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DETERMINING NODES, FINITE DIFFERENCE SCHEMES AND INERTIAL MANIFOLDS

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DETERMINING NODES, FINITE DIFFERENCE SCHEMES AND INERTIAL MANIFOLDS

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Abstract : – The aim of this paper is to present a connection between the concepts of *determining nodes* and *inertial manifolds* with that of finite difference and finite volumes approximations to dissipative partial differential equations. In order to illustrate this connection we consider the 1-D Kuramoto-Sivashinsky equation as a instructive paradigm. We remark that the results presented here apply to many other equations such as the 1-D complex Ginzburg-Landau equation, the Chafee-Infante equation, etc....

1. Introduction. In this paper we consider certain class of one dimensional dissipative evolution partial differential equations (P.D.E.'s) - that have an *Inertial Manifold* (I.M.). An I.M. for a dissipative evolution equation has the following properties :

(i) it is a finite dimensional Lipschitz manifold

(ii) which is positively invariant under the flow induced by the solutions of the equation

(iii) and it attracts all the solutions with an exponential rate

(cf. Constantin et al. (1988,1989), Foias et al. (1988d, 1989)).

So far, inertial manifolds were constructed in the phase space as graphs of functions. Typically, such a function determines the high Fourier modes (high wave numbers) in terms of the lower Fourier modes (lower wave numbers). In this paper we will present a different representation of the I.M.. More pricesly, we will show that the functions, which are points on the I.M., are determined in a unique fashion by their values in a fixed number of points in the domain *(nodes)*. That means that one can parametrize the I.M. in terms of nodal values of those functions which are on the I.M.. Also, we will see that the number of these points is comparable with the dimension of the I.M. (Theorem 3.1). We also show that a similar result is available if we consider the averaged values *(finite volumes)* of the functions at the points instead of the nodal values. We remark that in the latter case the number of points necessary for the parametriztion is less than that in the former case (Theorem 3.2). The above representation of I.M.'s enables us to introduce a new dynamical system of the evolution of the nodal values, and respectively the averaged nodal values (finite volumes), of the solutions which is equivalent to the dynamical system of the P.D.E.. We remark that the Kuramoto-Sivashinsky equation is just an illustrative example and that our results apply directly to many other dissipative equations such as

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the 1-D complex Ginzburg-Landau equation (cf. Doering et al. (1988), and Ghidaglia and Héron (1987)), the Chafee-Infante equation (cf. Jolly (1989)), etc... Let us mention here that this work was inspired from the work of Foias and Temam (1984) on the existence of finite number of determining nodes for the Navier-Stokes equations. This concept of determinig nodes is important from the practical point of view. This is because all the experimental data are, in general, collected from measurments at a finite number of points, such as the temperature, the velocity, etc.... However, we would like to emphasize that the number of determining nodes cannot always be very low (see Foias and Titi (1990)).

In order to approximate the evolution of nodal values it is natural, for instance, to use the semi-finite difference scheme. However, while introducing the semi-finite difference scheme one should keep in mind the dynamical features of the P.D.E. – especially the dissipation. In section 4 we present a dissipative semi-finite difference scheme of order $O(h^{3/2})$. It is remarkable that other schemes, which are of the same order, could lead to numerical artifacts as it is shown, computationally as well as analytically, in Foias et al. (1990).

In recent years a number of *approximate inertial manifolds* and their induced *approximate inertial forms* were introduced in literature (see e.g. Fabes et al. (1990), Foias et al. (1987, 1988b, 1988e,1989), Marion (1989), Temam (1988b), Titi (1988, 1990a)). Since all these approximate I.M.'s are based on a Galerkin type of approximation, they are sometimes called *nonlinear Galerkin methods*. It has been shown analytically that the nonlinear Galerkin schemes converge to the real solution (cf. Marion and Temam (1989)), and that they converge with a faster rate than the standard Galerkin approximation (cf. Marion and Titi (1990)). Also, they have been implemented in real computations (cf. Foias et al. (1988a), Jauberteau et al. (1990) and Jolly et al. (1990a, 1990b)), and gave some encouraging results.

The dissipative semi-finite difference scheme, introduced in section 4, is a small perturbation of evolution equation of the nodal values, so in view of the above one can consider it as an approximate inertial form for the Kuramoto-Sivashinsky equation. Moreover, we expect it to capture the "essential dynamics" of the Kuramoto-Sivashinsky by virtue of the recent work of Sell and Pliss (1990).

This paper is organized as follows. In section 2 we recall the Kuramoto-Sivashinsky equation and some of the relavent results. In section 3 we show that the dynamics on the I.M. is equivalent to that of the nodal values and the finite volumes, provided we take enough nodes. In section 4 we present a dissipative semi-finite difference approximation to the evolution of the nodal values. It is shown in section 5 that the dissipative semi-finite difference scheme has an I.M., in addition this I.M. enjoys the *exponential tracking property* (see e.g. Foias et al. (1989)).

2. Functional setting and preliminary results. As an illustrative example of our idea, we consider the one dimensional Kuramoto-Sivashinsky equation with periodic boundary condition, with period $L > 0$ (cf. Nicolaenko-Scheurer (1984), Nicolaenko et al. (1985) and references therein):

$$\text{(2.1)} \qquad \begin{cases} u_t + u_{xxxx} + u_{xx} + u_x u = 0 & \text{in} \quad (0,\infty) \times \mathbb{R} \\ u(t,x) = u(t,x+L) & \text{in} \quad (0,\infty) \times \mathbb{R}\,. \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R} \end{cases}$$

The problem (2.1) is known to be well-posed and has a regular global solution (cf. Nicolaenko-Scheurer (1984) and Tadmor (1986)). We denote the solution of (2.1) $u(t) = S(t)u_0$; $S(t)$ is a semigroup of nonlinear operators. Denote:

$$H^m_{\text{per}}((0,L)) = \Big\{ \varphi \in H^m((0,L)) : \varphi^{(k)}(0) = \varphi^{(k)}(L) \\ \text{for } k = 0,1,\ldots,m-1; \int_{(0,L)} \varphi(x)dx = 0 \Big\}$$

where $H^m((0,L))$ denotes the usual Sobolev space of index m, for $m \geq 1$. Denote the inner product in $L^2((0,L))$ by $(\cdot,\cdot)$ and the corresponding norm by

$$|\varphi| = \left(\int_{(0,L)} |\varphi(x)|^2 dx \right)^{1/2} \qquad \forall \varphi \in L^2((0,L)).$$

We set $D(A) = H^4_{\text{per}}((0,L))$ the domain of the operator $A = \frac{\partial^4}{\partial x^4}$. A is an unbounded self-adjoint positive operator. The functions

$$w_k(x) = \sin\left(\frac{2\pi k}{L} x\right), \quad v_k(x) = \cos\left(\frac{2\pi k}{L} x\right)$$

are eigenfunctions of the operator A with corresponding eigenvalues $\lambda_k = \left(\frac{2\pi k}{L}\right)^4$ for $k = 1,2,\ldots$. For every $u, v \in H^1_{\text{per}}((0,L))$ we denote by

$$B(u,v) = \frac{2}{3} u v_x + \frac{1}{3} u_x v.$$

It is clear that

$$(B(u,v),w) = -(B(u,w),v) \quad \forall u,v,w \in H^1_{\text{per}}((0,L)).$$

The equation (2.1) is then equivalent to the functional differential equation

$$\frac{du}{dt} + Au - A^{1/2}u + B(u,u) = 0 \quad \text{in } V',$$

where V' is the dual space of $V = D(A^{1/2})$ (see e.g. Temam (1988a)).

