and triplet states separately, and consequently we have a good knowledge about the scattering lengths for these respective states.

Moreover, for low energies, say, below 10 Mev in the centre-of-mass system, the S-matrix (and consequently $D(\omega)$ and $A(\omega)$) can be represented by only one phase shift (the S-wave in the case of singlet state and the so-called α-wave, the S- and D- wave admixture, in the triplet state), contributions of all other partial waves being negligibly small, and for the phase shift of the S-wave (or α-wave) the well-known effective range approximation holds.[9]

Therefore, if we choose ω_1 so small that for the left-hand side of (2·1) the effective range approximation holds and that the contribution of the fourth order potential to $U^{(s)}(\omega_1)-U^{(s)}(m)$ is sufficiently small in comparison with the second order, and if the integral of the last term is sufficiently convergent so that $A(\omega)$ for large value of ω (say, above 10 Mev in c. m. system) contributes to the integral very little, eq. (2·1) can be regarded as a relation of $g^2/4\pi$, the scattering length a and the effective range r (deuteron radius R can be represented by a function of a and r).

This is indeed the case, and we can determine the value of $g^2/4\pi$ from the scattering length in the triplet state and the binding energy B of the deuteron to a certain extent.

§ 3. Method of calculations and result

Denoting the phase shift of the S-wave (or the α-wave) by δ, the dispersive and absorptive part for the singlet state can be represented as follows:

$$D(\omega_c) = \frac{2\pi}{m}\frac{1}{k_c}\sin\delta\cos\delta$$

and

$$A(\omega_c) = \frac{2\pi}{m}\frac{1}{k_c}\sin^2\delta, \tag{3·1}$$

where the subscript c means "in the centre-of-mass system".

For the triplet state, where the effect of the tensor force is not negligible, we can write down the expressions for D and A in the same form as (3·1) provided that δ denotes the phase shift for α-wave, and the D and A the averaged values over the direction of the resultant spin.[10]

As we are mainly concerned with the non-relativistic energies, $k_c = k/2$, and the effective range approximation reads

$$k_c \cot\delta = -\frac{1}{a} + \frac{1}{2} r k_c^2. \tag{3·2}$$

On inserting (3·1) and (3·2) into (2·1), the integration of the last term can be carried out *analytically* without difficulty, and we have no need of a numerical integration of a singular function such as in (2·1) which has often been the source of an enormous error in the pion-nucleon case.

Let us consider the case of the neutron and proton in the singlet state. Inserting the numerical value $a = (23.69 \pm 0.06) \times 10^{-13}$ cm, and $r = 2.1 \sim 2.7 \times 10^{-13}$ cm, and taking $\omega_1 = 1.17$ Mev$+m$ in the laboratory system, we see that the term proportional to the nuclear force is extremely small (less than 1 per cent) compared with other two terms. This is perhaps due to the fact that the scattering length for this state is large and negative and the appearance of the virtual bound state makes the Born approximation (i. e. the Fourier transform of the nuclear potential) very much different from the scattering amplitude $D(\omega_1)$ for such a low energy. Owing to this fact and the large uncertainty of r, we cannot determine the value of the coupling constant from the singlet data, but we can see that our formula (2·1) is consistent with the value of $g^2/4\pi$ determined from other methods.

For the triplet state, the situation is much better. The scattering length is positive and much smaller than that for the singlet state, and the effective range r can be accurately determined by the relation

$$r = 2R^2\left(\frac{1}{R} - \frac{1}{a}\right)^*$$

* Strictly speaking, the quantity defined by this relation is not r of (3·2), but $\rho(0, -B)$ of Bethe.[11] Similarly, the r in the first term on the right-hand side of (2·1) should be replaced by $\rho(-B, -B)$. Errors caused by the identification of $\rho(0, -B)$ with r and $\rho(-B, -B)$ will be estimated in the next section.

and

$$R^{-1}=(mB)^{1/2}, \tag{3·3}$$

where B is the binding energy of the deuteron.

Inserting the experimental value for a and B known at present,

$$a=(5.377\pm0.023)\times10^{-13}\text{ cm},$$
$$B=2.226\pm0.004\text{ Mev}, \tag{3·4}$$

and taking $\omega_1=m+1.17$ Mev*, we see that the magnitude of the second term on the right-hand side of (2·1) is about 4 per cent of the left-hand side and that the experimental data (3·4) have an enough accuracy to determine the coupling constant.

Concerning the convergence of the integral in the last term of (2·1), we see that the most (98 per cent) of the contribution of the integral comes from the energy region below 10 Mev in the c. m. system, and therefore the use of the effective range approximation proves legitimate. Detailed discussions for this point will be given in the next section.

