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Determining Finite Volume Elements for the 2$\mathcal{D}$ Navier-Stokes Equations

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Abstract

We consider the 2$\mathcal{D}$ Navier-Stokes equations on a square with periodic boundary conditions. Dividing the square into N equal subsquares, we show that if the asymptotic behavior of the average of solutions on these subsquares (finite volume elements) is known, then the large time behavior of the solution itself is completely determined, provided N is large enough. We also establish a rigorous upper bound for N needed to determine the solutions to the Navier-Stokes equation in terms of the physical parameters of the problem.

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1 Introduction

It is well established that the long time behavior of solutions to the Navier-Stokes Equations (NSE), in bounded domains, has a finite number of *degrees of freedom*. Several detailed rigorous studies support this assertion. For example, it is known that the behavior as $t \to \infty$ of the solutions to the NSE is completely determined by the behavior of their projection on the space spanned by the first m eigenfunctions of the linear Stokes operator, for m sufficiently large. More precisely, if the asymptotic behavior of the first m modes of two solutions agree, then the entire solutions agree as $t \to \infty$, [10]. The corresponding modes are called *determining modes*. (Also along these lines see [24] for a slightly weaker result.) Later, in [9] an upper bound was established of the order $G(1 + \log(G))^{1/2}$ for the number of determining modes, where G, the Grashof number, is the analogue of the Reynolds number (see section 2 below). It is also known that the large time behavior of solutions is determined by their values on a discrete set of points [14]. More specifically, if two solution of the NSE agree on a sufficiently dense (finite) set of points, called a set of *determining nodes*, as $t \to \infty$, then they agree everywhere as time goes to infinity. Later in [23] it was found that an upper bound for the number of determining nodes is of the order $G^2(1 + \log(G))$. Moreover, it is well known that the NSE possess a compact global attractor. The best known upper bound for its Fractal as well as its Hausdorff dimension is of the order $G^{2/3}(1 + \log(G))^{1/3}$, given in [3]. More recently, it has been shown [22] that the NSE has an *inertial form*. That is, the large-time behavior of the Navier-Stokes equations is completely described by a finite dimensional system of ODEs.

These results are also important from a practical point of view. The existence of a finite number of determining modes implies that the high modes are enslaved, at least asymptotically, by the lower modes. Thus, one may seek the existence of a global function which gives the high modes of every solution in terms of the lower modes, asymptotically in time. Such a function has been shown to exist for several interesting partial differential equations (see , for example, [2] and the references therein). The graphs, in phase space, of such functions are called *Inertial Manifolds* (I.M.). In general, an I.M. is a smooth (Lipschitz) finite dimensional manifold which is positively invariant under the flow, and which attracts every bounded subset in phase space at an exponential rate [11]. It is clear that if the inertial manifold exists, then it contains the global attractor. Further, the reduction of the PDE to the I.M. gives a finite dimensional ODE called an inertial form. Though the existence of an I.M. for the NSE is still open, however, as mentioned above, the NSE does have an inertial form. In any case, these ideas have suggested new numerical schemes that may be appropriate for approximating the global attractor. For example, they have lead to the introduction of *approximate inertial manifolds* and associated *nonlinear Galerkin methods* [5], [6], [8], [12], [20], [27], [33], [34], and the references therein. A similar interpretation is possible for the determining nodes. Indeed, in case the PDE has an I.M. it has been shown in [15] that the induced dynamical system of the nodal values of the solutions is conjugate (equivalent) to the dynamical system of the PDE. In particular, it may be possible to express, approximately, the values of the solutions at certain points in terms of the values at other points (*cf.* [4], [15], [28], [31], [32]). In either case these results indicate that it may be possible to improve the numerical simulation of the NSE for long time intervals without increasing the number of modes used or increasing the resolution of the computational grid.