It is known that if we restrict ourselves to the invariant subspace of odd functions then the dynamical system defined by $S(t)$, the semigroup of solution operator, is dissipative. More precisely, let

$$H = \{ \varphi \in L^2((0,L)) : \varphi \text{ is odd, i.e., } \varphi(x) = -\varphi(L-x) \text{ a.e. in } (0,L) \},$$

then $S(t)H \subset H$ for all $t > 0$. Moreover, we have :

THEOREM 2.1. *There exist convex sets* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *bounded and closed in* H *and* $D(A^{1/4}) \cap H = H^1_{\text{per}}(0,L) \cap H$ *respectively, such that:*

(i) $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ *and* $S(t)(\mathcal{B}_0 \cap \mathcal{B}_1) \subset (\mathcal{B}_0 \cap \mathcal{B}_1)$ *for all* $t > 0$. *Moreover,*

(ii) *for every ball* $\mathcal{B} \subset H$ *centered at the origin with radius* $\rho > 0$, *there exists a time* $T^*(\rho) > 0$ *such that:*

$$S(t)\mathcal{B} \subset (\mathcal{B}_0 \cap \mathcal{B}_1) \quad \text{for all} \quad t > T^*(\rho).$$

(We call the set $(\mathcal{B}_0 \cap \mathcal{B}_1)$ *an absorbing set.)*

For the proof see e.g. Nicolaenko et al. (1985), Foias et al. (1988c) and Temam (1988a).

Since we do not know if a similar result holds for general initial data, and since we are interested in studying the long time dynamics of equation (2.1), we also restrict ourselves in this paper to the invariant subspace of odd functions H. In this case, and due to the dissipation property (existence of an absorbing set), it is known that the equation (2.1) has a compact global (universal) attractor which has a finite Hausdorff and Fractal dimensions (see e.g. Foias et al. (1988c), Hale (1988), Nicolaenko et al. (1985), and Temam (1988a)). Moreover, this attractor lies in a finite dimensional smooth (at least Lipschitz) invariant manifold that attracts every trajectory exponentially. This invariant manifold is called Inertial Manifold (I.M.). There are several techniques to construct I.M.'s for the Kuramoto-Sivashinsky equation, see for instance Constantin et al. (1988, 1989), Fabes et al. (1990), Foias et al. (1988c, 1988c), Mallet-Paret and Sell (1988). In this paper we will follow the *Spectral Barriers* method which was introduced by Constantin et.al. (1989). First we remark that for technical reasons one needs to prepare the equation (2.1) in order to construct its I.M.. Namely, one needs to truncate, in a smooth way, the nonlinear term outside of a "large" set, say $\mathcal{B}$, which contains the absorbing set $\mathcal{B}_0 \cap \mathcal{B}_1$ (e.g. one can choose $\mathcal{B}$ to be double the size of $\mathcal{B}_0 \cap \mathcal{B}_1$). Both equations, the prepared and the original, are identical inside $\mathcal{B}$, consequently, they will have the same long time dynamics (global attractor). In fact, since $\mathcal{B}_0 \cap \mathcal{B}_1$ is invariant then both equations will have the same flow inside $\mathcal{B}_0 \cap \mathcal{B}_1$. In general, there are few ways to prepare an equation, however, in principle, they are all similar. In this paper we will always refer to the preparation suggested by Constantin et al. (1989) for the Kuramoto-Sivashinsky equation, and we recall from there the following result:

THEOREM 2.2. *There exists a positive integer* $M (M \sim L^3)$ *such that the prepared Kuramoto-Sivashinsky equation has an inertial manifold* $\mathcal{M} \subset D(A)$ *of dimension* M. *Moreover, for every* $u_1, u_2 \in \mathcal{M}$ *we have*

$$|A^{1/2}(u_1 - u_2)|^2 \leq \lambda |u_1 - u_2|^2 \tag{2.2}$$

where $\lambda = \frac{\lambda_{M+1} + \lambda_M}{2}$, *(*$\lambda$ *is a spectral barrier).*

In the next theorem we recall the *exponential tracking property* or *the asymptotic completeness property* of I. M.'s from Foias et al. (1989) (see also Constantin et al. (1988, 1989)).

THEOREM 2.3. *For every solution $u(t)$ of (2.1) there exist a time $T^*(|u(0)|) > 0$ and a solution $v(t)$ of (2.1), which lies on the inertial manifold, such that*

$$|A^{1/4}(u(t+T^*) - v(t))| \leq C_1 e^{-\frac{\lambda_{M+1}}{2}t} \quad \text{for all} \quad t > 0 \quad , \tag{2.3}$$

where C_1 is a positive constant which depends on $|u(0)|$.

Let us denote by P_k the orthogonal projection from the Hilbert space H onto the subspace $H_k := \text{span}\{w_1, \ldots, w_k\}$. Then one seeks the I.M. as a graph of a global Lipschitz function

$$\Phi : H_M = \text{span}\{w_1, \ldots, w_M\} \to H_M^{\perp}.$$

The reduction of the Kuramoto-Sivashinsky equation to the I.M., inside the absorbing ball, is given by the inertial from

$$\frac{d}{dt}p + Ap - A^{1/2}p + P_M B(p + \Phi(p), p + \Phi(p)) = 0. \tag{2.4}$$

THEOREM 2.4. *Let $u(t)$ and $v(t)$ be any two solutions of equation (2.1) such that*

$$\lim_{t\to\infty} |P_k(u(t) - v(t))| = 0 \quad \text{for some} \quad k \geq M. \tag{2.5}$$

Then

$$\lim_{t\to\infty} |(u(t) - v(t))| = 0. \tag{2.6}$$

Proof. By the exponential tracking property, Theorem 2.3, there exist two solutions $u_{\mathcal{M}}$ and $v_{\mathcal{M}}$ of equation (2.1) which lie in $\mathcal{M}$ such that

$$\lim_{t\to\infty} |A^{1/4}(u(t + T^*(|u(0)|)) - u_{\mathcal{M}}(t))| = 0 \quad , \tag{2.7}$$

and

$$\lim_{t\to\infty} |A^{1/4}(v(t + T^*(|v(0)|)) - v_{\mathcal{M}}(t))| = 0 \quad . \tag{2.8}$$

Because $k \geq M$ we use (2.5), (2.7) and (2.8) to get

$$\lim_{t\to\infty} |P_M(u_{\mathcal{M}}(t - T^*(|u(0)|)) - v_{\mathcal{M}}(t - T^*(|v(0)|)))| = 0 \quad .$$

Since the I.M. in our case is a graph of a global Lipschitz function, Φ, we obtain

$$\lim_{t\to\infty} |u_{\mathcal{M}}(t - T^*(|u(0)|)) - v_{\mathcal{M}}(t - T^*(|v(0)|))| = 0 \quad . \tag{2.9}$$

Combine (2.7), (2.8) and (2.9) to get (2.6). □

We would like to mention here that a similar result regarding the existence of finite number of determining modes was first established for the Navier-Stokes equation by Foias and Prodi (1967), even though the existence of I.M.'s to the Navier-Stokes equations is still an open problem. Later an explicit estimate for the number of determining modes for the Navier-Stokes equation was given in Foias et al. (1983). Following the latter work Nicolaenko et al.(1985) established an explicit estimate for the number of determining mode for the Kuramoto-Sivashinshy equation. Since in their approach they take advantage of the nice upper bounds available for the time averaging of certain normes of the solutions, they get a smaller estimate for the number of determining modes than the one we present in Theorem 2.4. Since the I.M. in our case is constructed in the space of Fourier modes, Theorem 2.4 brings no surprises. Nevertheless, the idea of its proof, which is a nice application of the exponential tracking property, will be applied later in section 3.1 to the determining nodes and which is extendable to other parametrizations of the I.M..

