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Developing Conceptual and Procedural Knowledge of Mathematics

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Abstract
Mathematical competence rests on developing knowledge of concepts and of procedures (i.e. conceptual and procedural knowledge). Although there is some variability in how these constructs are defined and measured, there is general consensus that the relations between conceptual and procedural knowledge are often bi-directional and iterative. The chapter reviews recent studies on the relations between conceptual and procedural knowledge in mathematics and highlights examples of instructional methods for supporting both types of knowledge. It concludes with important issues to address in future research, including gathering evidence for the validity of measures of conceptual and procedural knowledge and specifying more comprehensive models for how conceptual and procedural knowledge develop over time.

Keywords: conceptual knowledge, procedural knowledge, mathematics, iterative relations, instruction

Introduction

When children practise solving problems, does this also enhance their understanding of the underlying concepts? Under what circumstances do abstract concepts help children invent or implement correct procedures? These questions tap a central research topic in the fields of cognitive development and educational psychology: the relations between conceptual and procedural knowledge. Delineating how these two types of knowledge interact is fundamental to understanding how knowledge development occurs. It is also central to improving instruction.

The goals of the current paper were: (1) discuss prominent definitions and measures of each type of knowledge, (2) review recent research on the developmental relations between conceptual and procedural knowledge for learning mathematics, (3) highlight promising research on potential methods for improving both types of knowledge, and (4) discuss problematic issues and future directions. We consider each in turn.

Defining Conceptual and Procedural Knowledge

Although conceptual and procedural knowledge cannot always be separated, it is useful to distinguish between the two types of knowledge to better understand knowledge development.

First consider conceptual knowledge. A concept is 'an abstract or generic idea generalized from particular instances' (Merriam-Webster's Collegiate Dictionary, 2012). Knowledge of concepts is often referred to as *conceptual knowledge* (e.g. Byrnes & Wasik, 1991; Canobi, 2009; Rittle-Johnson, Siegler, & Alibali, 2001). This knowledge is usually not tied to particular problem types. It can be implicit or explicit, and thus does not have to be verbalizable (e.g. Goldin Meadow, Alibali, & Church, 1993). The National Research Council adopted a similar definition in its review of the mathematics education research literature, defining it as 'comprehension of mathematical concepts, operations, and relations' (Kilpatrick, Swafford, & Findell, 2001, p. 5). This type of knowledge is sometimes also called conceptual understanding or principled knowledge.

At times, mathematics education researchers have used a more constrained definition. Star (2005) noted that: 'The term *conceptual knowledge* has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g. deeply and with rich connections)' (p. 408). This definition is based on Hiebert and LeFevre's definition in the seminal book edited by Hiebert (1986):

'Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network' (pp. 3–4).

After interviewing a number of mathematics education researchers, Baroody and colleagues (Baroody, Feil, & Johnson, 2007) suggested that conceptual knowledge should be defined as 'knowledge about facts, [generalizations], and principles' (p. 107), without requiring that the knowledge be richly connected. Empirical support for this notion comes from research on conceptual change that shows that (1) novices' conceptual knowledge is often fragmented and needs to be integrated over the course of learning and (2) experts' conceptual knowledge continues to expand and become better organized (diSessa, Gillespie, & Esterly, 2004; Schneider & Stern, 2009). Thus, there is general consensus that conceptual knowledge should be defined as knowledge of concepts. A more constrained definition requiring that the knowledge be richly connected has sometimes been used in the past, but more recent thinking views the richness of connections as a feature of conceptual knowledge that increases with expertise.

Next, consider procedural knowledge. A procedure is a series of steps, or actions, done to accomplish a goal. Knowledge of procedures is often termed *procedural knowledge* (e.g. Canobi, 2009; Rittle-Johnson et al., 2001). For example, 'Procedural knowledge … is 'knowing how', or the knowledge of the steps required to attain various goals. Procedures have been characterized using such constructs as skills, strategies, productions, and interiorized actions' (Byrnes & Wasik, 1991, p. 777). The procedures can be (1) algorithms—a predetermined sequence of actions that will lead to the correct answer when executed correctly, or (2) possible actions that must be sequenced appropriately to solve a given problem (e.g. equation-solving steps). This knowledge develops through problem-solving practice, and thus is tied to particular problem types. Further, 'It is the clearly sequential nature of procedures that probably sets them most apart from other forms of knowledge' (Hiebert & LeFevre, 1986, p. 6).