In this paper we investigate another way to characterize the degrees of freedom of the NSE, and that is the idea of *Determining Finite Volume Elements*, first introduced in [15]. We consider the 2D NSE for a viscous incompressible fluid filling a square $\Omega = (0, L) \times (0, L)$ with periodic boundary conditions imposed. The governing equations are

$$\left.\begin{array}{c} \frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \text{ in } \mathbb{R}^2 \times (0, \infty) \\ \nabla \cdot u = 0 \text{ in } \mathbb{R}^2 \times (0, \infty) \\ u(x_1, x_2, t) = u(x_1, x_2 + L, t) \\ u(x_1, x_2, t) = u(x_1 + L, x_2, t), \end{array}\right\} \tag{1.1}$$

where $f = f(x, t)$, the volume force, and ν, the kinematic viscosity, are given. We denote by $u = u(x, t)$ the velocity vector, and $p = p(x, t)$ the pressure which are the unknowns. Further, we assume that the integrals of u and f vanish on Ω at all time (i.e. u and f have mean zero in Ω).

We divide Ω into N equal squares of side $l = L/\sqrt{N}$, and label the squares by $Q_1, \ldots, Q_N$. We study the average values of solutions on the Q_j's. For this purpose set

$$\langle u \rangle_{Q_j} = \frac{N}{L^2} \int_{Q_j} u(x) dx$$

for every $1 \leq j \leq N$. We wish to see to what degree knowledge of the behavior of the local averages of the velocity vector characterizes the flow. We investigate elsewhere the implementation of these results in numerical simulations (see [15] and [7]).

The paper is organized as follows. In section 3 we investigate stationary solutions. We show that if the finite volume elements of two stationary solutions agree, for sufficiently large N, then the two solutions are equal. In this case, we establish an upper bound of order G for the number of subsquares needed for the finite volume elements to be determining.

Section 4 is devoted to the large time behavior of solutions. It is shown that if the behavior of the local averages of two solutions goes to zero as time goes to infinity, for sufficiently small subsquares, then the two solutions agree everywhere as $t \to \infty$. We also show that the number of subsquares needed for the finite volume elements to be determining, in this case, is of the order G^2. Notice that by using local averages instead of point wise values we remove the logarithmic correction terms as found in the upper bounds for the number of determining nodes (*cf.* [23]). We remark that similar estimates can be easily obtained for the $3D$ NSE provided the latter has global strong solutions in time. In particular, we know, in both the $2D$ as well as the $3D$ cases, if local averages of two stationary solutions agree on sufficiently small subsquares, then the two solutions agree everywhere in the domain.

In section 5 we extend our results to a reaction-diffusion equation. In addition, we show that the number of determining finite volume elements is of the same order as the dimension of the global attractor in this case.

2 Functional Setting and Preliminary Results

We set

$$\begin{aligned} \mathcal{V} &= \{u : \mathbb{R}^2 \mapsto \mathbb{R}^2, \text{ vector valued trigonometric polynomials} \\ &\quad \text{with period } L, \nabla \cdot u = 0, \text{ and } \int_\Omega u\,dx = 0\}. \end{aligned}$$

Further, we set

$$H = \text{the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2,$$

$$V = \text{the closure of } \mathcal{V} \text{ in } (H^1(\Omega))^2,$$

where $H^l(\Omega)$ ($l = 1, 2, \dots$) denote the usual L^2-Sobolev spaces. H is a Hilbert space with the inner product and norm

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx, \quad |u| = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2},$$

respectively, and $u(x) \cdot v(x)$ is the usual Euclidean scalar product. Thanks to the Poincaré Lemma, V is also a Hilbert space with inner product and norm

$$((u, v)) = \sum_{i,j}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \quad \|v\|^2 = \sum_{i,j}^{2} \int_\Omega \left|\frac{\partial v_i}{\partial x_j}\right|^2 dx,$$

respectively. Let P denote the orthogonal projection in $L^2(\Omega)^2$ onto H. We denote by A the Stokes operator

$$Au = -P\Delta u,$$

(notice that in the periodic case $Au = -\Delta u$) and the bilinear operator

$$B(u, v) = P((u \cdot \nabla)v)$$

for all u, v in $\mathcal{D}(A) = V \cap (H^2(\Omega))^2$. We recall that the operator A is positive self-adjoint operator with compact inverse. Thus there exists a complete orthonormal set w_j of eigenfunctions of A such that $Aw_j = \lambda_j w_j$ and $0 < \lambda_1 \le \lambda_2 \le \dots$, where $\lambda_1 = (\frac{2\pi}{L})^2$.