Remark 2.1. (i) Denote $\mathcal{M}(t) = S(t)(\mathcal{B}_0 \cap \mathcal{B}_1 \cap \mathcal{M})$. It is clear from Theorem 2.1 that $\mathcal{M}(t) \subset \mathcal{M}(s)$ for $t \geq s > 0$. Also, by applying the usual energy estimates and Sobolev imbedding theorems one can easily infer that

$$\mathcal{M}(t) \subset C^\infty_{\text{per}}([0,L]) = \Big\{ \varphi \in C^\infty([0,L]) : \varphi^{(k)}(0) = \varphi^{(k)}(L) \quad \text{for } k = 0,1,2,\dots \Big\}$$

(ii) Let $t_0 > 0$, then it is not difficult to show, by applying the methods of Foias and Temam (1979), that $S(t)|_{\mathcal{M}(t_0)}$ can be extended to a complex analytic function, with values in $D(A^2)$, in a band $\mathcal{S} \subset \mathbb{C}$ about the interval $(-t_0, \infty)$ (see e.g. Jolly et al. (1990a)).

3. Nodal values and finite volumes.

3.1 The evolution of nodal values. In this section we will derive a dynamical system which is defined by the evolution of the nodal values of the solution of (2.1) at N fixed nodes (i.e., at N fixed points in the interval $[0, L)$). We also verify that this dynamical system is equivalent to the flow on the invariant part of the I.M. which is contained in $\mathcal{B}_0 \cap \mathcal{B}_1$ (i.e., on $\mathcal{M} \cap \mathcal{B}_0 \cap \mathcal{B}_1$, see Theorem 2.1). As a result one concludes that the whole dynamics is determined by the evolution of these nodal values.

Define $\Theta_N : H^1_{\text{per}}((0,L)) \to \mathbb{R}^N$ as follows

$$\Theta_N(u) = (u(x_j))_{j=0}^{N-1}. \tag{3.1}$$

where $x_j = jh$ for $j = 0, 1, \dots, N-1$, and $h = \frac{L}{N}$.

We recall that $H^1((0,L))$ is continuously imbedded in the space $C^{0,1/2}([0,L])$ of Hölder continuous functions in $[0,L]$ with exponent $1/2$ (see e.g. Adams (1975), and Lions and Magenes (1972)). Therefore, the mapping Θ_N is well defined and the equation (3.1) makes sense. Notice that $\Theta_N(u)$ is a sampling of the periodic function $u(x)$. It is shown, in the next lemma, that if N is large enough then the sampling determines the points (the functions) on the inertial manifold in a unique way.

Let $h = \frac{L}{N}$ be as above, for every $\vec{\xi}, \vec{\eta} \in \mathbb{R}^N$ we define the inner product in $\mathbb{R}^N$:

$$\langle \vec{\eta}, \vec{\xi} \rangle = h \sum_{k=1}^{N} \xi_k \eta_k$$

and the corresponding norm :

$$|\vec{\xi}| = \left(h \sum_{k=1}^{N} \xi_k^2 \right)^{1/2} .$$

THEOREM 3.1. *Let N be large enough satisfying*

$$N > L\lambda^{1/4} \sim 2\pi(M+1) \tag{3.2}$$

where λ is as in Theorem 2.2. Then $\Theta_N|\mathcal{M}$ is a Lipschitz homeomorphism from $\mathcal{M}$ onto $\Theta_N(\mathcal{M})$, where $\mathcal{M}$ is endowed with the H topology and $\Theta_N(\mathcal{M})$ with that of $\mathbb{R}^N$.

Proof. Let $u_1, u_2 \in \mathcal{M}$, set $w = u_1 - u_2$. From Theorem 2.2 $w \in D(A)$. By the Sobolev imbedding theorem (see e.g. Adams (1975), and Lions and Magenes (1972)) we have

$$||w||_\infty \le C(L)|A^{1/2}w|,$$

(where $C(L)$ is a constant which depends only on L) using (2.2) we conclude

$$|\Theta_N(w)| \le hN^{1/2}||w||_\infty \le hC(L)(N\lambda)^{1/2}|w|,$$

thus, $\Theta_N|\mathcal{M}$ is Lipschitz continuous. Next, we verify that $\Theta_N|\mathcal{M}$ has a Lipschitz continuous inverse. Let $u_1, u_2 \in \mathcal{M}$ and set $w = u_1 - u_2$, then

$$|w|^2 = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} (w(x))^2 dx. \tag{3.3}$$

Denote $y_k = \frac{x_{k-1}+x_k}{2}$ for $k = 1, 2, \dots, N$.

Since $w \in D(A)$ (see Theorem 2.2) we have

$$(w(x))^2 = (w(x_{k-1}))^2 + 2\int_{x_{k-1}}^{x} w(y)w'(y)dy \quad \text{for } x \in (x_{k-1}, y_k).$$

and

$$(w(x))^2 = (w(x_k))^2 - 2\int_x^{x_k} w(y)w'(y)dy \quad \text{for } x \in (y_k, x_k).$$

Integrate the above equalities with respect to x over (x_{k-1}, y_k) and (y_k, x_k), respectively, to get

$$\int_{x_{k-1}}^{y_k} (w(x))^2 dx = \frac{L}{2N}(w(x_{k-1}))^2 + 2\int_{x_{k-1}}^{y_k} w(y)w'(y)(y_k - y)dy,$$
$$\int_{y_k}^{x_k} (w(x))^2 dx = \frac{L}{2N}(w(x_k))^2 - 2\int_{y_k}^{x_k} w(y)w'(y)(y - y_k)dy.$$

Add the above inequalities to obtain

$$\int_{x_{k-1}}^{x_k} (w(x))^2 dx \le \frac{L}{2N}((w(x_{k-1}))^2 + (w(x_k))^2) + \frac{L}{N}\int_{x_{k-1}}^{x_k} |w(y)|\ |w'(y)|dy,$$

apply the Cauchy-Schwarz inequality

$$\int_{x_{k-1}}^{x_k} (w(x))^2 dx \le \frac{L}{2N}((w(x_{k-1}))^2 + (w(x_k))^2) + \frac{L}{N}\left(\int_{x_{k-1}}^{x_k} (w(y))^2 dy\right)^{1/2}\left(\int_{x_{k-1}}^{x_k} (w'(y))^2 dy\right)^{1/2}.$$

