# Determining eigenvalues of a density matrix with minimal information in a single experimental setting

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Eigenvalues of a density matrix characterize well the quantum state's properties, such as coherence and entanglement. We propose a simple method to determine all the eigenvalues of an unknown density matrix of a finite-dimensional system in a single experimental setting. Without fully reconstructing a quantum state, eigenvalues are determined with the minimal number of parameters obtained by a measurement of a single observable. Moreover, its implementation is illustrated in linear optical and superconducting systems.

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The eigenvalues of a density matrix are fundamental quantities in quantum physics and characterize many quantum properties, such as coherence and entanglement. Typically, a function of the density matrix's eigenvalues allows us to examine features of a quantum state. The von Neumann entropy, for example, is defined as the Shannon entropy of eigenvalues of a density matrix and has different applications such as thermodynamic entropy [1], optimal compression rate of a quantum state [2], and entanglement measure [3]. Other entropies of eigenvalues, e.g., the Rényi entropy and the Tsallis entropy [4], can have curious applications in thermodynamics and statistical mechanics. The entanglement spectrum [5] depends on spectrum of a reduced density matrix and is useful for studying the ground-state properties of many-body quantum systems in low dimensions. Thus, the eigenvalues of a density matrix (and their functions) are probes into quantumness in various issues.

Developing a method for determining eigenvalues of an unknown quantum state is highly desirable, because one can experimentally test many theoretical ideas about sensing quantum features. A fundamental question here is whether the method is simple enough and implemented by a small number of experimental setting, ideally by a single setup. We first consider a simple approach. Reconstructing a density matrix via quantum-state tomography [6] leads to determining all the eigenvalues. In a $d$-level quantum system, however, we have $(d^2-1)$ numbers of free parameters to be fixed in the reconstruction, so it is highly redundant to determine $d$ numbers of the eigenvalues, where only $(d-1)$ numbers of free parameters (with the normalization) are necessary. Actually, without a full reconstruction of a quantum state, the eigenvalues of a density matrix are attainable. We only need to know the moments of a density matrix $\rho$, as seen in, e.g., Ref. [7]. There are experimental proposals [8–10] to directly measure $\mathrm{tr}\,\rho^k$ $(k=2,\ldots,d)$. Since the number of the unknown eigenvalues is equal to that of the measured quantities, this approach is regarded as an eigenvalue determination with minimum information. We call such a method *minimal*. An eigenvalue determination with minimum information is considered to be simple enough, since there is no redundancy. The minimality of information gain is important in quantum systems, from the viewpoint of the information-disturbance relation [11,12]. Extracting minimal information can lead to suppressing unnecessary disturbance in a quantum state.

Next, let us consider the implementability of eigenvalue determination, with a single setup. In proposed methods [8–10] with the minimality, the $k$th moment of $\rho$ is measured by the expectation value of an observable under an identically and independently distributed (i.i.d.) state, $\rho^{\otimes k}$. Thus, this approach requires $(d-1)$ kinds of the experimental settings to determine all the eigenvalues. A single-setup determination is built straightforwardly by a measurement of an information complete positive operator-valued measure (see, e.g., Ref. [13]). We recall that the information completeness is defined by the state-reconstructing ability from statistical measurement data. One can reconstruct a density matrix, via a single experimental setting for such a measurement [14]. However, as pointed out above, this method does not have minimality. An alternative way for obtaining all the moments of $\rho$ with a single setup is to use random unitary operations on a single system [15]. This method is applicable to photon qudits passing through a disordered medium [16], although one must guarantee the uniformity of the random operations.

