DEVELOPING AN UNDERSTANDING OF CHILDREN'S JUSTIFICATIONS FOR THE CIRCLE AREA FORMULA

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In this study we investigated eighth grade students' informal justification for the circle area formula to expand accounts of the measurement knowledge for middle-school age students. Data were collected during three paired interviews of a three-year teaching experiment. Here we describe schemes students exhibited as they operated on measurement tasks at a level we have described as "conceptual area measurer"; the tasks prompted the use of square units to quantify a figure that is not rectilinear. We found students could follow and rehearse a rationale for the validity of the circle area formula with substantive opportunities for movement and figural operations with units, or with decompositions from unit images that coordinated circle and rectangle images.

Keywords: Measurement, Learning Trajectories (or Progressions), Geometry and Geometrical and Spatial Thinking

Area measurement is an important part of elementary and middle school mathematics; unfortunately, many students do not have an adequate understanding of area measurement concepts (Outhred & Mitchelemore, 2000). Many elementary students can remember standard formulas for shapes such as rectangles; however, area measurement is still problematic (Lehrer, 2003). This could be because students are taught the area formula through rote memorization (Simon & Blume, 1994). "Rather than memorize particular formulas for certain shapes, they need to understand why the formulas work" (Strutchens, Martin, & Kenny, 2003). The Common Core State Standards for 7th grade recommends students be able to give an informal justification for the circle area formula (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). In this paper we set out to explore 8th grade students' understanding of the area formula for a circle, including their ability to apply and reason about area formulas for circles.

We expect that asking students to justify the use of the area formula for circles provides an effective context for assessing advances in students' area measurement knowledge. The purpose of this study is to describe and analyze students' thinking as they found the area of circles and developed an informal justification for the circle area formula by coordinating with the area of a triangle, and by tiling with squares. We also hoped to extend a hypothetical learning trajectory (HTL) on area measurement by addressing measures of non-rectilinear shapes.

Research Question

How do eighth grade students develop an informal justification for the circle area formula?

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Theoretical Framework

To investigate students' development of understanding for area measurement, we needed a tool to identify varying levels of understanding of area. Thus, we used a hypothetical learning trajectory (HLT) for area measurement developed by Barrett et al. (in press) that describes levels of sophistication. An HLT has three parts: an instructional goal, developmental progressions to characterize mental schemes and actions pertaining to the knowledge goal, and instructional activities to help students progress along the progression (Clements & Sarama, 2014). The instructional goals and activities follow from particular schemes and certain mental actions on objects characteristic of successive levels in the trajectory. An HLT on area measurement has guided the development of our tasks.

The following area HLT levels (Barrett et al., in press) are relevant for the present study because the tasks were designed to scaffold the students to a more sophisticated level. These three levels of thinking address the degree to which children may integrate and coordinate figural images and internal, conceptual images by analyzing parts of figures. By re-organizing a figure into essentially the same collection of components, yet within a different overall shape, children may accomplish the first, least sophisticated of these levels (ARCS). The more sophisticated levels are achieved as children abstract the measures of regions to define area measures as constructions that are products of other linear measures, measures which refer to highly indexed, linear collections of units arranged along a second, orthogonal dimension in arrays. The highest level indicates the most flexible, algebraic grasp of products from linear quantities, taken as inputs to a functional account of the area measurement.

- *Area row and Column Structurer* (ARCS): children at this level can decompose and recompose partial units to create whole units
- *Array Structurer* (AS): children at this level have an abstract understanding of the area formula for rectangles
- *Conceptual Area Measurer* (CAM): children at this level have an abstract and generalizable understanding of the rectangle area formula, they are able to restructure regions to find area, and can provide a justification for the restructuring of the shape.

In the interview sequence reported on here, we prompted students to connect the measure of circles to that of rectangles as a consequence of our reading of the mental actions on objects most likely to be enacted at the Conceptual Area Measurer level of the HLT and because we had found the students expressing related schemes of decomposition and recomposition of area measures.

