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Deterministic sparse FFT algorithms

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In this paper we consider sparse signals $\mathbf{x} \in \mathbf{C}^N$ which are known to vanish outside a support interval of length bounded by $m < N$. For the case that m is known, we propose a deterministic algorithm of complexity $\mathcal{O}(m \log m)$ for reconstruction of $\mathbf{x}$ from its discrete Fourier transform $\widehat{\mathbf{x}} \in \mathbf{C}^N$.

1 Introduction

Fast algorithms for the computation of the discrete Fourier transform of a vector of length N have been known for many years. These FFT algorithms have an algorithmic complexity of $\mathcal{O}(N \log N)$. Recently, there has been a stronger interest in Fourier algorithms for sparse vectors which can even achieve a sublinear complexity. Randomized sparse Fourier algorithms achieving a complexity of $\mathcal{O}(m \log N)$ resp. $\mathcal{O}(m \log m)$ for m-sparse vectors can e.g. be found in [2] resp. [3], [4]. An overview of the methods of randomized sparse Fourier transforms is given in [1].

In this paper, we present a deterministic FFT algorithm and restrict ourselves to vectors with a short support interval. Such vectors occur in different applications, such as in X-ray microscopy, where compact support is a frequently used a-priori condition in phase retrieval, as well as in computer tomography reconstructions.

Let $\mathbf{x} \in \mathbf{C}^N$. We define the *support length* $m = |\text{supp}\,\mathbf{x}|$ of $\mathbf{x}$ as the minimal integer m for which there exists a $\mu \in \{0, \ldots, N-1\}$ such that the components x_k of $\mathbf{x}$ vanish for all $k \notin I := \{(\mu + r) \mod N, \quad r = 0, \ldots, m-1\}$. The index set I is called *support interval* of $\mathbf{x}$. We always have $x_\mu \neq 0$ and $x_{\mu+m-1} \neq 0$, but there may be zero components of $\mathbf{x}$ within the support interval. Observe that if $m \leq \frac{N}{2}$, the support interval and hence the first support index μ of $\mathbf{x}$ is uniquely determined.

We define the discrete Fourier transform of a vector $\mathbf{x} \in \mathbf{C}^N$ by $\widehat{\mathbf{x}} = \mathbf{F}_N \mathbf{x}$, where the Fourier matrix $\mathbf{F}_N$ is given by $\mathbf{F}_N := (\omega_N^{jk})_{j,k=0}^{N-1}$, $\omega_N := \mathrm{e}^{-\frac{2\pi \mathrm{i}}{N}}$. In the following, we describe a deterministic algorithm for the reconstruction of $\mathbf{x}$ of length $N = 2^J$ from Fourier data $\widehat{\mathbf{x}} \in \mathbf{C}^N$. The algorithm is based on the idea that the (at most) m nonzero components of $\mathbf{x}$ can already be identified from a periodization of $\mathbf{x}$ of length $2^L \geq m$. Hence for the complete reconstruction it remains to determine the support interval (i.e., the first support index) of $\mathbf{x}$.

2 Reconstruction of $\mathbf{x}$ with short support interval

Let $N := 2^J$ for some $J > 0$. We define the periodizations $\mathbf{x}^{(j)} \in \mathbf{C}^{2^j}$ of $\mathbf{x}$ by

$$\mathbf{x}^{(j)} = (x_k^{(j)})_{k=0}^{2^j-1} = \left(\sum_{\ell=0}^{2^{J-j}-1} x_{k+2^j \ell} \right)_{k=0}^{2^j-1} \tag{1}$$

for $j = 0, \ldots, J$. Obviously, $\mathbf{x}^{(0)} = \sum_{k=0}^{N-1} x_k$ is the sum of all components of $\mathbf{x}$, $\mathbf{x}^{(1)} = (\sum_{k=0}^{N/2-1} x_{2k}, \sum_{k=0}^{N/2-1} x_{2k+1})^T$ and $\mathbf{x}^{(J)} = \mathbf{x}$. The discrete Fourier transform of the vectors $\mathbf{x}^{(j)}$, $j = 0, \ldots, J$, can be described in terms of $\widehat{\mathbf{x}}$. According to the following lemma, it can be obtained by just picking suitable components of $\widehat{\mathbf{x}}$.

Lemma 2.1 *For the vectors* $\mathbf{x}^{(j)} \in \mathbf{C}^{2^j}$, $j = 0, \ldots, J$, *in* (1)*, we have the discrete Fourier transform*

$$\widehat{\mathbf{x}}^{(j)} := \mathbf{F}_{2^j} \mathbf{x}^{(j)} = (\widehat{x}_{2^{J-j}k})_{k=0}^{2^j-1},$$

where $\widehat{\mathbf{x}} = (\widehat{x}_k)_{k=0}^{N-1} = \mathbf{F}_N \mathbf{x}$ *is the Fourier transform of* $\mathbf{x} \in \mathbf{C}^N$.

Assume that the Fourier data $\widehat{\mathbf{x}} = \mathbf{F}_N \mathbf{x} \in \mathbf{C}^N$ and $|\text{supp}\,\mathbf{x}| \leq m$ for some given m. Choose L such that $2^{L-1} < m \leq 2^L$. By Lemma 2.1 we have $\widehat{\mathbf{x}}^{(L+1)} = (\widehat{x}_{2^{J-(L+1)}k})_{k=0}^{2^{L+1}-1}$. Thus, we can compute $\mathbf{x}^{(L+1)}$ using inverse FFT of length 2^{L+1}.