The second term on the right-hand side of (2·1), the term containing the Fourier transform of the nuclear force, is expanded in a power series of the square of the coupling constant, and the term of the second order in g^2 is immediately given by (2·2). The evaluation of the term proportional to g^4 should be carefully done for, if we calculate it with pseudovector coupling theory the divergence difficulty which cannot be amalgamated into the mass nor the coupling constant appears. On the other hand, if we start with pseudoscalar coupling theory, the main contribution comes from nucleon pair terms which in almost all the cases known at present show a violent disagreement with experiment.

We have therefore decided to calculate it with pv coupling with momentum cut-off at 6μ which has shown in some cases tolerably good agreement with experiment. As the fourth order term itself is a minor correction to the second order term (about 6 per cent) a rough estimation will suffice. Although the Fourier transform of the fourth order potential in pv coupling theory without cut-off itself is a quadratically divergent quantity, it reduces to logarithmic one when the subtraction procedure as in (2·1) is made. Thus this term becomes fairly cut-off-independent, e. g. even if we take the cut-off momentum 3μ instead of 6μ, our result, $g^2/4\pi$, is changed only by 1 per cent. Therefore it is not unreasonable to suppose that our procedure, although crude may it seem, gives a fairly reliable result so far as $g^2/4\pi$ is concerned. All the terms of the orders higher than g^4 are neglected.

The result thus obtained is

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* This energy corresponds to the momentum $k_1=0.05\ mc$. The choice of the value for ω_1 is of course arbitrary so long as the approximation methods described above hold. Indeed, precisely the same result is obtained also when we put e. g. $k_1=0.01\ mc$.

$$g^2/4\pi = 0.080 \pm 0.007. \tag{3·5}$$

§ 4. Estimation of the errors due to the approximation method

The method of calculations described in the preceding section contains some approximations in certain respects. Since the term in our formula containing the coupling constant is represented by a small difference of large quantities, the accuracy of these approximations must be carefully examined.

In this section, we shall deal with this problem in detail, and find that our result (3·5), although surprisingly accurate may it seem, is subject to some uncertainties.

a) The effect of P-wave

Phase shifts for triplet P state are given by Worthington and others[12] for the case of proton-proton scattering. Charge independence of the nuclear forces prescribes that the same phase shifts apply also in our case. They are so small that when inserted in our formula averaged with respect to resultant spin directions give rise to no appreciable change in our result. This fact suggests that the effects of partial waves of higher angular momentum can be safely neglected.

b) As the integral in the last term of (2·1) extends to high energy region, this term may deviate from the original one if we take a more accurate formula

$$k_c \cot \delta = -\frac{1}{a} + \frac{1}{2} r k_c^2 - P r^3 k_c^4, \tag{4·1}$$

instead of (3·2).

The numerical value of P is not exactly known at present either experimentally or theoretically, but it is plausible that P does not exceed 0.05.[13] If we assume $P=0.05$, the value of the integration increases by 0.083 per cent and the value of $g^2/4\pi$ decreases by 1.3 per cent.

c) Differences between r and $\rho(-B, -B)$ and $\rho(0, -B)$

When the effective range approximation (3·2) exactly holds, there is no difference between r and $\rho(-B, -B)$ and $\rho(0, -B)$, but if we take into account the last term of (4·1), slight differences between them come forth. They are given by

$$\rho(-B, -B) = \left(1 + 2\frac{\rho^2}{R^2} P\right) \rho(0, -B)$$

and

$$r = \left(1 - 2\frac{\rho^2}{R^2} P\right) \rho(0, -B),$$

and change our result by an appreciable amount. Even if we put the value of P as small as 0.01, the first term on the right-hand side of (2·1) increases by 0.19 per cent, the other terms increasing very slightly, and the $g^2/4\pi$ increases by 9.4 per cent.

§ 5. Conclusion

(3·5) is our main result. This result has been obtained by putting $P=0$, and the probable error indicated here is purely of experimental origin. About 4/5 of it comes from the uncertainty of the triplet scattering length and the rest from that of the binding energy of the deuteron. Inclusion of the last term of (4·1) may change our result to an appreciable amount, and therefore we cannot determine the value of the coupling constant very accurately by our method.

However, it is to be remarked that our result agrees well with those obtained by quite different methods. The good agreement of our result obtained by putting $P=0$ with others may be an indication of the smallness of P. If we can obtain the value of P, in either experimental or theoretical way, the accuracy of our calculation will be much increased and another test of the consistency of the meson theory will be presented.

In the singlet state, where the effect of P is less important, more accurate measurement of the scattering length is desired.

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