Methodology

As part of a three-year longitudinal teaching experiment (Steffe & Thompson, 2000) on children's thinking and learning about length, area, and volume, we investigated four children's thinking on area of circles. The students were in eighth grade at a public school in the Midwest. For this report, we used data from three 25- to 30-minute semi-structured interviews with four students (Kari, Lindsey, Joey, and Tanner). We interviewed students in pairs, with Kari and Lindsey as partners and Joey and Tanner as partners. The interviews were videotaped and transcribed and the researchers then analyzed the data. The interviews took place during November and December of 2015.

For the first task, we asked students to find the area of a circle and tell us if they could explain why the circle area formula worked. We used this as an opportunity to see the prior knowledge the students had about area of circles. In the second task, we asked students to compare the area of a square radius to the area of a circle. We created a display with two orthogonal radii serving as adjacent sides of a square, having an area of one radius squared. In this approach, the interviewers

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encouraged the students to tile over the circle with cutout squares radii. We saw this activity as an opportunity for students to make a connection to the circle area formula (i.e., $A = \pi r^2$). We hoped students would either: (a) estimate that it would take between three and four square radii to match the area of the circle, or (b) use the formula to assert that approximately 3.14 square radii will always cover the circular region.

Next, students watched a video of a circle and its interior being transformed into a triangle (see Figure 1). We designed this video to help students relate the area of the circle to the area of the triangle. We expected to activate their scheme for decomposing and recomposing space.

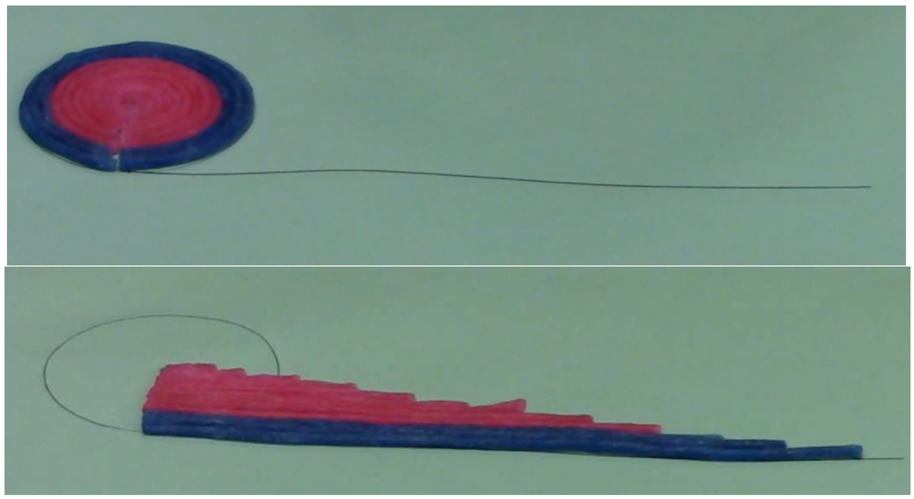

Figure 1. Circle Transformed to a Triangle.

For the third task, the students were asked to reflect on the transformation in the video and to find the area of a circle through relating the area of the circle to the area of a triangle. When they expressed the area of the triangle with an invented expression, we asked them to apply their invented expression to state the area of the circle without relying directly on the standard formula. In the fourth task, we asked students to provide an informal justification of the circle area formula, both by reviewing their prior work and by synthesizing what they had observed in the prior tasks.

Results and Discussion

Next we present descriptions of the students' work and responses for each of the four tasks. Following that, we sketch a generalized account of the reasoning we observed and the knowledge that was within their grasp given this set of tasks, and we comment on the relation of such knowledge to an existing framework characterizing relevant levels of sophistication for students' knowledge about area measurement.

Task 1: Find the area of this circle. If you use a formula, explain how you know that it will give you the correct area or where it comes from.

All four students correctly calculated the area of the circle using the standard area formula $A = \pi r^2$ during the first interview on this topic. Despite knowing the formula and how to compute the area, none of the four students gave an explanation about why the formula makes sense or its relation to that of a rectangle. One student, Lindsey, attempted to develop an explanation of the formula. She split the shape using a diameter and said the formula may have something to do with the height and the base.