As with conceptual knowledge, the definition of procedural knowledge has sometimes included additional constraints. Within mathematics education, Star (2005) noted that sometimes: 'the term procedural knowledge indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known (e.g. superficially and without rich connections)' (p. 408). Baroody and colleagues (Baroody et al., 2007) acknowledged that:

> *'some mathematics educators, including the first author of this commentary, have indeed been guilty of oversimplifying their claims and loosely or inadvertently equating "knowledge memorized by rote … with computational skill or procedural knowledge" (Baroody, 2003, p. 4). Mathematics education researchers (MERs) usually define procedural knowledge, however, in terms of knowledge type—as sequential or "step-by-step [prescriptions for] how to complete tasks" (Hiebert & Lefevre, 1986, p. 6' (pp. 116–117).*

Thus, historically, procedural knowledge has sometimes been defined more narrowly within mathematics education, but there appears to be agreement that it should not be.

Within psychology, particularly in computational models, there has sometimes been the additional constraint that procedural knowledge is implicit knowledge that cannot be verbalized directly. For example, John Anderson (1993) claimed: 'procedural knowledge is knowledge people can only manifest in their performance …. procedural knowledge is not reportable' (pp. 18, 21). Although later accounts of explicit and implicit knowledge in ACT-R (Adaptive Control of Thought—Rational) (Lebiere, Wallach, & Taatgen, 1998; Taatgen, 1999) do not repeat this claim, Sun, Merrill, and Peterson (2001) concluded that: ‘The inaccessibility of procedural knowledge is accepted by most researchers and embodied in most computational models that capture procedural skills’ (p. 206). In part, this is because the models are often of procedural knowledge that has been automatized through extensive practice. However, at least in mathematical problem solving, people often know and use procedures that are not automatized, but rather require conscious selection, reflection, and sequencing of steps (e.g. solving complex algebraic equations), and this knowledge of procedures can be verbalized (e.g. Star & Newton, 2009).

Overall, there is a general consensus that procedural knowledge is the ability to execute action sequences (i.e. procedures) to solve problems. Additional constraints on the definition have been used in some past research, but are typically not made in current research on mathematical cognition.

Measuring Conceptual and Procedural Knowledge

Ultimately, how each type of knowledge is *measured* is critical for interpreting evidence on the relations between conceptual and procedural knowledge. Conceptual knowledge has been assessed in a large variety of ways, whereas there is much less variability in how procedural knowledge is measured.

Measures of conceptual knowledge vary in whether tasks require implicit or explicit knowledge of the concepts, and common tasks are outlined in Table 1. Measures of implicit conceptual knowledge are often evaluation tasks on which children make a categorical choice (e.g. judge the correctness of an example procedure or answer) or make a quality rating (e.g. rate an example procedure as very-smart, kind-of-smart, or not-so-smart). Other common implicit measures are translating between representational formats (e.g. symbolic fractions into pie charts) and comparing quantities (see Table 1 for more measures).

Table 1: Range of tasks used to assess conceptual knowledge.