From the above and equality (3.3) we get

$$|w|^2 \le \frac{L}{2N}\sum_{k=1}^{N}((w(x_{k-1}))^2 + (w(x_k))^2) + \frac{L}{N}\sum_{k=1}^{N}\left[\left(\int_{x_{k-1}}^{x_k} (w(y))^2 dy\right)^{1/2}\left(\int_{x_{k-1}}^{x_k} (w'(y))^2 dy\right)^{1/2}\right].$$

We apply the Cauchy-Schwarz on the summation, and since $w(x)$ is periodic, we get:

$$|w|^2 \le |\Theta_N(w)|^2 + \frac{L}{N}|w|\ |A^{1/4}w|. \tag{3.4}$$

We interpolate in (3.4)

$$|A^{1/4}w| \le |w|^{1/2}|A^{1/2}w|^{1/2}$$

to obtain

$$|w|^2 \le |\Theta_N(w)|^2 + \frac{L}{N}|w|^{3/2}|A^{1/2}w|^{1/2}.$$

we substitute (2.2)

$$|w|^2 \leq |\Theta_N(w)|^2 + \frac{L}{N}\lambda^{1/4}|w|^2,$$

and because of (3.2) we reach

$$|w|^2 \leq \left(1 - \frac{L}{N}\lambda^{1/4}\right)^{-1} |\Theta_N(w)|^2,$$

which concludes our proof. □

We fix N large enough satisfying (3.2). It is clear from (3.2) and Theorem 2.2 that for large L one can choose $N \sim L^3$ to satisfy (3.2). To simplifyour notation we set Θ for Θ_N.

We fix $t_0 > 0$ and we set $\mathcal{M}_0 = \mathcal{M}(t_0)$ (see Remark 2.1). On $\Theta(\mathcal{M}_0)$ we define the semiflow

$$\Sigma(t)\vec{\xi}_0 = \Theta(S(t)(\Theta^{-1}(\vec{\xi}_0))) \quad \forall \vec{\xi}_0 \in \Theta(\mathcal{M}_0) \subset \mathbb{R}^N, \qquad \text{for } t > 0. \tag{3.4}$$

On account of Remark 2.1 it is easy to see that $\Sigma(t)$ is well defined for all $t > 0$. Moreover, by virtue of Lemma 3.1, $S(t)|_{\mathcal{M}_0}$ and $\Sigma(t)$ are conjugate dynamical systems (i.e. topologically equivalent).

Recall from Remark 2.1 that for every $u_0 \in \mathcal{M}_0, S(t)u_0 \in \mathcal{M}_0$, for all $t > 0$, and $S(t)u_0$ is analytic in t with values in $D(A^2)$. Also, since $D(A^2) \subset C^7([0,L])$ (by Sobolev imbedding theorem – see e.g. Adams (1975), and Lions and Magenes (1972)) then $u(t,x) = S(t)(u_0(x))$ is a classical solution of (2.1). Accordingly, if we set $\vec{\xi}(t) = \Sigma(t)\vec{\xi}_0$, for $\vec{\xi}_0 \in \Theta(\mathcal{M}_0)$, and $U(x;\vec{\xi}(t)) = \Theta^{-1}(\vec{\xi}(t))$, then U satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}U(x;\vec{\xi}(t)) + \frac{\partial^4}{\partial x^4}U(x;\vec{\xi}(t)) + \frac{\partial^2}{\partial x^2}U(x;\vec{\xi}(t)) \\ &\qquad + U(x;\vec{\xi}(t))\frac{\partial}{\partial x}U(x;\vec{\xi}(t)) = 0. \end{aligned} \tag{3.5}$$

Since $U(\cdot;\vec{\xi}(t)) \subset \mathcal{M}_0 \subset D(A^2)$ for all $t > 0$, then $U(\cdot;\vec{\xi}(t)) \in C^{7,1/2}([0,L])$ and it is uniformly bounded with the $C^7([0,L])$ norm, for all $t > 0$. In particular, equation (3.5) holds at $x = x_j$ for $j = 0, 1, \ldots, N$; and we get

$$\begin{aligned} &\frac{\partial}{\partial t}U(x_j;\vec{\xi}(t)) + \frac{\partial^4}{\partial x^4}U(x_j;\vec{\xi}(t)) + \frac{\partial^2}{\partial x^2}U(x_j;\vec{\xi}(t)) \\ &\qquad + U(x_j;\vec{\xi}(t))\frac{\partial}{\partial x}U(x_j;\vec{\xi}(t)) = 0 \end{aligned} \tag{3.6}$$

Equation (3.6) controls the evolution of the nodal values $U(x_j;\vec{\xi}(t))$ for $j = 0, 1, \ldots, N-1$. Since $\vec{\xi}(t) = (U(x_j;\vec{\xi}(t)))_{j=0}^{N-1}$, then (3.6) is equivalent to the reduction of (2.1) to the I.M., i.e. (3.6) is equivalent to the *inertial form* (2.4).

Next we show that the number of determining nodes for the Kuramoto-Sivashinsky equation is at most equal to M the dimension of the I.M..

COROLLARY 3.1. *Let $u(t)$ and $v(t)$ be any two solutions of equation (2.1) such that*

$$\lim_{t\to\infty} |(u(t,x_j) - v(t,x_j))| = 0 \quad \text{for} \quad j = 0,1,\dots,N-1, \tag{3.7}$$

where N satisfies (3.2). Then

$$\lim_{t\to\infty} |(u(t) - v(t))| = 0. \tag{3.8}$$

(i.e. $\{x_j\}_{j=0}^{N-1}$ are determining nodes).

Proof. By virtue of Theorem 2.3, there are two solutions $u_{\mathcal{M}}$ and $v_{\mathcal{M}}$ of equation (2.1) which lie in $\mathcal{M}$ for which (2.7) and (2.8) hold respectively. Since $H^1_{\text{per}}((0,L))$ is compactly imbedded in $L^\infty([0,L])$ (see e.g. Adams (1975), then (3.7), (2.7) and (2.8) imply

$$\lim_{t\to\infty} |u_{\mathcal{M}}(t-T^*(|u(0)|),x_j)-v_{\mathcal{M}}(t-T^*(|v(0)|),x_j))| = 0 \quad \text{for} \quad j=0,1,\dots,N-1 \quad . \tag{3.9}$$

But, $u_{\mathcal{M}}(t)$ and $v_{\mathcal{M}}(t)$ are solutions on the I.M.; therefore, by (3.9) and Theorem 3.1 we reach

$$\lim_{t\to\infty} |u_{\mathcal{M}}(t - T^*(|u(0)|)) - v_{\mathcal{M}}(t - T^*(|v(0)|))| = 0 \quad . \tag{3.10}$$

Combine (3.10) with (2.7) and (2.8) to get (3.8). □

We remark that the first result in this direction was established for the 2-D Navier-Stokes equation by Foias and Temam (1984). One can give an alternative proof to the above corollary, independent of the theory of I.M.'s, following Foias and Temam (1984).

3.2 The evolution of finite volumes. Let $\varphi \in H^1_{\text{per}}((0,L))$ we denote by :

$$\overline{\varphi}(x) = \frac{1}{h}\int_{x-h/2}^{x+h/2} \varphi(s)ds \quad , \tag{3.11}$$

the local average value, or the finite volumes of the function φ. We define the mapping $\overline{\Theta}_N : H^1_{\text{per}}((0,L)) \to \mathbb{R}^N$ as follows :

$$\overline{\Theta}_N(u) = (\overline{u}(x_j))_{j=0}^{N-1} \quad , \tag{3.12}$$

where $x_j = jh$ for $j = 0,1,\dots,N-1$ and $h = \frac{L}{N}$.