Although all of the students successfully found the area of the circle using the standard formula, they did not explain any part of the formula besides telling us the formula and that π was 3.14. This

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could be an indication that students were taught the formula through memorization alone. Students' responses to this task lead us to claim they were not yet operating at a Conceptual Area Measurer (CAM) level of the HLT for area measurement. We make this claim because they used the standard formula for area of a circle but did not demonstrate an abstract understanding of that formula, which is characteristic of the CAM level.

Task 2: Compare the area of the square radius to the area of the circle.

Kari and Lindsey made a guess that it would take four square radii to fill the circle. When they compared the area of the square radii with the area of the circle algebraically, they found the difference, not the ratio, which did not help them interpret how many square radii it takes to fill the circle. Later, using cut out square radii, they determined that it would take about two or three squares. When they were prompted to reflect back on the formula, Lindsey said, "oh π is 3.14 and so it would be 3 or a little bit more that would fit in the circle." Although this may have helped Kari and Lindsey interpret the formula, they still reported that they could not justify why the formula worked.

In the third interview, Kari and Lindsey were asked this question again. They were able to say it takes 3.14 square radii to fill the circle but they may not have taken the square as an object that occupied "radius square units" of the circle area. Instead, Lindsey dragged her finger along two sides of the square and said each showed a length of a radius. Students operating conceptually with area often use a sweeping motion within or across the area being discussed, (Dougherty, 2008). In contrast, Lindsey's gestures may indicate she was treating the sides operationally, as factors that would be used to feed into a calculation for a product of "radius squared". Later, Lindsey labeled another square shape with edge length of "radius" by writing "radius squared" on the interior, indicating an advance beyond the operational approach.

In the first interview Joey and Tanner concluded it would take three square radii to fill the circle. They did not relate the square radii to the standard formula. At the beginning of the third interview they went back to this task. The interviewer asked how many squares it would take to fill the circle? Joey said, "3 point something." The interviewer then asked them if the formula helps them answer that question? Tanner said, "3.14 because the radius square times π is the area of the circle." The interviewer explained to the students this was the beginning to understanding the standard formula for area of a circle but was not yet a justification for the formula.

Based on this interview, we claim Kari and Lindsey were able to interpret the formula as a statement of the number of square radii it would take to fill a circle but struggled to justify why. Joey and Tanner did not articulate a connection between the square radii and π at the end of the first interview, but at the end of the third interview they described the relationship clearly. Apparently this task allowed these students to develop an understanding of the circle area formula as an approximate value, in relation to the number of square radii units needed to cover the circle. Still, it did not help them justify why the formula specifies π square radius units.

Task 3: Finding the area of a circle by transforming it into a triangle

After watching the video of a circle being transformed into a triangle, Kari and Lindsey were given a page that had the circle and triangle shown in the video. They told the interviewer the triangle had the same area as the circle. They labeled the height of the triangle *r* because they could see the height of the triangle was concurrent with the segment showing a radius of the circle. However, they would not label the base of the triangle circumference, but the were willing to label it *C*. Lindsey said, "[we can] label it circumference of a circle, but not really because it is not circular." Kari agreed and said, "but if we rolled the circle out it would go to here" as she pointed to the end of the triangle. They agreed they could label it *C*. We think Lindsey and Kari did not view circumference as a measure but only as a name of part of a circle. They found the area of the triangle by first measuring the length of the base and height of the triangle and then multiplying the base (circumference), height (radius), and $\frac{1}{2}$. They checked to see if the triangle had the same area as the circle by using the standard area formula for a circle and found their calculation was about one square centimeter different. With the help of the interviewers they concluded the circle had the same area as the triangle; they found a difference between area measures of 18.75 and 19.63 but they needed to account for measurement error. They concluded they could find the area of the circle using the formula $A = Cr\frac{1}{2}$ (see Figure 2).