Type of task	Sample task	Additional citations
Implicit measures		
a. Evaluate unfamiliar procedures	Decide whether ok for puppet to skip some items when counting (Gelman & Meck, 1983)	(Kamawar et al., 2010; LeFevre et al., 2006; Muldoon, Lewis, & Berridge, 2007; Rittle-Johnson & Alibali, 1999; Schneider et al., 2009; Schneider & Stern, 2010; Siegler & Crowley, 1994)
b. Evaluate examples of concept	a. Decide whether the number sentence 3 = 3 makes sense (Rittle-Johnson & Alibali, 1999); b. 45 + 39 = 84, Does puppet need to count to figure out 39 + 45? (Canobi et al., 1998)	(Canobi, 2005; Canobi & Bethune, 2008; Canobi, Reeve, & Pattison, 2003; Patel & Canobi, 2010; Rittle-Johnson et al., 2001; Rittle-Johnson et al., 2009; Schneider et al., 2011)
c. Evaluate quality of answers given by others	Evaluate how much someone knows based on the quality of their errors, which are or are not consistent with principles of arithmetic (Prather & Alibali, 2008)	(Dixon, Deets, & Bangert, 2001; Mabbott & Bisanz, 2003; Star & Rittle-Johnson, 2009)
d. Translate quantities between representational systems	a. Represent symbolic numbers with pictures (Hecht, 1998) b. Place symbolic numbers on number lines (Siegler & Booth, 2004; Siegler, Thompson, & Schneider, 2011)	(Byrnes & Wasik, 1991; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Cobb et al., 1991; Hecht & Vagi, 2010; Hiebert & Wearne, 1996; Mabbott & Bisanz, 2003; Moss & Case, 1999; Prather & Alibali, 2008; Reimer & Moyer, 2005; Rittle-Johnson & Koedinger, 2009; Schneider et al., 2009; Schneider & Stern, 2010)
e. Compare quantities	Indicate which symbolic integer or fraction is larger (or smaller) (Hecht, 1998; Laski & Siegler, 2007)	(Durkin & Rittle-Johnson, 2012; Hallett et al., 2010; Hecht & Vagi, 2010; Laski & Siegler, 2007; Moss & Case, 1999; Murray & Mayer, 1988; Rittle-Johnson et al., 2001; Schneider et al., 2009; Schneider & Stern, 2010)
f. Invent principle-based shortcut procedures	On inversion problems such as 12 + 7–7, quickly stating the first number without computing (Rasmussen, Ho, & Bisanz, 2003)	(Canobi, 2009)

g. Encode key features	Success reconstructing examples from memory (e.g. a chess board or equations), with the assumption that greater conceptual knowledge helps people notice key features and chunk information, allowing for more accurate recall (Larkin, McDermott, Simon, & Simon, 1980)	(Matthews & Rittle-Johnson, 2009; McNeil & Alibali, 2004; Rittle-Johnson et al., 2001)
h. Sort examples into categories	Sort 12 statistics problems based on how they best go together (Lavigne, 2005)	Mainly used in other domains, such as physics
Explicit measures		
a. Explain judgements	On evaluation task, provide correct explanation of choice (e.g. '29 + 35 has the same numbers as 35 + 29, so it equals 64, too.' (Canobi, 2009)	(Canobi, 2004, 2005; Canobi & Bethune, 2008; Canobi et al., 1998, 2003; Peled & Segalis, 2005; Rittle-Johnson & Star, 2009; Rittle-Johnson et al., 2009; Schneider et al., 2011; Schneider & Stern, 2010)
a. Generate or select definitions of concepts	Define the equal sign (Knuth, Stephens, McNeil, & Alibali, 2006; Rittle-Johnson & Alibali, 1999)	(Star & Rittle-Johnson, 2009; Vamvakoussi & Vosniadou, 2004) (Izsák, 2005)
b. Explain why procedures work	Explain why ok to borrow when subtract (Fuson & Kwon, 1992)	(Berthold & Renkl, 2009; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Reimer & Moyer, 2005; Stock, Desoete, & Roeyers, 2007)
c. Draw concept maps	Construct a map that identifies main concepts in introductory statistics, showing how the concepts are related to one another (Lavigne, 2005)	(Williams, 1998)

Explicit measures of conceptual knowledge typically involve providing definitions and explanations. Examples include generating or selecting definitions for concepts and terms, explaining why a procedure works, or drawing a concept map (see Table 1). These tasks may be completed as paper-and-pencil assessment items or answered verbally during standardized or clinical interviews (Ginsburg, 1997). We do not know of a prior study on conceptual knowledge that quantitatively assessed how richly connected the knowledge was.

