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Determination of the rheological properties of thin plate under transient vibration

Abstract
The article deals with systematic analysis of the transient vibration of rectangular viscoelastic orthotropic thin 2D plate. The analysis is focused on specific deformation models of plate and for specific linear models of rheological properties. The viscoelastic isotropic and anisotropic (orthotropic) materials were investigated. The plate is loaded by general transient impulse. The time and coordinate dependencies of the fundamental quantities – displacements, rotations, stress and deformations in arbitrary points of plate under transient vibration, e. g. the analysis of stress and deformation waves in plates were investigated. The selected models are defined by constitutive equations for stress and deformation dependence. The isotropic and orthotropic model is considered. The analysis results depend on quality of the determined mechanical properties of the materials. The standard methods commonly used are time-consuming and not accurate enough. The new methodology of determining the material parameters directly from investigated plate is proposed and proven.

Keywords
Rheologic properties, thin plate, transient vibration

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1 INTRODUCTION

The theory of the plate vibration belongs to the most explored realm of mechanics of deformable plates. Authors have dealt with this realm more than 200 years. The hundred studies and monographs have been published till present time. The results of vibration investigation were obtained by many analytical, numerical and experimental methods. The most sophisticated analytical solution is focused on transient vibrations of thin elastic isotropic plate. The anisotropic plate e.g. orthotropic plate is less commonly discussed, e.g. [2]. The investigation of transient vibration of viscoelastic plate appears in the first half of the last century and this problem has been examined by significantly less articles [7]. The essential evolution of transient vibration of the viscoelastic plate begun after technology development of polymers (macromolecular materials) [6] and composites until second half of last century, first for Kirchhoff's model of plate [3, 4], later for Rayleigh model. The investigation of the viscoelastic anisotropic plate is progressing realm of mechanics now.

The article deals with the investigation of transient vibrations and stress wave propagation in viscoelastic orthotropic thin plate. The phenomenological models are commonly applied for description of material mechanical properties under transient states of viscoelastic bodies. The constitutive equations define relations of stress and deformation of investigation continuum by superposition of elastic and viscoelastic components (represented by spring and viscous damper respectively). The different models of viscoelastic heredity material were developed by various combinations (parallel or serial) of these elements, e.g. Voight-Kelvin, Maxwell, Zener etc [5].

The general problem is determination of the stress modulus of the viscoelastic orthotropic material (e.g. composite). The contemporary methods of stress modulus determination are based on static measurement of the modulus of individual elements. The final modulus of model is determined from conversion of measured modulus by volume fractions. This method is not satisfactory accurate and the results obtained for dynamic loading of the model are disputable.

These materials are wide used, e.g. in automotive industry, rail vehicles, planes, ships, space research, civil engineering etc. The composites and laminated materials have indisputable preferences (lower weight, higher strength, time stability etc.). It is obvious that the high requirements will be demanded to computational methods for these materials applications in the future. The presented contribution deals with one of the important assumptions for accuracy improvement of these calculations and determination of elasticity modulus of these materials under dynamic loading. The proposed method would be applicable for composite and sandwich construction and laminates.

2 SOLUTION

Most of the commonly used works for investigation of stress analysis in linear viscoelastic bodies solve constitutive equations in differential form. The features of these linear models for one axis tension are described in Alfrey [1] by equation

$$p_0\sigma + \sum_{r=1}^{n} p_r \frac{\partial^r \sigma}{\partial t^r} = q_0\varepsilon + \sum_{r=1}^{m} q_r \frac{\partial^r \varepsilon}{\partial t^r} \tag{01}$$

where $p \geq 0$, $q \geq 0$ are material parameters, ε - deformation, σ - stress, n – number of material parameters p, m – number of material parameters q.

If $m = n$ then it is trivial case

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} = ... = \frac{q_r}{p_r} \Rightarrow \sigma(t) = const.\varepsilon(t) \tag{01a}$$

When $m \neq n$, the important cases are for $m = (n-1)$ or $m = (n+1)$. These models correspond to fundamental phenomenological models of viscous continuum. It is necessary to experimentally obtain the values of *(m+n)* material parameters, when these models are applied. The differential equations of constitutive equations are more complicated under multi-axis tension.

For two-axis tension (normal stress σ_x, σ_y, in axis direction, and shear stress σ_{xy}), the constitutive equations for hereditary model of viscoelastic material could be expressed by equation that has analogous form to Alfrey's form for one-axis tension.

Normal stress

$$p_{0,i}\sigma_i + \sum_{r=1}^{n} p_{r,i}\frac{\partial^r \sigma_i}{\partial t^r} = q_{0,i}\varepsilon_i + \sum_{r=1}^{m} q_{r,i}\frac{\partial^r \varepsilon_i}{\partial t^r} + q_{0,j}\varepsilon_j + \sum_{r=1}^{m} q_{r,j}\frac{\partial^r \varepsilon_j}{\partial t^r} \tag{02}$$

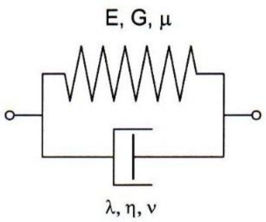
for $i, j \in \{x, y\}$ and $i \neq j$.