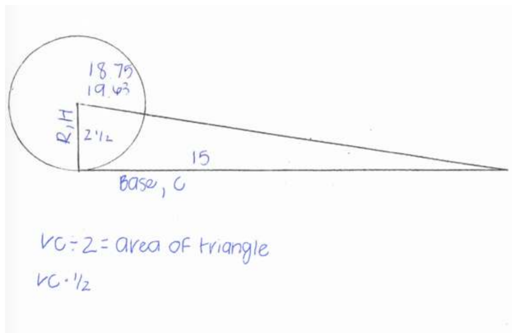

Figure 2. Kari and Lindsey's work for Task 3 on the picture of the circle to triangle transformation page given to them.

In the second interview, Kari and Lindsey, were given another circle and asked to calculate the area without using the standard formula. They drew a triangle that had an altitude the same as the radius of the circle and a base the same as the circumference. They used their invented formula $a = Cr\frac{1}{2}$ to find the area of the circle. After they found the area using their invented formula, they checked their answer with the standard formula (i.e., $a = \pi r^2$) and found the answer to be the same. This suggests a conceptual advance in that they expressed the area of a circle in terms of a related expression involving a triangle that was a reconstituted collection of parts of that circle.

Similarly, after Joey and Tanner watched the video of the circle being transformed into a triangle they found the area of the circle from the video (same page was given as Figure 2), by working with the triangle and using $a = bh\frac{1}{2}$. Joey and Tanner were then given a different circle and asked to find the area without using the standard formula. As we had hoped, their strategy was to try to create a triangle that would have the same area as the given circle. They knew the height of the triangle would be the radius, which they measured correctly. They were not sure how to calculate or measure the circumference of the circle, which they recognized would be the other side length needed for the triangle. With some prompting from the interviewer, they used a wikki-stix to find the distance around the circle. Once they found the measure of the circumference they found the area of the circle by find the area of their created triangle using the triangle area formula. Next, with guidance from the interviewer, Joey and Tanner wrote $\frac{rc}{2} = \frac{bh}{2} = r^2\pi$ on their paper. They then checked their answer by finding the area of the circle using the standard circle area formula. They had a difference in their answers by one square centimeter but concluded the difference was due to a measurement mistake.

We found evidence that students could follow our guidance and logic to interpret a more tangible formula for the area of a circle. Kari and Lindsey seemed to be able to complete the transformation of the circle into a triangle without guidance from the interviewer because they knew how to find the circumference of the circle. Joey and Tanner were able to complete this task with the help of the interviewer giving them a wikki-stix and later the interviewer told them the formula for finding the circumference of a circle.

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Task 4: Justifying the circle area formula

Kari and Lindsey began their informal justification for the circle area formula by drawing a circle and a radius. Lindsey said, "So the circumference is the circle and if you lay that out it would be this, which is the base of the triangle, which we are calling C, cause it is like the circumference of the circle. Then the height is like the radius. And to get the area of the triangle you do base times height which is like C times r … times a half." Kari said that the triangle had same area as the circle. They then set πr^2 equal to $Cr\frac{1}{2}$. They substituted $d\pi$ in for C, followed by $2r$ for d and simplified (see Figure 3).

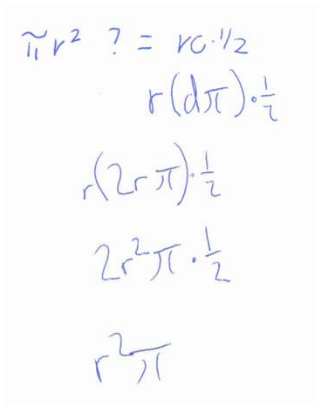

Figure 3. Kari and Lindsey's work on Task 4.

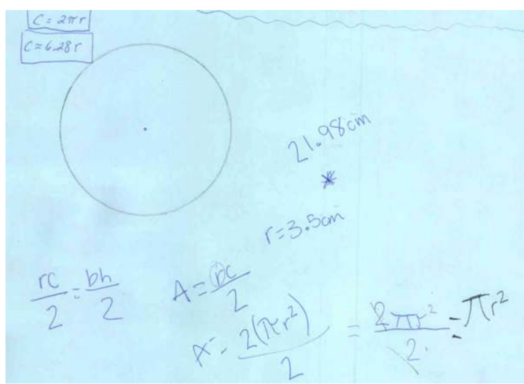

Figure 4. Tanner and Joey's work on Task 4.

