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Determining If Two Solid Ellipsoids Intersect

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Abstract

An analytical method is presented for determining if two ellipsoids share the same volume. The formulation involves adding an extra dimension to the solution space and examining eigenvalues that are associated with degenerate quadric surfaces. The eigenvalue behavior is characterized and then demonstrated with an example. The same method is also used to determine if two ellipsoids appear to share the same projected area based on an observers viewing angle. The following approach yields direct results without approximation, iteration, or any form of numerical search. It is computationally efficient in the sense that no dimensional distortions, coordinate rotations, transformations, or eigenvector computations are needed.

1 Introduction

As the U.S. Satellite Catalog transitions from general perturbations to special perturbations, the positional accuracy of each space object will be readily available in the form of a covariance matrix. These covariances can be used to determine probability of collision, radio-frequency interference, and/or incidental laser illumination. Because the probability calculations can be computationally burdensome, it is desirable to prescreen candidate objects based on user-defined thresholds. Specifically, each object can be represented by a covariance-based ellipsoid and then processed to determine if its uncertainty volume shares some space in common with anothers. Solid ellipsoids (or their projections) that do not intersect can be eliminated from further processing. This paper presents a simple analytical method to perform such screening.

To date, all ellipsoidal prescreening methods involve numerical searches [1]. For computational efficiency such prescreening is often reduced to spheres or "keep-out" boxes that have much larger volumes but allow for quick distance comparisons. The drawback to such screening is that these larger volumes cause many objects to become candidates for further (albeit unnecessary) processing. These methods result in increased downstream computational processing and/or increased operator workload to further assess potential satellite conjunctions.

The following method adds an extra dimension to the solution space. The subset of eigenvalues that are associated with intersecting degenerate quadric surfaces are then examined. The same method is also used to determine if two ellipsoids appear to share the same projected area based on viewing angle. The approach yields direct results without approximation, iteration, or any form of numerical search. It is computationally efficient in the sense that no dimensional distortions, coordinate rotations, transformations, or eigenvector computations are needed. This method expands the two-dimensional work of Hill [2] in his formulation of degenerate conics (i.e., the characteristic matrix is singular). It also furthers his work by examining the associated eigenvalue behavior.

This approach is not limited to Satellite Catalog applications. For computer graphics users such screening could be used to invoke a hidden line removal algorithm.

2 Ellipsoidal Formulation

Rogers and Adams [3] give various representational forms for an ellipsoid. Algebraically, the representation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0 \tag{1}$$

where A, B, C, D, E, F, G, H, J, and K are constants. In matrix form the same ellipsoid can be written as

$$XSX^T = 0 \tag{2}$$

where

$$X = [x \quad y \quad z \quad 1] \tag{3}$$

and

$$S = \frac{1}{2}\begin{pmatrix} 2A & D & F & G \\ D & 2B & E & H \\ F & E & 2C & J \\ G & H & J & 2K \end{pmatrix}. \tag{4}$$

The translation of the ellipsoid's center from the origin to $[X_0, Y_0, Z_0]$ can be accomplished by the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -X_0 & -Y_0 & -Z_0 & 1 \end{pmatrix} \tag{5}$$

where

$$X\,T\,S\,T^T X^T = 0. \tag{6}$$

Similarly, all points contained within the ellipsoid satisfy the constraint

$$X\,T\,S\,T^T X^T \leq 0. \tag{7}$$

Given a 3×3 covariance matrix C centered at $[X_0, Y_0, Z_0]$, the quadric representation of the ellipsoid would then be

$$X\,T \begin{pmatrix} Ci_{11} & Ci_{12} & Ci_{13} & 0 \\ Ci_{21} & Ci_{22} & Ci_{23} & 0 \\ Ci_{31} & Ci_{32} & Ci_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} T^T X^T = 0 \tag{8}$$

where Ci are the elements of the inverted covariance matrix.

3 Ellipsoidal Solution

For simplicity, assume a primary object is centered at the origin. An ellipsoid that corresponds to its positional covariance can be computed from the preceding, resulting in the equation

$$X\,A\,X^T = 0. \tag{9}$$

In the same manner a secondary object (center not colocated) and its ellipsoid can be appropriately translated relative to the primary object such that

$$X\,B\,X^T = 0. \tag{10}$$

If any X exists such that it satisfies Eqs. (9) and (10), then the primary and secondary ellipsoids intersect at that point. If some value of X satisfies the constraint for both objects as represented by Eq. (7), then that point lies inside both ellipsoids.