In this article, we propose a method for measuring the eigenvalues of a density matrix, equipped with both minimality and single-setup implementability. Our proposal for a $d$-level quantum system consists of a measurement of a single $d$-valued observable, or equivalently, a measurement of a projection-valued measure (PVM) measurement $(P_i)_{i=1}^d$, on a $d$-i.i.d. state, $\rho^{\otimes d}$. Our central idea is to build a specific unitary gate in a multipartite system. We will show that the characteristic polynomial of any density matrix $\rho$ is constructed in terms of probability distributions $\mathrm{tr}\,(P_i\rho^{\otimes d})$ as

$$\det(x-\rho)=\sum_{i=1}^{d} m_i(x)\,\mathrm{tr}\,(P_i\rho^{\otimes d}) \quad (x\in\mathbb{R}). \tag{1}$$

Here $m_i(x)$ $(i=1,\ldots,d)$ is a known polynomial of $x$, determined according to the choice of the single observable. From the PVM measurement $(P_i)_{i=1}^d$ on state $\rho^{\otimes d}$, we obtain the probability distribution $\mathrm{tr}\,(P_i\rho^{\otimes d})$. Then, we can calculate all the eigenvalues of $\rho$, via formula (1). This process can be efficiently performed by a classical computer, since the problem is a 1D root finding, simpler than a full matrix

 

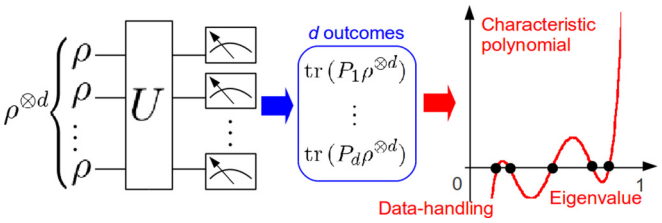


FIG. 1. (Color online) Schematic diagram of a single-observable-based method for an eigenvalue determination of a density matrix. First, one prepares an identically and independently distributed state, with finite copies of a target density matrix $\rho$. The number of the copies is equal to the dimension $d$ of a system. Next, $d$-outcome measurements with orthogonal projectors $(P_i)_{i=1}^d$ are performed by a global unitary gate $U$ and local measurements. Then, one numerically finds the root of the characteristic polynomial reconstructed from the measured data.

diagonalization. Figure 1 is the summary of our proposal. Moreover, we will show methods for implementing our proposal in physical systems.

One of the simple realizations of our approach is to use the antisymmetrizer (projector on a fermionic ground state). To see this, let us start with some mathematical ingredients. The unitary operator on $\mathcal{H}^{\otimes k}$ associated with a permutation $\sigma \in \mathfrak{S}_k$ over $k$ integers $\{1,\ldots,k\}$ is

$$U_\sigma |\phi_1\rangle \ldots |\phi_k\rangle = |\phi_{\sigma(1)}\rangle \ldots |\phi_{\sigma(k)}\rangle, \tag{2}$$

for $|\phi_i\rangle \in \mathcal{H}$ $(i = 1,\ldots,k)$. An $m$-cycle $c_m \in \mathfrak{S}_k$ is a permutation to cycle $m$ distinct integers from $\{1,\ldots,k\}$, with others being fixed. Noting that $\mathrm{tr}\,\rho = 1$, we have

$$\mathrm{tr}\,(U_{c_m}\rho^{\otimes k}) = \mathrm{tr}\,\rho^m \quad (m = 2,\ldots,k), \tag{3}$$

for any $m$-cycle $c_m$. Thus, each moment of $\rho$ is related to a physical process. An observable for determining $\mathrm{tr}\,\rho^m$ can be constructed by taking the Hermitian part of $U_{c_m}$, as seen in, e.g., Ref. [8]. For a general permutation $\sigma$, we can use the unique decomposition by cycles (see, e.g., Ref. [17]) to obtain

$$\mathrm{tr}\,(U_\sigma\rho^{\otimes k}) = \prod_{m=1}^{k} \mathrm{tr}\,(\rho^m)^{j_m(\sigma)}, \tag{4}$$

where $j_m(\sigma)$ is the number of the $m$ cycles in $\sigma$. For instance, we have $\mathrm{tr}\,(U_\sigma\rho^{\otimes 7}) = (\mathrm{tr}\,\rho^2)^2(\mathrm{tr}\,\rho^3)$ for $\sigma = (12)(34)(567)$, because $j_2(\sigma) = 2$ and $j_3(\sigma) = 1$, with others being zero.