The NSE, (1.1), is equivalent to the differential equation in H

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \tag{2.1}$$

where from now on $f = Pf$, and it is assumed that f satisfies $f \in L^\infty((0,\infty); H)$. That is $\sup_{t\geq 0}|f(t)| < \infty$. (For details see for example [1], [25] or [29].)

Let

$$F = \limsup_{t\to\infty} \left(\int_\Omega |f(t,x)|^2 dx\right)^{1/2}.$$

Following [9] we define the generalized Grashof number Gr as

$$Gr = \frac{F}{\lambda_1 \nu^2} = \frac{L^2 F}{4\pi^2\nu^2}.$$

The generalized Grashof number will play an analogous role as the Reynolds number, and will be our bifurcation parameter. In what follows all our estimates will be in terms of the generalized Grashof number. Notice that if f is time independent then Gr is the Grashof number $G = \frac{L^2|f|}{4\pi^2\nu^2}$

For questions related to existence, uniqueness, and regularity of solutions the reader is referred for instance to [1], [13], [17], [19], [21], [25], [29], and the references therein.

3 Stationary Solutions

Here we assume that the force, f, is time independent and that $Gr = G = |f|/\lambda_1\nu^2$. If G is sufficiently small then the dynamics of (1) is trivial [29, p.70], and in this case the global attractor consists of a unique exponentially stable stationary solution. We therefore suppose that $G >> 1$. However, we recall that ther exists volume forces such that the dynamics of the NSE remains trivial independent of G [26].

Lemma 3.1 *For every $w \in D(A)$ set*

$$\gamma(w) = \max_{1 \le j \le N} |\langle w \rangle_{Q_j}|.$$

Then

$$\begin{aligned} \|w\|_\infty = \sup_{x\in\Omega}|w(x)| &\le c_1\sqrt{6N}\gamma(w) + \frac{c_1 L}{2\sqrt{N}}|Aw| && (3.1)\\ |w| &\le \sqrt{2}L\gamma(w) + \frac{L^2}{6N}|Aw|, && (3.2)\\ \|w\| &\le \sqrt{6N}\gamma(w) + \frac{L}{2\sqrt{N}}|Aw|, && (3.3)\end{aligned}$$

where $c_1 = \frac{10+4\sqrt{2}}{\pi}$.

Proof. Applying the Poincaré inequality to $w(x)$ gives (see the appendix)

$$\int_{Q_j}|w(x)|^2dx \le \frac{L^2}{N}|\langle w \rangle_{Q_j}|^2 + \frac{L^2}{6N}\int_{Q_j}|\nabla w(x)|^2dx.$$

After summing over the j's we obtain

$$|w|^2 \le L^2\gamma^2(w) + \frac{L^2}{6N}\|w\|^2. \tag{3.4}$$

Now we interpolate in this last equation, $\|w\|^2 \le |w||Aw|$, and apply Young's inequality to obtain (3.2). Equation (3.3) is obtained in a similar fashion. To obtain (3.1) we use Agmon's inequality, $\|w\|_\infty^2 \le c_1^2|w||Aw|$, and another application of Youngs's inequality. For the bound on c_1 see [9] □

We need one more fact about the NSE in two dimensions with periodic boundary conditions; namely, the nonlinear term satisfies the identity

$$(B(w,w), Aw) = 0 \qquad \forall w \in \mathcal{D}(A) \tag{3.5}$$

(*cf.* [1], [29]). Differentiating this last expression with respect to w in the direction of u we obtain the useful identity

$$(B(u,w),Aw)+(B(w,u),Aw)+(B(w,w),Au)=0 \tag{3.6}$$

for all $u,w\in\mathcal{D}(A)$ (see [3]).

