Determining Feynman Integrals with Only Input from Linear Algebra

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We find that all Feynman integrals (FIs), having any number of loops, can be completely determined once linear relations between FIs are provided. Therefore, FI computation is conceptually changed to a linear algebraic problem. Examples up to five loops are given to verify this observation. As a by-product, we obtain a powerful method to calculate perturbative corrections in quantum field theory.

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Introduction.—Feynman integrals (FIs) encode key information of quantum field theories. The study of FIs is important both for exploring mysteries of quantum field theories and for their phenomenological application. Integrating over some variables is found to be a necessary step to determine FIs in all known systematic methods. This seems to be a reasonable phenomenon, as FIs themselves are defined by integrating over loop momenta. However, because it is usually hard to perform integration in a systematic and efficient way, is it possible to totally bypass integration in determining FIs?

Available systematic methods to compute FIs can be divided into direct methods and indirect methods. Direct methods include sector decomposition [1–13], the Mellin-Barnes representation [8–13], loop-tree duality [14–23], and so on, where one computes FIs by directly performing integration over some variables. Indirect methods compute FIs indirectly by solving corresponding equations, which include difference equations [24–27] and differential equations [28–40]. To uniquely determine the solution, boundary information is needed in these indirect methods. Unfortunately, the only known systematic way to obtain boundary information is to use direct methods to calculate them. Therefore, integration is still necessary in these indirect methods.

The auxiliary mass flow (AMF) method [38–40] is a kind of differential equation method, which computes FIs by setting up and solving differential equations (DEs) with respect to an auxiliary mass term η (called η-DEs). The virtue of AMF is that its boundary conditions at $\eta \to \infty$ are simply vacuum bubble integrals, which can be more easily calculated by using other methods [41–50].

In this Letter, we make the following observation. Boundary information for AMF, which can always be cast to single-mass vacuum FIs, can be related to propagator integrals (p integrals) with one less loop. Then, p integrals can again be calculated by using the AMF method, with the input of new boundary information having one less loop. By using this strategy iteratively, we eventually do not need any input for boundary information in the AMF framework. Thus, it is surprising to find that integration is totally bypassed in determining FIs.

As a result of our observation, FIs can be completely determined once linear relations between FIs are provided; these relations are used to decompose all FIs to a small set of bases, called master integrals (MIs), and to set up η-DEs of these MIs. We note that numerically solving ordinary differential equations (like η-DEs) is a well-known mathematical problem [51]. Therefore, the problem of integrating over loop momenta is now conceptually changed to a linear algebraic problem of exploring the linear space of FIs.

In the rest of this Letter, we first review the AMF method and emphasize its input. We then describe our method to compute boundary conditions within the AMF framework, without any unknown information. Some examples are in order to verify this method. Finally, we propose a powerful way to calculate perturbative corrections within dimensional regularization.

Before continuing, let us first give a brief introduction to FIs. A family of FIs is defined by the following integrals with various values of $\vec{\nu}$,

$$I_{\vec{\nu}} = \int \left(\prod_{i=1}^{L} \frac{d^D \ell_i}{i\pi^{D/2}}\right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}, \tag{1}$$

where L is the number of loops; ℓ_i are loop momenta; D is the dimensionality of ℓ_i; $\mathcal{D}_1, \ldots, \mathcal{D}_K$ are inverse propagators, with $\nu_1, \ldots, \nu_K$ being integers; and $\mathcal{D}_{K+1}, \ldots, \mathcal{D}_N$ are irreducible scalar products introduced for completeness,

with $\nu_{K+1},\ldots,\nu_N$ being nonpositive integers. It has been proven that a family of FIs forms a finite-dimensional linear space [52]. In other words, any FI in a given family can be decomposed into a linear combination of MIs, which is a finite set of bases of the linear space formed by the family of FIs. Coefficients in this decomposition are rational functions of all natural variables, like D, Mandelstam variables, masses, and the η introduced in AMF. Information of the linear space is completely encoded in these decompositions, or linear relations between FIs. Decomposition of FIs is usually realized by integration-by-parts (IBP) reduction, which has been extensively studied [24,53–73]. Having IBP reduction relations, we then only need to study MIs.