The resulting vector $\mathbf{x}^{(L+1)}$ has already the same support length as $\mathbf{x}$, since $|\text{supp}\,\mathbf{x}| \leq m \leq 2^L$, and for each $k \in \{0, \ldots, 2^{L+1}-1\}$ the sum in

$$x_k^{(L+1)} = \sum_{\ell=0}^{2^{J-L-1}-1} x_{k+2^{L+1}\ell} \tag{2}$$

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contains at most one nonvanishing term. Therefore, the support of $\mathbf{x}^{(L+1)}$ and its first index $\mu^{(L+1)}$ are uniquely determined. For reconstruction of the complete vector $\mathbf{x}$ it is now sufficient to determine the first support index $\mu^{(J)} = \mu$ of the support interval of $\mathbf{x}$. Then the components of $\mathbf{x}$ are given by

$$x_{(\mu^{(J)}+k)\bmod N} = \begin{cases} x^{(L+1)}_{(\mu^{(L+1)}+k)\bmod 2^{L+1}} & k = 0,\dots,m-1, \\ 0 & k = m,\dots,N-1. \end{cases} \tag{3}$$

By the following theorem (cf. Theorem 3.1 in [5]), it is possible to obtain $\mu^{(J)}$ and hence to recover $\mathbf{x}$ from the vector $\mathbf{x}^{(L+1)}$ and one additional Fourier component.

Theorem 2.2 *Let $\mathbf{x} \in \mathbb{C}^N$, $N = 2^J$, have support length m (or a support length bounded by m) with $2^{L-1} < m \le 2^L$. For $L < J-1$, let $\mathbf{x}^{(L+1)}$ be the 2^{L+1}-periodization of $\mathbf{x}$. Then $\mathbf{x}$ can be uniquely recovered from $\mathbf{x}^{(L+1)}$ and one nonzero component of the vector $(\widehat{x}_{2k+1})_{k=0}^{N/2-1}$.*

3 Sparse FFT Algorithm

We summarize the reconstruction of $\mathbf{x}$ from Fourier data $\widehat{\mathbf{x}}$ in the following algorithm.

Algorithm 3.1 *(Sparse FFT for vectors with short support)*

Input: $\widehat{\mathbf{x}} \in \mathbb{C}^N$, $N = 2^J$, $|\operatorname{supp}\mathbf{x}| \le m < N$.

- *Compute L such that $2^{L-1} < m \le 2^L$, i.e., $L := \lceil \log_2 m \rceil$.*
- *If $L = J$ or $L = J-1$, compute $\mathbf{x} = \mathbf{F}_N^{-1}\widehat{\mathbf{x}}$ using an FFT of length N.*
- *If $L < J-1$:*

1. *Choose $\widehat{\mathbf{x}}^{(L+1)} := (\widehat{x}_{2^{J-(L+1)}k})_{k=0}^{2^{L+1}-1}$ and compute $\mathbf{x}^{(L+1)} := \mathbf{F}_{2^{L+1}}^{-1}\widehat{\mathbf{x}}^{(L+1)}$ using an FFT of length 2^{L+1}.*
2. *Determine the first support index $\mu^{(L+1)} \in \{0,\dots,2^{L+1}-1\}$ of $\mathbf{x}^{(L+1)}$ such that $x^{(L+1)}_{\mu^{(L+1)}} \ne 0$ and $x^{(L+1)}_k = 0$ for $k \notin \{(\mu^{(L+1)}+r)\bmod 2^{L+1},\ r = 0,\dots,m-1\}$.*
3. *Choose a Fourier component $\widehat{x}_{2k_0+1} \ne 0$ of $\widehat{\mathbf{x}}$ and compute the sum*

$$a := \sum_{\ell=0}^{m-1} x^{(L+1)}_{(\mu^{(L+1)}+\ell)\bmod 2^{L+1}}\, \omega_N^{(2k_0+1)(\mu^{(L+1)}+\ell)}.$$

4. *Compute $b := \widehat{x}_{2k_0+1}/a$ that is by construction of the form $b = \omega^p_{2^{J-L-1}}$ for some $p \in \{0,\dots,2^{J-L-1}-1\}$, and find $\nu \in \{0,\dots,2^{J-L-1}-1\}$ such that $(2k_0+1)\,\nu = p \bmod 2^{J-L-1}$.*
5. *Set $\mu^{(J)} := \mu^{(L+1)} + 2^{L+1}\nu$, and $\mathbf{x} := (x_k)_{k=0}^{N-1}$ with entries*

$$x_{(\mu^{(J)}+\ell)\bmod N} := \begin{cases} x^{(L+1)}_{(\mu^{(L+1)}+\ell)\bmod 2^{L+1}} & \ell = 0,\dots,m-1, \\ 0 & \ell = m,\dots,N-1. \end{cases}$$

Output: $\mathbf{x}$.

Our algorithm has an arithmetical complexity of $\mathcal{O}(m\log m)$. This can be seen as follows: In the first step, an FFT algorithm of this complexity is performed. All further steps require at most $\mathcal{O}(m)$ operations. Moreover, the algorithm needs less than $4m$ Fourier values.

The results can be found in a more detailed version in [5] where we also propose an algorithm for noisy input data as well as numerical results.

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