THEOREM 3.2. *Let N be large enough satisfying*

$$N > \frac{L}{2\pi}\lambda^{1/4} \sim (M+1) \tag{3.13}$$

where λ is as in Theorem 2.2. Then $\overline{\Theta}_N|\mathcal{M}$ is a Lipschitz homeomorphism from $\mathcal{M}$ onto $\overline{\Theta}_N(\mathcal{M})$, where $\mathcal{M}$ is endowed with the topology of H and $\overline{\Theta}_N(\mathcal{M})$ with that of $\mathbb{R}^N$.

Proof. The idea of the proof is similar to that of Theorem 3.1. Let $u_1, u_2 \in \mathcal{M}$, set $w = u_1 - u_2$. In particular, it is very easy to see by using the integral mean value theorem that

$$|\overline{\Theta}_N(w)| \leq hN^{1/2}||w||_\infty \leq hC(L)(N\lambda)^{1/2}|w|,$$

thus, $\overline{\Theta}_N|\mathcal{M}$ is Lipschitz continuous.

Now, we would like to show that $\overline{\Theta}_N$ has a Lipschitz continuous inverse. Let w be as before, from the Poincaré inequality we have :

$$\int_{x_j-h/2}^{x_j+h/2} |w(x) - \overline{w}(x_j)|^2 dx \leq \left(\frac{h}{2\pi}\right)^2 \int_{x_j-h/2}^{x_j+h/2} |w'(x)|^2 dx \quad ,$$

hence,

$$\int_{x_j-h/2}^{x_j+h/2} |w(x)|^2 dx \leq h|\overline{w}(x_j)|^2 + \left(\frac{h}{2\pi}\right)^2 \int_{x_j-h/2}^{x_j+h/2} |w'(x)|^2 dx \quad ,$$

we sum the above inequalities with respect to j to get

$$|w|^2 \leq |\overline{\Theta}_N(w)|^2 + \left(\frac{h}{2\pi}\right)^2 |w'|^2 \quad . \tag{3.14}$$

As in Theorem 3.1 we interpolate to obtain

$$|w|^2 \leq |\overline{\Theta}_N(w)|^2 + \left(\frac{h}{2\pi}\right)^2 |w||A^{1/2}w| \quad ,$$

we substitute (2.2) to conclude our proof. □

One can intreprat Theorems 3.1 and 3.2 as parametrizations of the I.M. in terms of the nodal values and the finite volume, respectively. However, as indicated by (3.2) and (3.13) the parametrization based on the local averages is better in terms of the number of parameters necessary for the representation.

We fix N large enough satisfying (3.13), then on $\overline{\Theta}_N(\mathcal{M}_0)$ we define the semiflow

$$\overline{\Sigma}(t)\vec{\eta}_0 = \overline{\Theta}_N(S(t)(\overline{\Theta}_N^{-1}(\vec{\eta}_0))) \quad \forall \vec{\eta}_0 \in \overline{\Theta}_N(\mathcal{M}_0) \subset \mathbb{R}^N, \quad \text{for } t > 0.$$

By virtue of Theorem 3.2 $S(t)|_{\mathcal{M}_0}$ and $\overline{\Sigma}(t)$ are conjugate dynamical systems. Moreover, if we set $\vec{\eta}(t) = \overline{\Sigma}(t)\vec{\eta}_0$, for $\vec{\eta}_0 \in \overline{\Theta}_N(\mathcal{M}_0)$, and $U(x;\vec{\eta}(t)) = \overline{\Theta}_N^{-1}(\vec{\eta}(t))$, then U satisfies

$$\begin{aligned}\frac{\partial}{\partial t}U(x;\vec{\eta}(t)) + \frac{\partial^4}{\partial x^4}U(x;\vec{\eta}(t)) + \frac{\partial^2}{\partial x^2}U(x;\vec{\eta}(t))\\ + U(x;\vec{\eta}(t))\frac{\partial}{\partial x}U(x;\vec{\eta}(t)) = 0 \quad .\end{aligned} \tag{3.15}$$

By taking the averages in (3.15) and by using

$$\begin{aligned}\overline{U(x;\vec{\eta}(t))\frac{\partial}{\partial x}U(x;\vec{\eta}(t))} &= \frac{1}{2}\overline{\frac{\partial}{\partial x}U^2(x;\vec{\eta}(t))}\\ &= \frac{1}{2}\left(U(x+h/2;\vec{\eta}(t)) + U(x-h/2;\vec{\eta}(t))\right)\frac{\partial}{\partial x}\overline{U}(x;\vec{\eta}(t)) \quad ,\end{aligned}$$

we get that the evolution of the finite volumes, $\vec{\eta}(t) = (\overline{U}(x_j;\vec{\eta}(t)))_{j=0}^{N-1}$, satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}\overline{U}(x_j;\vec{\eta}(t)) + \frac{\partial^4}{\partial x^4}\overline{U}(x_j;\vec{\eta}(t)) + \frac{\partial^2}{\partial x^2}\overline{U}(x_j;\vec{\xi}(t)) + \overline{U}(x_j;\vec{\eta}(t))\frac{\partial}{\partial x}\overline{U}(x_j;\vec{\eta}(t))\\ &+ \frac{1}{2}\left(U(x_j+h/2;\vec{\eta}(t)) - 2\overline{U}(x_j;\vec{\eta}(t)) + U(x_j-h/2;\vec{\eta}(t))\right)\frac{\partial}{\partial x}\overline{U}(x_j;\vec{\eta}(t)) = 0\end{aligned} \tag{3.16}$$

Here again equation (3.16) is equivalent to reduction of (2.1) to the I.M., i.e. it is equivalent to the inertial form (2.4).

4. Semi-finite difference approximation. It was remarked in section 3.1 that

$$\xi_j(t) = U(x_j;\vec{\xi}(t)) \quad \text{for } j = 0,1,\ldots,N-1. \tag{4.1}$$

Since $U(x;\vec{\xi}(t))$ is an odd function of x (i.e., $U(x;\vec{\xi}(t)) = -U(L-x;\vec{\xi}(t))$, see section 2) then it is clear from (4.1) that

$$\begin{cases}\xi_j &= -\xi_{N-j} \quad \text{for} \quad j = 1,2,\ldots,N-1\\ \text{and} & \\ \xi_0 &= 0\end{cases} \quad . \tag{4.2}$$

Also, since $U(x;\vec{\xi}(t))$ is a periodic function (i.e., $U(x+L;\vec{\xi}(t)) = U(x;\vec{\xi}(t))$, we can extend $\vec{\xi}(t)$ periodically to a "double infinite" sequence such that

$$\xi_{j+N} = \xi_j \text{ for } j = 0,\pm1,\pm2,\ldots. \tag{4.3}$$

With this in mind, we use the centered difference operators to approximate equation (3.6). Namely,

$$\frac{\partial^2}{\partial x^2}U(x_j;\vec{\xi}(t)) \approx \frac{\xi_{j+1} - 2\xi_j + \xi_{j-1}}{h^2} \tag{4.4}$$

$$\frac{\partial^4}{\partial x^4}U(x_j;\vec{\xi}(t)) \approx \frac{\xi_{j+2} - 4\xi_{j+1} + 6\xi_j - 4\xi_{j-1} + \xi_{j-2}}{h^4}. \tag{4.5}$$

and

$$U(x_j;\vec{\xi}(t))\frac{\partial}{\partial x}U(x_j;\vec{\xi}(t)) \approx \frac{1}{3}\xi_j\frac{(\xi_{j+1}-\xi_{j-1})}{2h} + \frac{1}{3}\frac{(\xi_{j+1})^2-(\xi_{j-1})^2}{2h}. \tag{4.6}$$

for $j = 0, \pm 1, \pm 2, \ldots$; where $h = \frac{L}{N}$.