Furthermore, because FIs containing linear propagators can be determined by FIs containing only quadratic propagators [74], we will not consider linear propagators anymore.

Auxiliary mass flow method.—To determine $I_{\vec{\nu}}$ defined in Eq. (1), in the AMF method, one introduces an auxiliary family of integrals defined by

$$\tilde{I}_{\vec{\nu}}(\eta)=\int\left(\prod_{i=1}^{L}\frac{d^D\ell_i}{\mathrm{i}\pi^{D/2}}\right)\frac{\tilde{\mathcal{D}}_{K+1}^{-\nu_{K+1}}\cdots\tilde{\mathcal{D}}_N^{-\nu_N}}{\tilde{\mathcal{D}}_1^{\nu_1}\cdots\tilde{\mathcal{D}}_K^{\nu_K}}. \tag{2}$$

Without loss of generality, we assume $\nu_1>0$ and $\mathcal{D}_1=\ell_1^2-m^2+\mathrm{i}0^+$, where m can be zero. We can then choose the propagator mode [40] to set $\tilde{\mathcal{D}}_i=\mathcal{D}_i$ for $i>1$ and modify the mass term for $i=1$ by

$$\tilde{\mathcal{D}}_1=\ell_1^2-m^2-\eta. \tag{3}$$

The original $I_{\vec{\nu}}$ can be obtained by taking $\eta\to\mathrm{i}0^-$,

$$I_{\vec{\nu}}=\lim_{\eta\to\mathrm{i}0^-}\tilde{I}_{\vec{\nu}}(\eta). \tag{4}$$

Let us denote MIs of the auxiliary family by $\vec{\tilde{J}}(\eta)$ and denote its dimension by n. Using IBP reduction, $(\partial/\partial\eta)\vec{\tilde{J}}(\eta)$ can again be expressed as linear combinations of $\vec{\tilde{J}}(\eta)$, which results in a system of closed η-DEs,

$$\frac{\partial}{\partial\eta}\vec{\tilde{J}}(\eta)=A(\eta)\vec{\tilde{J}}(\eta), \tag{5}$$

where $A(\eta)$ is an $n\times n$ matrix with entries rationally depending on η. Supposing that we already have boundary conditions in hand, we can solve the η-DEs numerically [38,51] to obtain $\vec{\tilde{J}}(\eta)$ and thus their limit $\vec{\tilde{J}}(\mathrm{i}0^-)$. As $\tilde{I}_{\vec{\nu}}(\eta)$ can be expressed as linear combinations of $\vec{\tilde{J}}(\eta)$ using IBP reduction, all original FIs $I_{\vec{\nu}}$ (and certainly also their MIs) are eventually determined.

An advantage of AMF is that boundary conditions at $\eta\to\infty$ can be systematically calculated. In this limit, nonzero contributions only come from integration regions where linear combinations of loop momenta are either of $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$ [75,76]. In each of these limited number of regions, a general propagator can be expressed as

$$\frac{1}{(\ell_L+\ell_S+p)^2-m^2-\kappa\eta},$$

where ℓ_L is the $\mathcal{O}(\sqrt{\eta})$ part of the loop momenta, ℓ_S is the $\mathcal{O}(1)$ part of the loop momenta, p is a linear combination of external momenta, m is the mass, and $\kappa=0$ or 1. Then, if $\ell_L\neq 0$ or $\kappa\neq 0$, we can simplify the propagator by

$$\frac{1}{(\ell_L+\ell_S+p)^2-m^2-\kappa\eta}\sim\frac{1}{\ell_L^2-\kappa\eta}. \tag{6}$$

Otherwise, the propagator is unchanged. After the above simplification, the new FIs at the boundary are either single-mass vacuum FIs or simpler FIs compared with the original FIs. For the latter cases, we can compute them again using AMF, which needs even simpler FIs as input for boundary conditions.

By using AMF iteratively, to determine any L-loop FI, we eventually only need single-mass vacuum FIs with no more than L loops as additional input, besides IBP reductions. Diagrams of some typical single-mass vacuum FIs are shown in Fig. 1.