Shear stress

$$p_{0,xy}\sigma_{xy} + \sum_{r=1}^{n} p_{r,xy}\frac{\partial^r \sigma_{xy}}{\partial t^r} = q_{0,xy}\varepsilon_{xy} + \sum_{r=1}^{m} q_{r,xy}\frac{\partial^r \varepsilon_{xy}}{\partial t^r} \tag{3}$$

The constitutive equation could be expressed in form by technical parameters (considered basic simplest models).

Voigt-Kelvin model



Normal stress for isotropic material

$$\sigma_i = \frac{E}{1-\mu^2}(\varepsilon_i + \mu\,\varepsilon_y) + \frac{\lambda}{1-\mu^2}\left(\frac{\partial \varepsilon_i}{\partial t} + \mu\frac{\partial \varepsilon_j}{\partial t}\right) \tag{4a}$$

Normal stress for orthotropic material

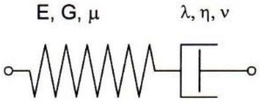
$$\sigma_i = \frac{E_i}{1-\mu_{ij}\mu_{ji}}(\varepsilon_i + \mu_{ji}\,\varepsilon_j) + \frac{\lambda_i}{1-\mu_{ij}\mu_{ji}}\left(\frac{\partial \varepsilon_i}{\partial t} + \mu_{ji}\frac{\partial \varepsilon_j}{\partial t}\right) \tag{4b}$$

Shear stress

$$\sigma_{xy} = G\varepsilon_{xy} + \eta\frac{\partial \varepsilon_{xy}}{\partial t} \tag{4c}$$

where E, G are the Young and Kirchhoff modules, λ, η are the coefficients of normal and tangential (shear) viscosity, μ, ν are the elastic and viscous Poisson ratios.

Maxwell model



Normal stress for isotropic material

$$\sigma_i + \frac{\lambda}{E}\frac{\partial \sigma_i}{\partial t} = \frac{E}{1-\mu^2}\delta\left(\frac{\partial \varepsilon_i}{\partial t} + \mu\frac{\partial \varepsilon_j}{\partial t}\right) \tag{5a}$$

Normal stress for orthotropic material

$$\sigma_i + \frac{\lambda_i}{E_i}\frac{\partial \sigma_i}{\partial t} = \frac{E_i}{1-\mu_{ij}\mu_{ji}}\delta_i\left(\frac{\partial \varepsilon_i}{\partial t} + \mu_{ji}\frac{\partial \varepsilon_j}{\partial t}\right) \tag{5b}$$

Shear stress

$$\sigma_{xy} + \frac{\eta}{G}\frac{\partial \sigma_{xy}}{\partial t} = \eta\frac{\partial \varepsilon_{xy}}{\partial t}, \qquad \text{where} \quad \delta = \frac{E}{\lambda}, \qquad \delta_i = \frac{E_i}{\lambda_i}. \tag{5c}$$

It is possible to obtain the equations for stress components in integral form by solution of differential equation (5a) for σ_i, or (5c) for σ_{xy} respectively.

Normal stress for isotropic material

$$\sigma_i = \frac{E}{1-\mu^2}\int_0^t \left(\frac{\partial \varepsilon_i(\tau)}{\partial \tau} + \mu\frac{\partial \varepsilon_j(\tau)}{\partial \tau}\right)e^{-\delta_i(t-\tau)}d\tau \tag{6a}$$

After integration by parts, we got

$$\sigma_i = \frac{E}{1-\mu^2}\left[\varepsilon_i(t) + \mu\,\varepsilon_j(t) - \delta_i\int_0^t \left(\frac{\partial \varepsilon_i(\tau)}{\partial \tau} + \mu\frac{\partial \varepsilon_j(\tau)}{\partial \tau}\right)e^{-\delta_i(t-\tau)}d\tau\right] \tag{6b}$$

Normal stress for orthotropic material

$$\sigma_i = \frac{E_i}{1-\mu_{ij}\mu_{ji}}\int_0^t \left(\frac{\partial \varepsilon_i(\tau)}{\partial \tau} + \mu_j\frac{\partial \varepsilon_j(\tau)}{\partial \tau}\right)e^{-\delta_i(t-\tau)}d\tau \tag{6c}$$

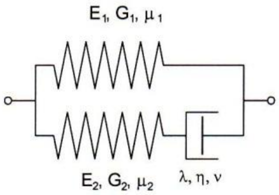
After integration by parts

$$\sigma_i = \frac{E_i}{1-\mu_{ij}\mu_{ji}}\left[\varepsilon_i + \mu_j\varepsilon_j - \delta_i\int(\varepsilon_i(\tau) + \mu_j\varepsilon_j(\tau)e^{-\delta_i(t-\tau)})\,d\tau\right] \tag{6d}$$