Theorem 3.2 *Let u,v be two stationary solutions of the 2D NSE satisfying*

$$\langle u\rangle_{Q_j}=\langle v\rangle_{Q_j} \quad \textit{for } 1\le j\le N.$$

Then $u=v$ provided $N\ge[(10+4\sqrt{2})\sqrt{2}\pi]G$.

Proof. Set $w=u-v$. Then w solves the equation

$$\nu Aw+B(u,w)+B(w,u)-B(w,w)=0.$$

Taking the inner product with Aw and using (3.5) we obtain

$$\nu|Aw|^2=-(B(w,w),Au).$$

Now we have the estimate

$$|(B(w,w),Au)|\le\sqrt{2}\|w\|_\infty\|w\||Au| \tag{3.7}$$

(*cf.* [1], [29]). Hence,

$$\nu|Aw|^2\le\sqrt{2}\|w\|_\infty\|w\||Au|.$$

One can easily show by using $(B(u,u),Au)=0$ that $|Au|\le|f|/\nu$. From (3.1), (3.3) (notice $\gamma(w)=0$ in this case), we get

$$|Aw|^2\left(\nu - \frac{\sqrt{2}c_1L^2|f|}{4\nu N}\right) \le 0.$$

It follows $|Aw| = 0$ (i.e. $u = v$), provided

$$N \ge c_1\sqrt{2}\pi^2 G.$$

□

4 Large Time Behavior

In this section we describe our results concerning the behavior for $t \to \infty$ of the solutions to the NSE. We recall the following generalized version of Gronwall's inequality. This version was first used in [9] to estimate the number of modes needed to determine the solutions to the NSE.

Lemma 4.1 *Let α be a locally integrable real valued function on $(0,\infty)$, satisfying for some $0 < T < \infty$ the following conditions:*

$$\liminf_{t\to\infty}\int_t^{t+T}\alpha(\tau)d\tau = \gamma > 0$$

$$\limsup_{t\to\infty}\int_t^{t+T}\alpha^-(\tau)d\tau = \Gamma < \infty,$$

where $\alpha^- = \max\{-\alpha, 0\}$. Further, let β be a real valued measurable function defined on $(0,\infty)$ such that $\beta(t) \to 0$ as t goes to infinity. Suppose that ξ is an absolutely continuous non-negative function on $(0,\infty)$ such that

$$\frac{d}{dt}\xi + \alpha\xi \le \beta, \quad a.e. \ on \ (0,\infty).$$

Then $\xi(t) \to 0$ as $t \to \infty$.

Let u, v solve respectively the Navier-Stokes equations

$$\frac{du}{dt} + \nu Au + B(u, u) = f$$

$$u(0) = u_0,$$

$$\frac{dv}{dt} + \nu Av + B(v, v) = g$$

$$v(0) = v_0,$$

where f, g are given forces in $L^\infty(0, \infty; H)$. Further, we suppose that $\lim_{t\to\infty} |f - g| = 0$.

Theorem 4.2 *In addition to the above assumptions suppose that*

$$\lim_{t\to\infty} (\langle u\rangle_{Q_j} - \langle v\rangle_{Q_j}) = 0,$$

for $1 \le j \le N$. *If* $N \ge [(10 + 4\sqrt{2})2]^2 Gr^2$, *then*

$$\lim_{t\to\infty} \|u(\cdot, t) - v(\cdot, t)\|_{L^\infty(\Omega)} = 0.$$

Proof. The proof is similar to the one given for the determining nodes in [23]. We therefore only give a sketch here.