Since $U(\cdot, \vec{\xi}(t))$ is uniformly bounded in $C^{7,1/2}([0,L])$, for all $t > 0$, then all the errors in (4.4)-(4.6) are of order $0(h^2)$ uniformly in t. Consequently, we will use the right hand side of (4.4)-(4.6) to approximate the system (3.6) by:

$$\frac{d}{dt}\xi_j + \frac{(\xi_{j+2}-4\xi_{j+1}+6\xi_j-4\xi_{j-1}+\xi_{j-2})}{h^4} + \frac{(\xi_{j+1}-2\xi_j+\xi_{j-1})}{h^2} + \frac{\xi_j(\xi_{j+1}-\xi_{j-1})+\xi_{j+1}^2-\xi_{j-1}^2}{6h} = 0, \tag{4.7}$$

for $j = 0, 1, \ldots, N-1$, subject to (4.3).

Remark 4.1. Because of the particular choice of discretization of the nonlinear term in (4.6) (see also (4.8) and (4.9)), we will be able to show later that the system (4.7) has a global solution for all $t \in \mathbb{R}_+$. By virtue of the uniqueness theorem of ordinary differential equations, it is clear that if $\vec{\xi}(0)$ satisfies (4.2) then also $\vec{\xi}(t)$, the solution of (4.7), satisfies (4.2) for all $t \in \mathbb{R}_+$. Furthermore, it will be shown later that in this case (i.e., when (4.2) holds) the system (4.7) is dissipative and has an absorbing ball (Theorem 4.1). We would like to remark that for discetizations of the nonlinearity , of order $O(h^2)$, which are different from that in (4.6), such as :

$$U(x_j;\vec{\xi}(t))\frac{\partial}{\partial x}U(x_j;\vec{\xi}(t)) \approx \xi_j\frac{(\xi_{j+1}-\xi_{j-1})}{2h} \quad ,$$

or

$$U(x_j;\vec{\xi}(t))\frac{\partial}{\partial x}U(x_j;\vec{\xi}(t)) \approx \frac{1}{2}\frac{(\xi_{j+1})^2-(\xi_{j-1})^2}{2h} \quad ,$$

the corresponding semi-discrete system is not dissipative, and in certain cases it might blow up in finite time as it is indicated computationally as well as analytically in Foias et al. (1990).

Remark 4.2. (i) Let $\vec{\eta}_0 \in \overline{\Theta}_N(\mathcal{M}_0)$; then by a similar argument to the above, one can show that

$$\left(U(x_j+h/2;\vec{\eta}(t)) - 2\overline{U}(x_j;\vec{\eta}(t)) + U(x_j-h/2;\vec{\eta}(t))\right)\frac{\partial}{\partial x}\overline{U}(x_j;\vec{\eta}(t)) = O(h^2) \quad . \tag{4.8}$$

Therefore, on account of (3.16),(4.4)-(4.6) and (4.8), one can consider the system (4.7) as a semi-discret finite difference approximation to (3.12) with $\vec{\xi}$ replaced by $\vec{\eta}$. However, in this case one can take $N \approx M$ while in the case of (3.6) $N \approx 2\pi M$.

(ii) Since the vector field in (4.7) is a small perturbation, of order $L^{1/2}h^{3/2}$ of the vector fields in (3.6) and (3.16), we expect, in view of the recent work of Sell and Pliss (1990), that the "essential dynamics" of (4.7) and the equations (3.6) and (3.16) to be isomorphic, for h small enough. This means that the finite difference scheme (4.7) provides a good approximation to the qualitative dynamics of (2.1). Therefore, in this case, the finite difference scheme in (4.7) gives a qualitative approximation to the dynamics (for related results concerning approximating the dynamics of the Navier-Stokes equations; see e.g. Constantin et al. (1984), Heywood and Rannacher (1986) and Titi (1987, 1990b)).

(iii) On account of the above, (4.7) represents an approximate inertial form to equation (2.1). There are several methods that have been used for the construction of approximate inertial manifolds and their associate approximate inertial forms. Almost all these methods are based on a nonlinear Galerkin type of approximation, see e.g. Fabes et al. (1990), Foias et al. (1987, 1988a, 1988b, 1988d, 1989), Marion (1989), Marion and Temam (1989), Sell and Pliss (1990), Temam (1988b) and Titi (1988,1990a).

We denote by

$$S^N_{\text{odd,per}} = \{ \text{ all the double infinite odd periodic sequences of period } N(\text{ i.e., satisfy (4.2) and (4.3) })\}.$$

We will represent the elements of $S^N_{\text{odd,per}}$ by N-dimensional vectors $\vec{\xi} = (\xi_i)_{i=0}^{N-1}$ with the understanding that $\vec{\xi}$ satisfies (4.2) and extendable by (4.3).

PROPOSITION 4.1. *Let $B^h : S^N_{\text{odd,per}} \times S^N_{\text{odd,per}} \to S^N_{\text{odd,per}}$ be defined as follows: for every $\vec{\xi}, \vec{\eta} \in S^N_{\text{odd,per}}$*

$$B^h_k(\vec{\xi},\vec{\eta}) = \frac{\xi_k(\eta_{k+1} - \eta_{k-1}) + \xi_{k+1}\eta_{k+1} - \xi_{k-1}\eta_{k-1}}{6h} \tag{4.8}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Then

$$\sum_{k=0}^{N-1} B^h_k(\vec{\xi},\vec{\eta})\eta_k = \langle B^h(\vec{\xi},\vec{\eta}), \vec{\eta}\rangle = 0. \tag{4.9}$$

(The proof is immediate and it will be omitted.)

PROPOSITION 4.2. *Let $\Delta_h : \mathbb{R}^N \to \mathbb{R}^N$ be the matrix*

$$\Delta_h = \frac{-1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & & -1 & 2 & -1 \\ -1 & 0 & & 0 & -1 & 2 \end{pmatrix}.$$

Set $\omega = e^{\frac{2\pi i}{N}}$, then the real and the imaginary parts of the vector $(1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k})$ are eigenvectors of Δ_h with corresponding eigenvalue

$$\mu_k = \frac{2}{h^2}\left(1 - \cos\left(\frac{2\pi}{N}k\right)\right) = \frac{4}{h^2}\sin^2\left(\frac{\pi}{N}k\right) \qquad for \qquad k = 0, 1, \dots N-1.$$

(The proof is immediate; hence it will be omitted.)