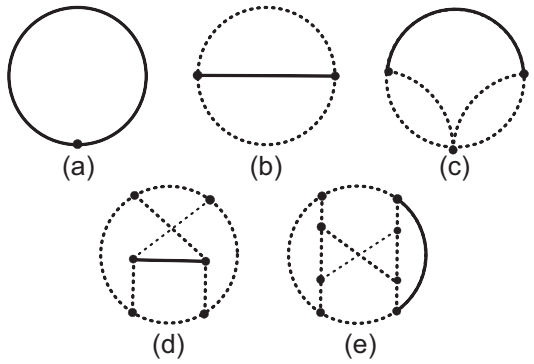

FIG. 1. Some typical Feynman diagrams of single-mass vacuum FIs up to five loops, where solid lines denote massive propagators and dotted lines denote massless propagators.

Determining single-mass vacuum Feynman integrals.—Now, let us assume that $I_{\vec{\nu}}$ defined in Eq. (1) are single-mass vacuum FIs, with $\mathcal{D}_1=\ell_1^2-m^2+\mathrm{i}0^+$ as the only massive propagator and $\nu_1>0$. Without loss of generality, we set $m^2=1$ in the rest of this Letter.

Let us define a massless p integral

$$\hat{I}_{\vec{\nu}'}(\ell_1^2)=\int\left(\prod_{i=2}^{L}\frac{d^D\ell_i}{\mathrm{i}\pi^{D/2}}\right)\frac{\mathcal{D}_{K+1}^{-\nu_{K+1}}\cdots\mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2}\cdots\mathcal{D}_K^{\nu_K}} \tag{7}$$

with $\vec{\nu}'=(\nu_2,\ldots,\nu_N)$, where ℓ_1 is its "external momentum" and ℓ_1^2 is the only mass scale. Based on dimensional counting, we have

$$\hat{I}_{\vec{\nu}'}(\ell_1^2) = (-\ell_1^2)^{\frac{(L-1)D}{2}-\nu+\nu_1}\hat{I}_{\vec{\nu}'}(-1), \tag{8}$$

where $\nu = \sum_{i=1}^{N}\nu_i$. The original integral $I_{\vec{\nu}}$ is then factorized into two parts and can be evaluated as

$$\begin{aligned} I_{\vec{\nu}} &= \int \frac{d^D\ell_1}{i\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2}-\nu+\nu_1}}{(\ell_1^2-1+i0^+)^{\nu_1}}\hat{I}_{\vec{\nu}'}(-1) \\ &= \frac{\Gamma(\nu-LD/2)\Gamma(LD/2-\nu+\nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)}\hat{I}_{\vec{\nu}'}(-1), \end{aligned} \tag{9}$$

which determines an L-loop single-mass vacuum FI $I_{\vec{\nu}}$ by an $(L-1)$-loop massless p integral $\hat{I}_{\vec{\nu}'}(-1)$. This relation is well known.

The key observation is as follows: The $(L-1)$-loop massless p integral $\hat{I}_{\vec{\nu}'}(-1)$ can be computed via AMF discussed in the last section, which requires single-mass vacuum FIs with no more than $(L-1)$ loops as additional input, besides IBP reductions. Therefore, we find that, with linear algebra provided by IBP reductions, single-mass vacuum FIs with L loops are determined by those with less than L loops. This works iteratively until the boundary at $L=1$. Vacuum FIs with $L=1$ are completely determined by the relation (9) by noticing that the value of the zero-loop p integral is simply 1.

We eventually arrive at a surprising conclusion that all single-mass vacuum FIs, and therefore all FIs, can be completely determined once linear algebraic relations between different FIs are provided. This conclusion is valid for any number of loops L and arbitrary dimensionality D.

Examples.—To better understand the above observation, let us compute some FIs.

One of the simplest examples is the two-loop single-mass vacuum integral shown in Fig. 1(b), defined by

$$I_{(1,1,1)} = \int \left(\prod_{i=1}^{2}\frac{d^D\ell_i}{i\pi^{D/2}}\right)\frac{1}{(\ell_1^2-1)\ell_2^2(\ell_1+\ell_2)^2}, \tag{10}$$

where the Feynman prescription $i0^+$ for each denominator is suppressed. The relation (9) gives

$$I_{(1,1,1)} = \frac{\Gamma(3-D)\Gamma(D-2)}{-\Gamma(1)\Gamma(D/2)}\hat{I}_{(1,1)}(-1), \tag{11}$$

with

$$\hat{I}_{(1,1)}(-1) = \int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{\ell_2^2(\ell_2+p)^2}, \tag{12}$$

where p^μ satisfies $p^2=-1$.