In the third interview with Joey and Tanner, they were asked to justify the circle area formula. To reply, they revisited previous work and recalled this equation: $\frac{rc}{2} = \frac{bh}{2}$. They substituted $2\pi r$ for the circumference leaving them with $A = \frac{r2\pi r}{2}$. They simplified this equation and were left with the standard circle area formula. This algebra was not easy for them and it took them some time to decide if $r \times r$ was $2r$ or r^2 (see Figure 4).

All four students took for granted the equivalence of the circle and the triangle and used it in their defense for the informal justification of the formula for area of a circle. At first they all wanted to use the standard formula they knew for area of a circle to justify their invented formula. The students were able to restructure the circle into a triangle and relate the shape to the area of a triangle and provide justification for the transformation and the circle area formula. We note here that the operation used in the video display of representing the circle and the triangle with an identical set of strips, and reorganizing to show the transition between the two figures involves an over-simplification, but we believe it was productive. This scenario disregards the contortion of the inner and outer edges of these unitary strips. However, as we approach the limiting case for the width of the strips this distortion becomes negligible. Thus, our imagined conservation of a collection of strips serves as a thought experiment more than a comprehensive argument. Nonetheless, it is conceptually sound, as the knowledge can be expected to mature when a student gains a sophistication level suited to learning the principles of the calculus, later in their academic career. We note too, that others have experimented with similar spatial morphing, which children may see as space conserving (e.g., Lehrer, Jenkins, & Osana, 1998; Kara, 2013).

Conclusion and Implications

Prior to the interviews, the students were not able to explain the standard circle area formula and at the end of the interview they were able to produce an informal justification for the circle area formula. We claim, these tasks supported students' development of a more meaningful understanding of finding the area of a circle. With scaffolding, the four students were preforming at the CAM level because they were able to restructure a circle into a triangle to find an area measure using algebraic expressions and operations. As well, they were able to provide an informal justification of this transformation and defend the circle area formula.

After the first task we found the students were able to compute the area, but they were not able to explain or interpret the formula. Our findings from this task support those of Outhred and Mitchelemore (2000) that elementary and middle school students do not have a sufficient understanding of measurement and those of Lehrer (2003) that students can remember standard formulas but not have a understanding of the formula or area measurement.

To our surprise, these students used the standard formula to justify their invented formula instead of using their invented formula to justify the standard formula. This could be because they had been taught the standard formula and did not learn about the formula as measurement from units and unit iteration for covering such an object. We conjecture that if they were to invent their own formula first, they could use their formula to develop the standard formula, which would drastically alter their conception of the standard formula. They may have conceived of the standard formula as an arbitrary construction that the teacher was merely relaying to them, and moreover, assumed it did not have a practical basis in physical measures with units of area. Students often accept statements like this as valid statements (theorems in action) for practical use without testing or challenging them. Setting measurement activities as empirical tasks is unusual, especially the measure of the circle, given that a formulaic computation is available, requiring only the measure of a radius. By problematizing the formula for measuring area, we found that students in 8th grade were capable of taking a novel unit square, the unit with a side length of one circle radius, and using it to measure both the circle and the triangle for area. By relating a circle to a triangle through a physical transformation we were able to relate the area formula for a circle to the area formula for a triangle. By working with the imagistic and the symbolic representations at the same time we claim students were able to recognize an informal argument to justify the formula for measuring the area of a circle.

Thus, we found that students in Grade 8 are able to recognize the validity and figural veracity of the standard formula for computing the measurement of the area of a circle in terms of its radius. This finding informs and allows us to adapt the row of the area HLT for Conceptual Area Measurer, indicating that students at a conceptual understanding of area measuring in terms of arbitrary square units should also be expected to recognize extensive algebraic manipulation of the area formula to relate it to the formula for finding the area of a triangle and add the four tasks to the HLT. Future research will be completed with these students to see they are able to describe and complete an informal justification for the circle area formula on their own.

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