Set $w = u - v$. Then w solves the equation

$$\frac{dw}{dt} + \nu Aw + B(u, w) + B(w, u) - B(w, w) = f - g.$$

Taking the inner product of (2.1) with Aw and using Equations (3.1), (3.5), (3.6), and (3.7) we get

$$\frac{1}{2}\frac{d\|w(t)\|^2}{dt} + \left(\nu\frac{|Aw|^2}{\|w\|^2} - \frac{c_1 L}{\sqrt{2N}}\frac{|Aw|}{\|w\|}|Au|\right)\|w\|^2 \le c_1\sqrt{12N}\gamma(w)\|w\||Au| + |f - g||Aw|.$$

We apply Lemma 4.2. Set

$$\beta(t) = c_1\sqrt{12N}\gamma(w)\|w\||Au| + |f - g||Aw|.$$

Using the fact that $|Au|, |Av|$ are bounded for $t \gg 1$, [16], and the assumptions on f, g and u, v, we have that $\beta(t) \to 0$ as $t \to \infty$. Set

$$\alpha = \nu\frac{|Aw|^2}{\|w\|^2} - \frac{c_1 L}{\sqrt{2N}}\frac{|Aw|}{\|w\|}|Au|.$$

It follows from *a priori* estimates on the time average of $|Au|$ (see [23]), namely,

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}|Au|^2 d\tau \le \frac{F^2}{T\nu^3\lambda_1} + \frac{F^2}{\nu^2}$$

for every $T > 0$, that

$$\limsup_{t\to\infty}\int_t^{t+T}\alpha^-(\tau)d\tau < \infty. \tag{4.1}$$

Similarly,

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau > 0 \tag{4.2}$$

holds provided $N \ge 4\pi^2 c_1^2 Gr^2$, where c_1 is as in Lemma 3.1. Thus Lemma 4.1 applies and $\|w\| \to 0$ as $t \to \infty$. Using the appropriate interpolation inequalities one can get convergence in stronger norms.□

5 A Reaction-Diffusion Equation

Consider on $\Omega = (0, L) \times (0, L)$ the two dimensional Chafee-Infante equation

$$\begin{aligned} \tfrac{du}{dt} - d\Delta u - b_1 u + b_2 u(t,x)\cdot u(t,x)u(t,x) &= 0 \\ u(0,x) &= u_0(x), \end{aligned} \tag{5.1}$$

where $u(t,x) \in \mathbb{R}^2$ and with periodic boundary conditions imposed. Hereafter d, b_1, b_2, etc denote positive constants. For existence and uniqueness of solutions to (5.1) see for example [18].

Theorem 5.1 *Let u, v solve (5.1) be such that*

$$\lim_{t\to\infty}(\langle u\rangle_{Q_j} - \langle v\rangle_{Q_j}) = 0,$$

for $1 \le j \le N$. If

$$N > \frac{1}{6}\frac{L^2 b_1}{d}.$$

Then

$$\lim_{t\to\infty} \|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(\Omega)} = 0.$$

Proof. Set $w = u - v$. Then w solves the equation

$$\frac{dw}{dt} - d\Delta w - b_1 w + b_2\{u(t,x)\cdot u(t,x)u(t,x) - v(t,x)\cdot v(t,x)v(t,x)\} = 0.$$

Taking the $L^2(\Omega)$ inner product with w gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 &+ d\|w\|^2 - b_1|w|^2 \\ &+ b_2\int_\Omega (u(t,x)\cdot u(t,x)u - v(t,x)\cdot v(t,x)v)\cdot(u(t,x) - v(t,x))dx = 0. \qquad (5.2)\end{aligned}$$

A calculation shows that for any vectors $a, b \in \mathbb{R}^2$

$$((a\cdot a)a - (b\cdot b)b)\cdot(a - b) \ge 0.$$

This implies that the last term in Equation (5.2) is positive and so

$$\frac{1}{2}\frac{d}{dt}|w|^2 + d\|w\|^2 - b_1|w|^2 \le 0.$$

Using Equation (3.4) we obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \left(d - \frac{b_1 L^2}{6N}\right)\|w\|^2 \le b_1 L^2 \gamma^2(w).$$

Now if we require $N > \frac{1}{6}\frac{L^2 b_1}{d}$

the result follows after an application of Lemma 4.1. □

Theorem 5.2 *Suppose $L^2 b_1/4\pi^2 d \ge 1$. Then the Hausdorff dimension of the attractor for Equation (5.1) satisfies*

$$\pi\left(\sqrt{\frac{b_1 L^2}{4\pi^2 d}} - \frac{\sqrt{2}}{2}\right)^2 - 1 \le dim\mathcal{A} \le \frac{\sqrt{2}(2+\sqrt{2})}{2\pi}\frac{b_1 L^2}{d} + 1.$$

If $L^2 b_1/4\pi^2 d < 1$, then $dim\mathcal{A}=0$.