Equation (9) can be multiplied by a scalar constant λ with no loss in generality:

$$X\,(\lambda A)\,X^T = 0. \tag{11}$$

Subtracting Eqs. (11) and (10),

$$X\,(\lambda A - B)\,X^T = 0. \tag{12}$$

As explained by Hill [2], λ is chosen so that the matrix $(\lambda A - B)$ is singular. Because A is the characteristic matrix of an ellipsoid, it is invertible and can be used to alter Eq. (12) to produce

$$X\,A\,(\lambda I - A^{-1}B)\,X^T = 0. \tag{13}$$

This representation is more readily recognized as an eigenvalue formulation and also lends itself well to many mathematical software packages.

Substituting selected eigenvalues into Eq. (13) will produce characteristic matrices that represent degenerate quadric surfaces. If the X subset assumption holds regarding overlapping objects, then these surfaces must also pass through the points shared by the primary and secondary ellipsoids. It can be deduced [4] that if the ellipsoids just intersect (i.e., share a single point in common) then that solution vector must also be an eigenvector of $A^{-1}B$. The converse is not true as not all eigenvectors of $A^{-1}B$ will satisfy the ellipsoidal constraints of Eqs. (9) and (10). Eigenvectors with a zero in their last component are considered inadmissible because this formulation has been framed in a four-dimensional space with the last dimension fixed as shown in Eq. (3). An admissible eigenvector can be tested by simply scaling it to produce a one in the last component and then determining if it meets the ellipsoidal conditions as represented by matrices A and B.

When the primary and secondary ellipsoids overlap, then a family of solutions describes the intersection. For such cases two of the eigenvalues become complex. This is demonstrated in the Appendix and proven [4].

4 Observed Eigenvalue Behavior

To gain an understanding of the eigenvalues when the ellipsoids do not just intersect at a single point, the locus of values is plotted for various cases by altering size, shape, orientation, and location. Figure 1 is representative of all cases tested. In each set of cases, the two ellipsoids are initially defined to be completely outside each other. There are always two negative, real eigenvalues that produce admissible eigenvectors. The vectors do not satisfy Eqs. (9) and (10), and no point is shared in common between the ellipsoids.

The primary ellipsoid is then continually increased in size until it just intersects the secondary. This means that only a single, unique point satisfies Eqs. (9) and (10). The two eigenvalues move towards each other until they meet (repeated). At this point the admissible eigenvectors give the solution for ellipsoids intersecting at a point.

The scaling then continues so that both ellipsoids share some volume in common. The two admissible eigenvalues become complex conjugates.The real portion of the eigenvectors satisfies the inequality for both ellipsoids as defined in Eq. (7). The location indicated by these vectors is always shown to be inside both ellipsoids; therefore, they intersect.

As the primary ellipsoid continues to grow, it eventually intersects the

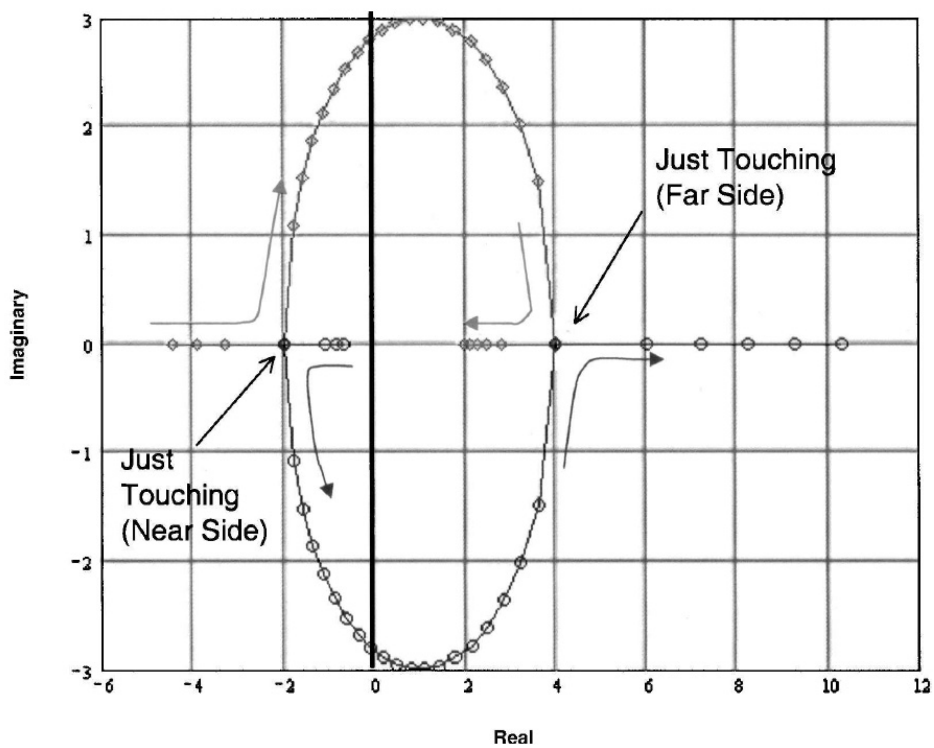

Figure 1: Representative locus of admissible eigenvalues

far side of the secondary. The two admissible eigenvalues again become real and repeated, but are positive instead of negative. Again, those eigenvectors define the exact point where the ellipsoids intersect.