COROLLARY 4.1. *The matrix $(-\Delta_h)$ is a symmetric nonnegative definite. Moreover, for every $\vec{\xi} \in S^N_{\text{odd,per}}$ we have*

$$\langle(-\Delta_h)\vec{\xi}, \vec{\xi}\rangle = h\sum_{k=0}^{N-1}\left[\left(-\Delta_h\vec{\xi}\right)_k \xi_k\right] \geq \mu_1|\vec{\xi}|^2 \tag{4.11}$$

where $\mu_1 = \frac{2}{h^2}\left(1 - \cos\left(\frac{2\pi}{N}\right)\right)$. $\left(\text{Notice that for } h \ll 1, \mu_1 \geq \frac{2\pi^2}{L^2}\right)$.

Proof. First, we can easily check that

$$\langle -\Delta_h\vec{\xi}, \vec{\xi}\rangle = h\sum_{k=0}^{N-1}\left(\frac{\xi_{k+1} - \xi_k}{h}\right)^2 \geq 0 \tag{4.12}$$

and equality holds if and only if $\vec{\xi}$ is parallel to eigenvector $(1, 1, \dots, 1)^T$. Notice that if $\vec{\xi} \in S^N_{\text{odd,per}}$ then $\vec{\xi}$ is perpendicular to $(1, 1, 1, \dots, 1)^T$. Hence, the rest of the proof follows as a result of Proposition 4.2 and the above observation. □

By using the notation of Proposition 4.1 and Proposition 4.2 the system (4.7) is equivalent to the equation

$$\begin{cases} \frac{d}{dt}\vec{\xi} + \Delta_h^2\vec{\xi} + \Delta_h\vec{\xi} + B^h(\vec{\xi}, \vec{\xi}) = 0 \\ \vec{\xi}(0) = \vec{\xi}_0 \in S^N_{\text{odd,per}} \end{cases}. \tag{4.13}$$

Next we want to show that equation (4.13) has a global solution for all time, and that it possesses an absorbing ball in $\mathbb{R}^N$. But first we need the following preliminary results.

PROPOSITION 4.3. *Let* $\vec{\xi} \in S^N_{\text{odd,per}}$, *then*

$$|\xi_{k+1} - \xi_k|^2 \le h^2 L |(-\triangle_h)\vec{\xi}|^2, \quad \textit{for all } k = 0,1,2,\dots,N-1. \tag{4.14}$$

In particular, we have

$$|\xi_1|^2 = |\xi_{N-1}|^2 \le h^2 L |(-\triangle_h)\vec{\xi}|^2. \tag{4.15}$$

Proof. Let $\vec{\xi} \in S^N_{\text{odd,per}}$, and let $0 \le k, j \le N-1$. Then

$$(\xi_{k+1} - \xi_k) - (\xi_{j+1} - \xi_j) \le \left| \sum_{\ell=j+1}^{k} [(\xi_{\ell+1} - \xi_\ell) - (\xi_\ell - \xi_{\ell-1})] \right|.$$

We sum with respect to j for $0 \le j \le N-1$ to get

$$(\xi_{k+1} - \xi_k) \le \sum_{\ell=0}^{N-1} |\xi_{\ell+1} - 2\xi_\ell + \xi_{\ell-1}|,$$

we apply the Cauchy-Schwarz inequality to obtain

$$|\xi_{k+1} - \xi_k|^2 \le N h^4 \sum_{\ell=0}^{N-1} \left| \frac{\xi_{\ell+1} - 2\xi_\ell + \xi_{\ell-1}}{h^2} \right|^2$$

which gives (4.14) (by recalling $Nh = L$). Notice that (4.15) follows immediately from (4.12) because $\xi_0 = \xi_N = 0$. □

LEMMA 4.1. *Let* $\vec{\eta} \in S^N_{\text{odd,per}}$ *be such that*

$$\begin{cases} \eta_j &= jh - \frac{L}{2} \quad j = 1,\dots,N-1, \\ \eta_0 &= \eta_N = 0 \end{cases} \tag{4.16}$$

and let h *be small enough such that (4.20) below holds. Then for every* $\vec{\xi} \in S^N_{\text{odd,per}}$ *we have*

$$\langle \triangle_h^2 \vec{\xi}, \vec{\xi} \rangle + \langle \triangle_h \vec{\xi}, \vec{\xi} \rangle + \langle B^h(\vec{\xi}, \vec{\eta}), \vec{\xi} \rangle \ge \frac{1}{4} |(-\triangle_h)\vec{\xi}|^2 : \tag{4.17}$$

which implies that the linear operator

$$\triangle_h^2 + \triangle_h + B^h(\cdot, \vec{\eta})$$

is coercive.

Proof. From (4.8) we have

$$\langle B^h(\vec{\xi},\vec{\eta}),\vec{\xi}\rangle = \frac{1}{6}\sum_{k+1}^{N}\left[\xi_k^2\left(\eta_{k+1}-\eta_{k-1}\right)+\xi_k\left(\xi_{k+1}\eta_{k+1}-\xi_{k-1}\eta_{k-1}\right)\right]$$

because of the periodicity of $\vec{\xi}$ and $\vec{\eta}$ we have

$$\langle B^h(\vec{\xi},\vec{\eta}),\vec{\xi}\rangle = \frac{1}{6}\sum_{k=1}^{N}\left[\xi_k^2\left(\eta_{k+1}-\eta_{k-1}\right)+\xi_k\xi_{k=1}\left(\eta_{k+1}-\eta_k\right)\right]. \tag{4.18}$$

From (4.16) we have

$$\eta_{j+1}-\eta_{j-1} = \begin{cases} 2h & 2\le j\le N-2 \\ 2h-\frac{L}{2} & j=1 \\ \frac{L}{2}-2h & j=N-1 \end{cases}.$$

and

$$\eta_{j+1}-\eta_j = \begin{cases} h & 1\le j\le N-2 \\ h-\frac{L}{2} & j=N-1 \\ \frac{L}{2}-h & j=N \end{cases}.$$

Therefore, (4.18) yields

$$\langle B^h(\vec{\xi},\vec{\eta}),\vec{\eta}\rangle = \frac{1}{6}\left[\sum_{k=1}^{N}\left(2\xi_k^2+\xi_k\xi_{k+1}\right)h-\frac{L}{2}\xi_1^2+\left(\frac{L}{2}-4h\right)\xi_{N-1}^2\right].$$

Since $\xi_1^2=\xi_{N-1}^2$ we get

$$\begin{aligned}\langle B^h(\vec{\xi},\vec{\eta}),\vec{\eta}\rangle &\ge \frac{1}{3}|\vec{\xi}|^2-\frac{2}{3}\xi_{N-1}^2+\frac{1}{6}\sum_{k=1}^{N}\xi_k\xi_{k+1} \\ &\ge \frac{1}{2}|\xi|^2-\frac{2h}{3}\xi_{N-1}^2-\frac{h}{6}\sum_{k=1}^{N}(\xi_{k+1}-\xi_k)^2\end{aligned}$$

by (4.12) and (4.15) we obtain

$$\langle B^h(\vec{\xi},\vec{\eta}),\vec{\xi}\rangle \ge \frac{1}{2}|\vec{\xi}|^2-\frac{2}{3}h^3L|\triangle_h\vec{\xi}|^2+\frac{h^2}{6}\langle\triangle_h\vec{\xi},\vec{\xi}\rangle. \tag{4.19}$$