Proof. To show the lower bound linearize Equation (5.1) about the steady state solution $u = 0$. Then it can be shown zero is a hyperbolic stationary solution, and the dimension of the unstable manifold is larger than or equal to the number of eigenvalues of $-\Delta$ on Ω satisfying $\lambda_k < b_1/d$ (see [30]). In our case the eigenvalues are of the form $\frac{4\pi^2}{L^2}(l^2 + k^2)$ where l, k are integers. We then need to calculate how many integers satisfy

$$l^2 + k^2 < \frac{L^2 b_1}{4\pi^2 d}.$$

To do this we follow the proof of Proposition 4.14 in [1]. There it is shown that $N_r = \{k \in \mathbb{Z}^2 |\ k \neq 0, |k| \le r\}$ satisfies

$$\pi\left(\sqrt{\frac{rL^2}{4\pi^2}} - \frac{\sqrt{2}}{2}\right)^2 \le N_r + 1.$$

To obtain the upper bound we use the trace formula as in [3]. In particular, these estimates come down to determining a constant such that

$$\sum_{j=1}^{m} \lambda_j \ge \frac{c_1'' m^2}{L^2}$$

(see [30], p.300). Again from Proposition 4.14 of [1] we find that

$$\lambda_j \geq \frac{4\pi}{(2+\sqrt{2})L^2} j.$$

This implies that $c_1'' = 2\pi/(2+\sqrt{2})$. We have also that

$$\dim \mathcal{A} - 1 \leq \frac{\sqrt{2}}{c_1''} \frac{b_1 L^2}{d}$$

again see ([30] p.302). □

Remark 5.3 Notice that the bounds for the lower dimension of the attractor, the number of determining finite volume elemen' and the upper dimension of the attractor are of the same order as $b_1 L^2/d \to \infty$.

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6 Appendix

Here we give a brief sketch of a proof for the Poincaré inequality used in Lemma 3.1. Let $\Omega = (0,l) \times (0,l)$. We first show the one dimensional version. For this let $v \in C_0^\infty(\mathbb{R})$. Integrating the equality

$$v(x) - v(y) = \int_y^x v'(z)dz$$

over y and changing the order of integration one finds

$$l(v(x) - \langle v \rangle) = \int_0^l v'(z)\varphi(z)dz,$$

where $\varphi(z) = \begin{cases} z & z \leq x \\ z - l & z > x \end{cases}$ and $\langle v \rangle = \frac{1}{l}\int_0^l v(x)dx$. It follows after squaring both sides, integrating over x, and using the Cauchy-Schwarz inequality that

$$|v|^2 \leq l\langle v \rangle^2 + \frac{l^2}{6}\|v\|^2.$$

To obtain the two dimensional version let $u(x_1, x_2) \in C_0^\infty(\mathbb{R}^2)$. Now apply the one dimensional version to $u(x_1, x_2)$ holding x_1 fixed. Then integrate over x_1 to obtain

$$\int_0^l \int_0^l u^2(x_1, x_2)dx_2dx_1 \leq \frac{1}{l}\int_0^l \left(\int_0^l u(x_1, x_2)dx_2\right)^2 dx_1 + \frac{l^2}{6}\int_0^l \int_0^l u_{x_2}^2(x_1, x_2)dx_2dx_1.$$

To handle the first term in the above inequality again apply the one dimensional version to $v(x_1) = \int_0^l u(x_1, x_2)dx_2$. After some algebra it follows that

$$|u|^2 \leq l^2|\langle u \rangle|^2 + \frac{l^2}{6}\|u\|^2.$$

Since $C_0^\infty(\mathbb{R}^2)|_{\overline{\Omega}}$ is dense in $H^1(\Omega)$, the inequality follows.

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