Scaling beyond this point always gives two positive real admissible eigenvalues that move away from each other. In all cases tested it means that some portion of the primary surface has entered and exited the secondary ellipsoid, but does not mean that the primary has completely engulfed the secondary (Fig. 2). A simplified mathematical explanation for eigenvalue behavior is presented in the Appendix. The complete, n-dimensional, mathematical proof was done by Chan [4] to verify these observations.

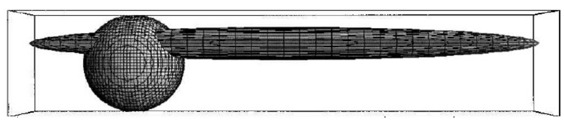

Figure 2: Complete penetration of one ellipsoid by another

5 Simple Ellipsoidal Example

This example involves a primary ellipsoid that is four units long on the x axis and two units long on the y and z axes. The secondary is six units long on the x axis, four on the y, and eight on the z with its center at $[7, 0, 0]$. The primary should just intersect the secondary on the near side when scaled by two and just intersect the far side when scaled by five. The intersecting will occur on the x axis.

The initial A and B matrices are

$$A = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{14}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -7 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.111 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.063 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} 0.111 & 0 & 0 & -0.778 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.063 & 0 \\ -0.778 & 0 & 0 & 4.444 \end{pmatrix}. \tag{16}$$

Scaling the primary ellipsoid by a factor of n is done by simply multiplying the last element of A by n^2 .

Table 1 shows the history of the eigenvalues and their interpretations.

Scale	1	2	3	4	5	6
Eig 1	-0.114	-0.333	$-0.025 + 0.221i$	$0.083 + 0.114i$	0.133	0.276
Eig 2	-3.886	-0.333	$-0.025 - 0.221i$	$0.083 - 0.114i$	0.133	0.045
Vector	N/A	$\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 5.429 - 2.556i \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 7.429 - 2.969i \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	N/A
Notes	Outside	Touch	Overlap	Overlap	Touch	Past

Table 1: Effects of scaling on eigenvalues and eigenvectors

6 Coordinate Reduction Through Projection

Although two ellipsoids might not share the same space, when viewed from certain angles one might appear to cover or overlap the other. Analysis of such circumstances is necessary to prevent accidental laser illumination if a secondary object is in or near the line of sight of the primary. Equally important is determining the possibility of radio-frequency interference on a secondary object. For computer graphics users such analysis would indicate when to invoke a hidden line removal algorithm. Coordinate rotations are accomplished through the following matrix representation:

$$X\, R\, S\, R^T X^T = 0 \tag{17}$$

where rotation about the x axis of angle α produces

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

rotation about the y axis of angle β yields

$$R_y = \begin{pmatrix} \cos(\beta) & 0 & -\sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

and rotation about the z axis of angle θ is

$$R_z = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

The individual matrices can be multiplied to produce an overall rotation matrix R. The reader is cautioned to pay close attention to the signs of the sine terms; this is necessary for a positive right hand rule convention. Also, the order of multiplication is important to ensure the desired overall coordinate rotation.

Coordinate reduction is done by means of a simple orthographic projection in the rotated space to eliminate one component. The choice of coordinate for reduction is a matter of personal preference. The new z component was chosen for this work, resulting in

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$X\,(P\,R\,S\,R^T P^T)X^T = 0. \tag{22}$$

When the projection is completed, the expression in parentheses becomes singular. To proceed, it is necessary to reduce the dimension of the state vector and associated formulation as will be explained in the next section.

It is still necessary to translate the resultant based on the new coordinate frame. To do so, a new translation vector is computed and inserted into the translation matrix

$$[X_1 \quad Y_1 \quad Z_1 \quad 1] = [X_0 \quad Y_0 \quad Z_0 \quad 1]\, R \tag{23}$$

$$T_{new} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -X_1 & -Y_1 & -Z_1 & 1 \end{pmatrix}. \tag{24}$$