The antisymmetrizer on $\mathcal{H}^{\otimes k}$ is defined by

$$A_k = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \mathrm{sgn}(\sigma) U_\sigma \quad (k = 1,\ \ldots,\ d), \tag{5}$$

where $\mathrm{sgn}(\sigma) = \pm 1$ is the sign of $\sigma$. $A_k$ is a projection operator on $\mathcal{H}^{\otimes k}$ $(A_k = A_k^2 = A_k^\dagger)$ and has a natural extension on $\mathcal{H}^{\otimes d}$ by $A_k \otimes I^{d-k}$. Hereafter, we use the same symbol $A_k$ on $\mathcal{H}^{\otimes d}$ and define $A_0$ to be the identity operator on $\mathcal{H}^{\otimes d}$. Using Eqs. (4) and (5), we obtain

$$\mathrm{tr}\,(A_k\rho^{\otimes k}) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\prod_{m=1}^{k}(\mu_m)^{j_m(\sigma)}, \tag{6}$$

with $\mu_m = (-1)^{m-1}\,\mathrm{tr}\,\rho^m$. Here we have used $\mathrm{sgn}(\sigma) = \prod_{m=1}^{k}[\mathrm{sgn}(c_m)]^{j_m(\sigma)}$ and $\mathrm{sgn}(c_m) = (-1)^{m-1}$ for an $m$-cycle $c_m$. Thus, a projective measurement about $A_k$ includes the moments of $\rho$, up to the $k$th order.

Now, we show a way to reconstruct the characteristic polynomial of $\rho$, with Eq. (6). Let us write $\mathrm{tr}\,(A_k\rho^{\otimes k})$ as $a_k$, and formally define $a_0 = 1$. A straightforward calculation of the right-hand side of Eq. (6) leads to the Newton-Girard formula [18]

$$a_k = \frac{1}{k}\sum_{m=1}^{k}\mu_m a_{k-m} \quad (k = 1,\ldots,d). \tag{7}$$

Thus, the sequence of $\{a_k\}_{k=1}^d$ is equivalent to that of the coefficients of the characteristic polynomial (i.e., elementary symmetric polynomials). To sum up, we obtain

$$\det(x-\rho) = \mathrm{tr}\,[M(x)\rho^{\otimes d}], \tag{8}$$

with $M(x) = \sum_{k=0}^{d}(-1)^k x^{d-k}A_k$. This result is notable, because the characteristic polynomial is described by a single quantum observable $M(x)$. However, we still need to remove the dependence on the continuous variable $x$, to make a single-setup approach possible. The key is the following structure of the antisymmetrizers. Since a permutation procedure in $A_k$ is a part of $A_l$ when $k < l$ $(\mathfrak{S}_k \subset \mathfrak{S}_l)$, we find that

$$A_k A_l = A_l A_k = A_l \quad (k < l). \tag{9}$$

Furthermore, the projective property of $A_k$ leads to an eigensubspace $\mathcal{A}_k = A_k\,\mathcal{H}^{\otimes d}$. Therefore, we obtain the inclusion relation for $\mathcal{A}_k$:

$$\mathcal{A}_d \subset \mathcal{A}_{d-1} \subset \cdots \subset \mathcal{A}_2. \tag{10}$$