By means of (4.19) and the Cauchy-Schwarz inequality we have

$$\langle \Delta_h^2 \vec{\xi}, \vec{\xi} \rangle + \langle \Delta_h \vec{\xi}, \vec{\xi} \rangle + \langle B^h(\vec{\xi}, \vec{\eta}), \vec{\xi} \rangle$$
$$\geq |\Delta_h \vec{\xi}|^2 + \frac{1}{2}|\vec{\xi}|^2 - \left(1 + \frac{h^2}{6}\right) |\Delta_h \vec{\xi}|\ |\vec{\xi}| - \frac{2}{3} h^3 L |\Delta_h \vec{\xi}|^2$$

by Young's inequality we get

$$\geq \left(\frac{1}{2} - \frac{h^2}{6} - \frac{h^4}{36} - \frac{2}{3}h^3 L\right) |\Delta_h \vec{\xi}|^2$$

which implies (4.17) provided h is small enough satisfying

$$(3 - h^2)^2 - 24h^3 L \geq 0. \tag{4.20}$$

Notice that (4.20) is verified if $N \gg L^{4/3}$. □

THEOREM 4.1. *The system (4.13) has a global solution for all $t \geq 0$. Moreover, there exists $r_0 > 0$, given by (4.24), such that for every solution $\vec{\xi}(t)$ of (4.13) we have a $t^*(|\vec{\xi}(0)|)$ such that*

$$|\vec{\xi}(t)| \leq r_0, \text{ for all } t \geq t^*. \tag{4.21}$$

(i.e., the system has an absorbing set).

Proof. Replace $\vec{\xi}$ by $\vec{\xi} + \vec{\eta}$ where $\vec{\eta}$ satisfies (4.16), then equation (4.13) becomes

$$\frac{d\vec{\xi}}{dt} + \Delta_h^2 \vec{\xi} + \Delta_h \vec{\xi} + B^h(\vec{\xi}, \vec{\eta}) + B^h(\vec{\eta}, \vec{\xi}) + B^h(\vec{\xi}, \vec{\xi}) = \vec{f}. \tag{4.22}$$

where $\vec{f} = -B^h(\vec{\eta}, \vec{\eta}) - \Delta_h^2 \vec{\eta} - \Delta_h \vec{\eta}$.

Take the scalar product of (4.22) with $\vec{\xi}$ and use (4.9) and (4.17) to obtain

$$\frac{1}{2}\frac{d}{dt}|\vec{\xi}|^2 + \frac{1}{2}|\Delta_h \vec{\xi}|^2 \leq |\vec{f}|\ |\vec{\xi}|.$$

Apply (4.11) to get

$$\frac{1}{2}\frac{d}{dt}|\vec{\xi}|^2 + \frac{1}{2}\mu_1^2 |\vec{\xi}|^2 \leq \frac{|\vec{f}|^2}{\mu_1^2} + \frac{\mu_1^2}{4}|\vec{f}|^2$$

hence, by Gronwall's inequality we get

$$|\vec{\xi}(t)|^2 \leq e^{-\frac{1}{2}\mu_1^2 t} |\vec{\xi}(0)|^2 + 4\frac{|\vec{f}|^2}{\mu_1^4}\left(1 - e^{-\frac{1}{2}\mu_1^2 t}\right) \tag{4.23}$$

for all $t > 0$.

Thanks to (4.23) one can easily show that (4.13) has a global solution. Notice that (4.21) is a direct consequence of (4.23) for

$$r_0 = 2\frac{|\vec{f}|}{\mu_1^2}\quad. \tag{4.24}$$

□

5. Inertial manifolds for the semi-finite difference approximation. In this section we will show that the dissipative semi-finite difference scheme in (4.7) or (4.13) has an inertial manifold, provided N is large enough.

THEOREM 5.1. *Let N be large enough satisfying (3.13) and the conditiones mentioned in the proof below. Then the system (4.7) has an inertial manifold of dimension $k_0 = [N/4]+1$. Moreover, this inertial manifold enjoys the exponential tracking property, with rate of attraction $e^{-\frac{\Lambda_{k_0+1}}{2}t}$, where Λ_k are the eigenvalues of the operator $\triangle_h^2$ given in (5.1).*

Proof. We will not go through all the details of the proof. To complete the details see for instance Constantin et al. (1988,1989) or Foias et al. (1988d,1989). Since the system (4.7) is dissipative, as it was shown in Theorem 4.1, we will only show that the operator $\triangle_h^2$ has large spectral gaps, and satisfies what is known as the *gap condition.*

Recall from Proposition 4.2 the eigenvalues of the operator $\triangle_h^2$

$$\Lambda_k = \mu_k^2 = \frac{16}{h^4}\sin^4\left(\frac{\pi}{N}k\right). \tag{5.1}$$

Using some basic trigonometric inequalities we get

$$\Lambda_{k+1} - \Lambda_k = \frac{64}{h^4}\cos\left(\frac{\pi}{2N}\right)\cos\left(\frac{\pi}{N}k + \frac{\pi}{2N}\right)\sin\left(\frac{\pi}{2N}\right)\sin\left(\frac{\pi}{N}k + \frac{\pi}{2N}\right) \quad .$$

Assuming that $N \gg 1$ then it is easy to see

$$\Lambda_{k+1} - \Lambda_k \geq \frac{64}{Nh^4}\left(1 - \frac{\pi^2}{8N^2}\right)\cos\left(\frac{\pi}{N}k + \frac{\pi}{2N}\right)\sin\left(\frac{\pi}{N}k + \frac{\pi}{2N}\right) \quad . \tag{5.2}$$

If we choos $k_0 = [N/4]+1$, then (5.2) implies

$$\Lambda_{k_0+1} - \Lambda_{k_0} \geq \frac{64N^3}{L^4}\left(1 - \frac{\pi^2}{8N^2}\right)\left(\frac{\sqrt{2}}{2} - \frac{3\pi}{2N}\right)^2 \quad . \tag{5.3}$$

Also, (5.1) gives

$$\Lambda_{k_0+1} \leq \frac{16N^4}{L^4}\left(\frac{\sqrt{2}}{2} + \frac{2\pi}{N}\right)^2 \quad . \tag{5.4}$$

Therefore for $N \gg 1$ (5.3) and (5.4) imply

$$\frac{\Lambda_{k_0+1} - \Lambda_{k_0}}{{\Lambda_{k_0+1}}^{1/2}} \geq \frac{16N}{L^2}\left(1 - \frac{\pi^2}{8N^2}\right)\left(\frac{\frac{\sqrt{2}}{2} - \frac{3\pi}{2N}}{\frac{\sqrt{2}}{2} + \frac{2\pi}{N}}\right)^2 \quad . \tag{5.4}$$

Following any of the proofs in the above references, the spectral gap in (5.4) can be made arbitrary large, by choosing $N \gg 1$, to satisfy all the conditions required for the existence of an I.M..

Following Foias et al. (1989) one can show the exponential tracking property. □

Remark 5.1. In view of Theorem 5.1 one can follow the works of Foias et al. (1987, 1988b, 1989), Marion and Temam (1989) and Titi (1988,1990a), and introduce approximate inertial manifolds for the semi-finite difference scheme (4.7), and implement the combined finite difference and approximate inertial manifold schemes in real computations.

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