Shear stress

$$\sigma_{xy} = G\int_0^t \frac{\partial \varepsilon_{xy}(\tau)}{\partial \tau} e^{-\delta_{xy}(t-\tau)} d\tau \,, \qquad \text{where} \qquad \delta_{xy} = \frac{G}{\eta} \tag{6e}$$

Zener model



Normal stress for isotropic material

$$\sigma_i + \frac{\lambda_2}{E_2}\frac{\partial \sigma_i}{\partial t} = \frac{E_1}{1-\mu_1^2}(\varepsilon_i + \mu_2 \varepsilon_j) + + \frac{\lambda_2}{E_2}\frac{E_2 + E_1}{1-\mu_2^2}\left(\frac{\partial \varepsilon_i}{\partial t} + \mu_2 \frac{\partial \varepsilon_j}{\partial t}\right) \tag{7a}$$

Normal stress for orthotropic material

$$\sigma_i + \frac{\lambda_i}{E_{2i}}\frac{\partial \sigma_i}{\partial t} = \frac{E_{1i}}{1-\mu_{1ij}\mu_{1ji}}(\varepsilon_i + \mu_{xy}\varepsilon_j) + \frac{\lambda_i}{E_{2i}}\frac{E_{2i}+E_{1i}}{1-\mu_{2ji}\mu_{2ij}}\left(\frac{\partial \varepsilon_i}{\partial t} + \mu_{2j}\frac{\partial \varepsilon_j}{\partial t}\right) \tag{7b}$$

Shear stress

$$\sigma_{xy} + \frac{\eta}{G_2}\frac{\partial \sigma_{xy}}{\partial t} = \varepsilon_{xy} G_1 + \eta\left(1 + \frac{G_1}{G_2}\right)\frac{\partial \varepsilon_{xy}}{\partial t} \tag{7c}$$

We get the equations for stress components in integral form by solution of differential equation (7b) for σ_i, for orthotropic material and zero initial conditions.

$$\sigma_i = \frac{E_{1i}}{1-\mu_{1ij}\mu_{1ji}}(\varepsilon_i + \mu_{1ji}\varepsilon_j) + \frac{E_{2i}}{1-\mu_{2ji}\mu_{2ij}}\left[\varepsilon_i + \mu_{2ji}\varepsilon_j - \delta_i \int (\varepsilon_2(\tau) + \mu_{2ji}\varepsilon_j(\tau)) e^{-\delta_i(t-\tau)} d\tau\right] \tag{7d}$$

The specific material parameters E, G, μ, λ, η are necessary to determine by experiments. More often it is determined from simple bodies with static experiments and then it is applied to bodies with more complicated shapes and complicated loading process. The number of parameters arises from particular cases mentioned above.

The relationship between stress and deformation for hereditary viscoelastic materials is possible to express by Duhamel integral. Integral operator for one-axis tension expresses the fundamental constitutive equation.

$$\sigma(t) = \int_0^R R(t-\tau)\frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau \,, \quad \text{respective} \quad \varepsilon(t) = \int_0^J J(t-\tau)\frac{\partial \sigma(t)}{\partial t} d\tau \tag{08}$$

The relationship of both expressions in the differential (Alfrey) and integral (Volterra) form is possible to formulate after Laplace transformation (s – parameter of transformation)

$$R(s) = \frac{\sum_{i=1}^{n} p_i \sigma^{(i)}(s)}{s \sum_{i=1}^{m} q_i \varepsilon^{(i)}(s)} \tag{9a}$$

$$J(s) = \frac{\sum_{i=1}^{m} q_i \varepsilon^{(i)}(s)}{s \sum_{i=1}^{n} p_i \sigma^{(i)}(s)} \tag{9b}$$

where $R(s).J(s) = s^{-2}$, and $R(t\text{-}\tau).J(\tau) = t$.

The Boltzmann principle of superposition allows to write by adjustment of (1) for $p_0 = 1$ substituting into (8)

$$\sigma(t) = q_0\, \varepsilon(t) + \int_0^t K(t-\tau)\, \varepsilon(\tau)\, d\tau \tag{10}$$

The relationship of both solutions is obvious from comparison of (5a), (7a), (8) and (10).

Conclusion

It is more comfortable and less time-consuming to use constitutive equations in proposed form (approximation of time function, relaxation module $K(t - \tau)$), than to determine of many parameters.

From equation (10) arise the necessity of determination of parameter q_0 and function $K(t){=}Ae^{-\delta t}$, which approximate function of viscous component relaxation (module), parameters A and δ. The approximation of this function is less difficult than determination of material parameters (E, G, μ, λ, η).

In case of transient states investigation for thin viscoelastic plate vibration is possible to derive equations for stress components or displacement (time dependent) by integral expression of constitutive equations (8) and (10) respectively. The required parameters could be obtained from comparison of stress or displacement components find out by experimental and analytical methods.

The relaxation module function could be experimentally determined directly on investigated body. The methodology for experimental investigation of this function was proposed and it is under patent application.

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