We note that $\mathcal{A}_1 = \mathcal{H}^{\otimes d}$. Thus, we have an orthogonal decomposition of $\mathcal{H}^{\otimes d}$ as $\oplus_{i=1}^{d}\mathcal{B}_i$, with $\mathcal{B}_1 = \mathcal{A}_d$ and $\mathcal{B}_i = \mathcal{A}_{d-i+1} - \mathcal{A}_{d-i+2}$ $(i = 2,\ldots,d)$. The projection operators $P_i$ onto $\mathcal{B}_i$ are then defined by $P_1 = A_d$ and $P_i = A_{d-i+1} - A_{d-i+2}$ $(i = 2,\ldots,d)$. They constitute our PVM measurements $(P_i)_{i=1}^d$. By definition, we find that $A_i = \sum_{j=1}^{d-i+1} P_j$. By substituting this formula into Eq. (8), we obtain the practical formula (1) for eigenvalue determination, with

$$m_i(x) = \sum_{k=0}^{d-i+1}(-1)^k x^{d-k}. \tag{11}$$

Since the dimension of $\mathcal{A}_k$ is $D(k) = d^{d-k}\,{}_dC_k$, the dimension of $\mathcal{B}_i$ (i.e., rank $P_i$) is $D(d-i+1) - D(d-i+2)$, where we define $D(d+1)$ to be zero.

We stress that in our approach the use of the antisymmetrizer $A_k$ is not essential. For instance, the symmetrizer on $\mathcal{H}^{\otimes k}$ (projector on a bosonic ground state) leads to a similar formula to Eq. (7), with the parallel argument above. Hence, the symmetrizer works for the eigenvalue determination. The virtue of the antisymmetrizer is a direct connection to the coefficients of the characteristic polynomial with the expectation values of the antisymmetrizers, as shown in Eq. (8).

We also notice that the number of target-state copies may decrease if only partial information on the eigenvalues is needed. In particular, if the number of (approximately) zero

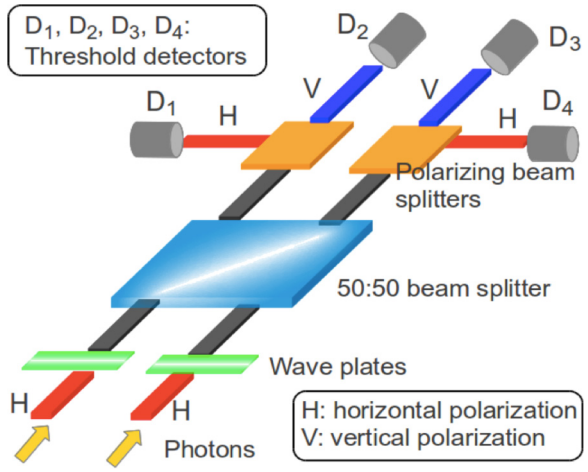


FIG. 2. (Color online) Proposal for an eigenvalue determination with linear optical qubits.

eigenvalues, $d_0(< d)$, is known a priori (alternatively can be estimated), the present method with $\rho^{\otimes(d-d_0)}$ leads to $(d-d_0)$ predominant eigenvalues. Any prior information other than $d_0$ is not required.

Now, we illustrate the present method in physical systems. We first consider qubits ($d=2$ and $\mathcal{H}=\text{span}\{|0\rangle,|1\rangle\}$). The relevant projectors are $P_1=|\Psi_-\rangle\langle\Psi_-|$ and $P_2=I^{\otimes 2}-|\Psi_-\rangle\langle\Psi_-|$, with $|\Psi_-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$. The first example is a linear optical qubit. A quantum state is encoded by horizontal (H) and vertical (V) polarization (i.e., $|\text{H}\rangle=|0\rangle$ and $|\text{V}\rangle=|1\rangle$). A two-photon interferometer, the Bell-state analyzer (see, e.g., Refs. [19,20]), leads to an implementation of the projectors, as seen in Fig. 2. Although the main part of the eigenvalue determination is the 50 : 50 beam splitter and the subsequent single-photon detectors, this setting includes a method to prepare a 2-i.i.d. state as well. Let us first explain this preparation stage. The initial photons are horizontally polarized. A series of wave plates is set on each input arm in the interferometer. On every trial, one randomly changes a combination of the wave plates to make an element of SU(2) [21]. This procedure probabilistically produces different unitary gates on a polarization state. Thus, one can examine the eigenvalue determination of $\rho=\sum p_r V_r|\text{H}\rangle\langle\text{H}|V_r^\dagger$, with $\sum p_r=1$, $p_r\geqslant 0$, and $V_r\in\text{SU}(2)$. Next, we turn to the determination part. After the interference at the 50 : 50 beam splitter, either antibunching or bunching occurs [19]. From the bosonic character of photons, the antibunching is related to an antisymmetric property with respect to the polarization (i.e., $(|\text{HV}\rangle-|\text{VH}\rangle)/\sqrt{2}$), whereas bunching is relevant to symmetric states. Hence, using two polarizing beam splitters and four threshold detectors ($\text{D}_1$, $\text{D}_2$, $\text{D}_3$, and $\text{D}_4$), we obtain $\text{tr}(P_i\rho^{\otimes 2})$. When either ($\text{D}_1$, $\text{D}_3$) or ($\text{D}_2$, $\text{D}_4$) detects photons, this event is counted as $P_1=1$. The others correspond to the case $P_2=1$.