To calculate the one-loop p integral $\hat{I}_{(1,1)}(-1)$ via the AMF method, we introduce auxiliary integrals

$$\tilde{I}_{(1,0)}(\eta) = \int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{\ell_2^2-\eta}, \tag{13}$$

$$\tilde{I}_{(1,1)}(\eta) = \int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{(\ell_2^2-\eta)(\ell_2+p)^2}, \tag{14}$$

which are MIs of the corresponding auxiliary family. Denoting $\vec{\tilde{J}} = (\tilde{I}_{(1,0)}, \tilde{I}_{(1,1)})^T$, η-DEs can be obtained using IBP reductions,

$$\frac{\partial}{\partial\eta}\vec{\tilde{J}}(\eta) = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 \\ \frac{1-\epsilon}{-\eta(1+\eta)} & \frac{1-2\epsilon}{1+\eta} \end{pmatrix}\vec{\tilde{J}}(\eta). \tag{15}$$

As $\eta\to\infty$, only the integration region $|\ell_2| \sim \mathcal{O}(\sqrt{\eta})$ gives a nonzero contribution. Thus, we have

$$\begin{aligned} \tilde{I}_{(1,0)}(\eta) &= \eta^{D/2-1}\int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{\ell_2^2-1} \\ &= \eta^{D/2-1}(-1)\Gamma(1-D/2), \end{aligned} \tag{16}$$

where in the last step the relation (9) has been used, and

$$\begin{aligned} \tilde{I}_{(1,1)}(\eta) &\overset{\eta\to\infty}{=} \int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{(\ell_2^2-\eta)\ell_2^2} \\ &= \eta^{D/2-2}\int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{(\ell_2^2-1)\ell_2^2} \\ &= \eta^{D/2-2}\int \frac{d^D\ell_2}{i\pi^{D/2}}\frac{1}{\ell_2^2-1} \\ &= \tilde{I}_{(1,0)}(\eta)\frac{1}{\eta}, \end{aligned} \tag{17}$$

where scaleless integrals are omitted in the third line.

By solving the η-DEs in Eq. (15) together with boundary conditions at $\eta\to\infty$ in Eqs. (16) and (17), $\hat{I}_{(1,1)}(-1) = \tilde{I}_{(1,1)}(i0^-)$ is determined. We thus obtain the desired FI $I_{(1,1,1)}$ using the relation (11).

Clearly, the same procedure can be used to compute any FI. Let us give the result of another example shown in Fig. 1(e), which is one of the most complicated five-loop single-mass vacuum FIs. Following the above-described procedure, we can compute all MIs in this family to a very high precision, with only input from the IBP reductions. The result of the corner integral with 10-digit precision is given by

$$\begin{aligned} &-2.073\,855\,510\epsilon^{-2} - 7.812\,755\,312\epsilon^{-1} - 17.258\,828\,64 \\ &+ 717.680\,884\,5\epsilon + 8190.876\,448\epsilon^2 + 78\,840.29\,598\epsilon^3 \\ &+ 566\,649.1116\epsilon^4 + 390\,171\,3.802\epsilon^5 + 237\,023\,84.71\epsilon^6, \end{aligned} \tag{18}$$

where we have set $D = 4 - 2\epsilon$ with only 9 orders in ϵ expansion shown, although more orders and digits can be easily obtained. The first seven terms of the expansion agree with those obtained in Ref. [48], and other terms are new.

A new method to calculate perturbative corrections.—An important feature of our strategy is that the FIs we calculate can have arbitrary dimensionality. On the one hand, this makes our strategy applicable for a general theory, e.g., nonrelativistic theory with dimensionality equal to 3. On the other hand, by sampling different dimensionality around a fixed value, say $4 - 2\epsilon$ with some small values of ϵ, we can fit the Laurent expansion with respect to ϵ to any desired order, which is actually the way we obtain the results in Eq. (18).