Combining all terms in the correct order produces

$$X\, T_{new}\, P\, R\, S\, R^T P^T T_{new}^T X^T = 0. \tag{25}$$

7 Elliptical Formulation and Solution

As one would expect, determining if two ellipses intersect (or if one lies entirely within the other) is identical to the ellipsoidal formulation reduced by one dimension. In matrix form the new z component resulting from coordinate rotation is eliminated, and the equations are reduced by one dimension such that

$$X = [x \quad y \quad 1]. \tag{26}$$

An ellipsoid described by the rotated 4×4 A matrix is projected into the new $x - y$ plane by removing the third row and column to produce the 3×3 AP matrix. The relationship

$$X\, A\, P\, X^T = 0 \tag{27}$$

now describes the primary objects projected ellipse in the new, dimensionally reduced frame. The same projection and reduction is done for the secondary object to determine the BP matrix

$$X\,B\,P\,X^T = 0. \tag{28}$$

If any X exists such that it satisfies Eqs. (28) and (29), then the primary and secondary projections intersect at that point. If some value of X satisfies the constraint for both projections as represented by Eq. (7), then that point lies inside both ellipses.

The evaluation is identical to the ellipsoidal one, observing the admissible eigenvalue behavior of $AP^{-1}BP$ to determine if the ellipses shared the same space. If two are negative real and different, then the ellipses share no area in common. If two are negative real and identical, then they just intersect on the secondarys side nearest the origin. If two are complex conjugates, the ellipses intersect at two points. If two are positive real and identical, then they share area and just intersect on the far side. If all are positive real, then one penetrates or engulfs the other.

8 Conclusions

A simple analytical method has been developed to determine if two ellipsoids share the same volume. This method can be used to alert operators of existing or impending conjunctions. The formulation involves adding an extra dimension to the solution space and examining the admissible eigenvalues. The admissible eigenvalues are examined to determine if any volume is shared. If volume is shared, a subset of the eigenvalues defines degenerate quadric surfaces that pass through the points of intersection. The same method is used to determine if two ellipsoids appear to share the same projected area based on viewing angle. This approach yields direct results without approximation, iteration, or any form of search.

Appendix: Single Dimensional Analysis

The mathematical underpinnings for the assertions of eigenvalue behavior in two and three dimensions are proven here for a single dimension; the

n-dimensional proof is found in Chan [4]. All objects can be scaled and rotated so that the primary is centered at the origin with unit dimensions. The primary ellipsoid becomes a sphere; the primary ellipse a circle. By selecting the proper viewing geometry, two ellipsoids that do not intersect can be projected to two ellipses that do not intersect; these ellipses can then be projected to two lines that do not intersect. This process reduces the problem to a single dimension.

For a single dimension the primary object is a line ranging from -1 to $+1$ with its "surface" represented by the endpoints. The secondary is also a line ranging from $(x_0 - a)$ to $(x_0 + a)$. Scaling can be accomplished so that the only case needing consideration is when $x_0 > 0$ and $a > 0$. Algebraically, these endpoints can be expressed as

$$x^2 = 1. \tag{A1}$$

$$a^{-2}(x - x_0)^2 = 1. \tag{A2}$$

In matrix form these become

$$\begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \tag{A3}$$

$$\begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix} \begin{pmatrix} a^{-2} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0. \tag{A4}$$

The eigenvalues of $(\lambda A - B)$ are solved using

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A5}$$

$$B = \begin{pmatrix} \frac{1}{a^2} & \frac{-x_0}{a^2} \\ \frac{-x_0}{a^2} & \frac{(x_0)^2}{a^2} - 1 \end{pmatrix} \tag{A6}$$

$$\lambda = \frac{-(x_0)^2 + a^2 + 1 \pm \sqrt{(a+1-x_0)(a+1+x_0)(a-1-x_0)(a-1+x_0)}}{2a^2}. \tag{A7}$$

Figure A1 helps in visualizing all possible values, both real and complex, of the solution.

Figure A1: One-dimensional analysis $(a > 0, x_0 > 0)$

Figure A1 shows that when $(x_0 - a) > 1$ the lines do not intersect. Placing this constraint into Eq. (A7) will always produce negative, real, distinct eigenvalues.

Increasing the value a and/or decreasing the value x_0 such that $(x_0 - a) = 1$ allows the lines to just intersect on the positive (near) side. The eigenvalues repeat with a value of $-1/a$.

Continuingto increase a or decrease x_0 such that $-1 < (x_0 - a) < 1$ and $(x_0 + a) > 1$ causes the lines to overlap, but not completely. The eigenvalues will always be complex conjugates under these conditions.

Should $(x_0 - a) = -1$ and $(x_0 + a) > 1$ then the lines overlap and just intersect on the negative (far) side. The eigenvalues repeat with a value of $+1/a$.

In the event that $-1 < (x_0 - a) < 1$ while $(x_0 + a) \leq 1$, then the secondary line is completely inside the primary, and the eigenvalues are positive, real, and distinct.

For the final case $(x_0 - a) < -1$ the primary line is completely inside the secondary, and the eigenvalues are again positive, real, and distinct.

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