The main idea in our proposal is to use a projective measurement on a multicopy of a target quantum state. This technique is applied in photon-qubit experiments [22–24] to measure *nonlinear* observabales such as concurrence. In other words, the present work studies such experimental techniques, from the viewpoint of minimality and single-setup implementability.

Next, we examine a solid-state system, especially a superconducting qubit [25]. Let us consider a controlled gate $U_{ZZ}=\exp[-i(\pi/4J)H_{ZZ}]$, with $H_{ZZ}=J\sigma_z\otimes\sigma_z$. The $2\times 2$ Pauli matrices are $\sigma_{x,y,z}$. The qubit-coupling Hamiltonian $H_{ZZ}$ is realized in various systems such as flux qubits [26] and transmon qubits [27,28]. This control gate with single-qubit gates leads to

$$U_{\text{D}}=(I\otimes U_{\text{H}})(Z_{-\pi/4}\otimes Z_{-\pi/4})U_{ZZ}(I\otimes U_{\text{H}}), \tag{12}$$

where $U_{\text{H}}=Y_{-\pi/8}Z_{\pi/2}Y_{\pi/8}$, $Z_\theta=\exp(i\theta\sigma_z)$, and $Y_\theta=\exp(i\theta\sigma_y)$. We find that $U_{\text{D}}|\Phi_-\rangle=|00\rangle$, $U_{\text{D}}|\Psi_+\rangle=|01\rangle$, $U_{\text{D}}|\Phi_-\rangle=|10\rangle$, and $U_{\text{D}}|\Psi_-\rangle=|11\rangle$, up to overall phases, where $|\Phi_\pm\rangle=(|00\rangle\pm|11\rangle)/\sqrt{2}$ and $|\Psi_+\rangle=(|01\rangle+|10\rangle)/\sqrt{2}$. If both of the qubits are detected as $|1\rangle$ after performing $U_{\text{D}}$, the projector $P_1$ is done.

We refer to the extendability of our proposal useful for the implementation in general systems. Using a superoperator $\Xi$ such that $\Xi(\rho^{\otimes d})=\rho^{\otimes d}$, and its adjoint $\Xi^*$, $\text{tr}\,[\Xi^*(A)\,B]=\text{tr}\,[A\,\Xi(B)]$, we find that an expansion of the characteristic polynomial is not unique. In contrast to $M(x)$ in Eq. (8), the observable $\Xi^*(M)(x)$ can involve operators other than the antisymmetrizers. Thus, one may perform our proposal, not sticking to the PVM measurement. Let us apply this technique to the eigenvalue determination in a linear optical qutrit ($d=3$). Our qutrit is a superposition of 3-path (or mode) single-photon states. The corresponding bosonic creation operators are $a_\ell^\dagger$ ($\ell=1,2,3$). To represent a 3-i.i.d. state $\rho^{\otimes 3}$, we need two additional spatial modes $b_\ell^\dagger$ and $c_\ell^\dagger$, each of which has another spatial mode index $\ell$ for expressing a qutrit state, like $a_\ell^\dagger$. To simplify the notations, we will denote $b_\ell^\dagger$ ($c_\ell^\dagger$) as $a_{\ell+3}^\dagger$ ($a_{\ell+6}^\dagger$). In our setting, a 3-i.i.d. qutrit enters an interferometer. For each $\ell$, a mixing among $a_\ell^\dagger$, $a_{\ell+3}^\dagger$, and $a_{\ell+6}^\dagger$ occurs:

$$\begin{pmatrix} a_\ell^\dagger \\ a_{\ell+3}^\dagger \\ a_{\ell+6}^\dagger \end{pmatrix} \to \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix}\begin{pmatrix} a_\ell^\dagger \\ a_{\ell+3}^\dagger \\ a_{\ell+6}^\dagger \end{pmatrix}. \tag{13}$$

This transformation (a qutrit quantum Fourier transformation [29]) does not alter the bosonic canonical commutation realtions and can be built up by beam splitters and phase shifters [30]. The interferometer has nine output ports, each of which is connected to a threshold detector $D_\alpha$ ($\alpha=1,\ldots,9$). In other words, the detector $D_\alpha$ is clicked if photons live in the $\alpha$th output mode after the transformation (13). We regard a triplet ($D_\beta$, $D_{\beta+1}$, $D_{\beta+2}$) as a single detector $\tilde{D}_\beta$ ($\beta=1,4,7$). Thus, when at least one element of the triplet is clicked, the logical value of this *coarse-graining* measurement is true. We can find that three distinct events occur at the outputs in the total apparatus: "bunching" (one of the three *coarse-graining* detectors is clicked), "antibunching" (all the *coarse-graining* detectors are clicked), and others. The elements of the corresponding positive operator-valued measure (not PVM) are $Q_1=(2/3)S_3$ for bunching, $Q_2=(S_3/3)+A_3$ for antibunching, and $Q_3=I^{\otimes 3}-Q_1-Q_2$ for residues, with the three-body symmetrizer $S_3$. Now, we take $\Xi$ as the three-body symmetrizing superoperator, $\Xi(A)=(1/3!)\sum_{\sigma\in\mathfrak{S}_3}U_\sigma A U_\sigma^\dagger$. Then, we find that $\Xi^*(M)(x)=\sum_{k=1}^3 m_k'(x)Q_k$. The polynomials $m_k'(x)$ can be obtained by straightforward calculations

and do not depend on $\rho$, like $m_i(x)$. In this way, we can reconstruct the characteristic polynomial of a qutrit density matrix, using $m'_k(x)$ and the measurement probabilities $\mathrm{tr}\,(Q_k\rho^{\otimes 3})$.

Finally, we compare our proposal to an approach proposed by Keyl and Werner [31]. They found a single observable for an eigenvalue determination, via a group-theoretic approach (see also Ref. [32]). Different from ours, the outcome of the observable in a single-shot measurement is an estimator of the eigenvalues. Their method can be considered to be minimal and implementable in a single setup. However, to obtain high accuracy, their approach requires an $N$-i.i.d. state $\rho^{\otimes N}$, with $N \to \infty$. One has to perform a measurement of a multipartite observable in a many-body system, whose particle number depends on a given accuracy. In contrast, our observable is fixed, once the dimension of a target system is set. Thus, our method could be much simpler, from a technical point of view.

In summary, we showed a simple method for measuring the eigenvalues of a density matrix of a $d$-level system in a single setup. We also implemented our proposal in linear optical and superconducting systems. In the present formulation, a characteristic polynomial is reconstructed via quantum measurements. The resultant polynomial is straightforwardly calculated by classical computers. This approach is also applicable to evaluating the energy spectrum of a physical system, like Ref. [33]. Thus, our proposal can be used for a practical assessment of quantum features in a physical system.

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