If we apply the above strategy directly to physical processes, we arrive at a new and powerful method to calculate perturbative corrections—for example, the next-to-next-to-leading-order (NNLO) QCD correction to a top-antitop quark pair fully inclusive production cross section in lepton colliders $e^+e^- \to \gamma^* \to t\bar{t} + X$, which have previously been calculated in Refs. [77–79]. In our method, we calculate a bare cross section (before renormalization) with a numerical value of ϵ and then renormalize it in the standard $\overline{\text{MS}}$ scheme, with the same value of ϵ. To show the numerical result, we choose the center-of-mass energy s, renormalization scale μ, and the top quark mass as $\mu = \sqrt{s} = 1$ and $m_t^2 = 1/8$. We ignore contributions from internal top quark loops and those from photons interacting with the other five types of quarks because these contributions are very small. Then, if we set $\epsilon = 0.001$, the NNLO correction gives

$$\sigma_{0.001}^{\text{NNLO}}/(\alpha\alpha_s^2) = 9.261823090, \tag{19}$$

where only 10 digits are shown. Because the cross section is a physical quantity that is free of divergence, $\sigma_\epsilon^{\text{NNLO}}$ can give an estimation of the total cross section up to an $\mathcal{O}(\epsilon)$ error. Now let us calculate the cross section with another value $\epsilon = 0.0011$, which gives

$$\sigma_{0.0011}^{\text{NNLO}}/(\alpha\alpha_s^2) = 9.262629688. \tag{20}$$

The fact that $\sigma_{0.001}^{\text{NNLO}}$ and $\sigma_{0.0011}^{\text{NNLO}}$ have a relative difference at the $\mathcal{O}(1/1000)$ level confirms two things. First, the $\sigma_\epsilon^{\text{NNLO}}$ calculated here is free of $1/\epsilon^n$ divergence; otherwise, the difference should be at the $\mathcal{O}(1)$ level. Second, $\sigma_{\epsilon_1}^{\text{NNLO}} = \sigma_{\epsilon_2}^{\text{NNLO}} + \mathcal{O}(\epsilon_1 - \epsilon_2)$ is justified. Therefore, we can fit a linear function of ϵ by combining values of $\sigma_{0.001}^{\text{NNLO}}$ and $\sigma_{0.0011}^{\text{NNLO}}$ to provide a better estimation of σ_0^{NNLO},

$$\sigma_0^{\text{NNLO}}/(\alpha\alpha_s^2) \approx 9.2537 + \mathcal{O}(\epsilon^2), \tag{21}$$

which becomes closer to the exact result 9.253454354. By calculating one more value of ϵ for each, we can further improve the estimation with uncertainty suppressed by one higher order in ϵ.

In this method, we do not need to manipulate a Laurent expansion of ϵ during the intermediate stage of the calculation; thus, the computational time can usually be reduced by several times. This improvement of efficiency is very important for cutting-edge problems. Actually, using this method, we have successfully calculated the above-mentioned $t\bar{t}$ production to next-to-next-to-next-to-leading order for the first time, which will be presented elsewhere [80].

Summary and outlook.—By combining the recently proposed AMF method and Eq. (9), we find that all FIs, with any number of loops and arbitrary dimensionality, can be completely determined once linear relations between FIs are provided. This interesting observation conceptually changes FI computation to an algebraic problem. We have explicitly verified this observation using some examples up to five loops.

For phenomenological purposes, many general FIs need to be calculated. The mainstream method to compute FIs can be divided into two steps. In the first step, one reduces all FIs to MIs, and in the second step, one calculates the MIs. Both steps are found to be very difficult for current cutting-edge problems. With our strategy, IBP reduction becomes the only obstacle for FI calculation. Our strategy has been implemented using the package AMFlow [81], which can automatically calculate general FIs, with any number of loops, to high precision, as long as IBP reduction is successful. These features make our method unique compared to other available methods of FI computation.

Because FIs with any dimensionality can now be calculated, we can determine physical processes directly with a given small value of ϵ, the dimensional regulator. In this way, we can significantly improve the efficiency of perturbative calculations. Furthermore, our method is applicable for a general theory, like nonrelativistic theory with dimensionality equal to 3.

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