Clearly, there are a large variety of tasks that have been used to measure conceptual knowledge. A critical feature of conceptual tasks is that they be relatively unfamiliar to participants, so that participants have to derive an answer from their conceptual knowledge, rather than implement a known procedure for solving the task. For example, magnitude comparison problems are sometimes used to assess children's conceptual knowledge of number magnitude (e.g. Hecht, 1998; Schneider, Grabner, & Paetsch, 2009). However, children are sometimes taught procedures for comparing magnitudes or develop procedures with repeated practice; for these children, magnitude comparison problems are likely measuring their procedural knowledge, not their conceptual knowledge.

In addition, conceptual knowledge measures are stronger if they use multiple tasks. First, use of multiple tasks meant to assess the same concept reduces the influence of task-specific characteristics (Schneider & Stern, 2010). Second, conceptual knowledge in a domain often requires knowledge of many concepts, leading to a multi-dimensional construct. For example, for counting, key concepts include cardinality and order-irrelevance, and in arithmetic, key concepts include place value and the commutativity and inversion principles. Although knowledge of each is related, there are individual differences in these relationships, without a standard hierarchy of difficulty (Dowker, 2008; Jordan, Mulhern, & Wylie, 2009).

Measures of procedural knowledge are much less varied. The task is almost always to solve problems, and the outcome measure is usually accuracy of the answers or procedures. On occasion, researchers consider solution time as well (Canobi, Reeve, & Pattison, 1998; LeFevre et al., 2006; Schneider & Stern, 2010). Procedural tasks are familiar—they involve problem types people have solved before and thus should know procedures for solving. Sometimes the tasks include near transfer problems—problems with an unfamiliar problem feature that require either recognition that a known procedure is relevant or small adaptations of a known procedure to accommodate the unfamiliar problem feature (e.g. Renkl, Stark, Gruber, & Mandl, 1998; Rittle-Johnson, 2006).

There are additional measures that have been used to tap particular ways in which procedural knowledge can be known. When interested in how well automatized procedural knowledge is, researchers use dual-task paradigms (Ruthruff, Johnston, & van Selst, 2001; Schumacher, Seymour, Glass, Kieras, & Meyer, 2001) or quantify asymmetry of access, that is, the difference in reaction time for solving a practiced task versus a task that requires the same steps executed in the reverse order (Anderson & Fincham, 1994; Schneider & Stern, 2010). The execution of automatized procedural knowledge does not involve conscious reflection and is often independent of conceptual knowledge (Anderson, 1993). When interested in how flexible procedural knowledge is, researchers assess students' knowledge of multiple procedures and their ability to flexibly choose among them to solve problems efficiently (e.g. Blöte, Van der Burg, & Klein, 2001; Star & Rittle-Johnson, 2008; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Flexibility of procedural knowledge is positively related to conceptual knowledge, but this relationship is evaluated infrequently (see Schneider, Rittle-Johnson & Star, 2011, for one instance).

To study the relations between conceptual and procedural knowledge, it is important to assess the two independently. However, it is important to recognize that it is difficult for an item to measure one type of knowledge to the exclusion of the other. Rather, items are thought to predominantly measure one type of knowledge or the other. In addition, we believe that continuous knowledge measures are more appropriate than categorical measures. Such measures are able to capture the continually changing depths of knowledge, including the context in which knowledge is and is not being used. They are also able to capture variability in people's thinking, which appears to be a common feature of human cognition (Siegler, 1996).

Relations Between Conceptual and Procedural Knowledge

Historically, there have been four different theoretical viewpoints on the causal relations between conceptual and procedural knowledge (cf. Baroody, 2003; Haapasalo & Kadijevich, 2000; Rittle-Johnson & Siegler, 1998). *Concepts-first views* posit that children initially acquire conceptual knowledge, for example, through parent explanations or guided by innate constraints, and then derive and build procedural knowledge from it through repeated practice solving problems (e.g. Gelman & Williams, 1998; Halford, 1993). *Procedures-first views* posit that children first learn procedures, for example, by means of explorative behaviour, and then gradually derive conceptual knowledge from them by abstraction processes, such as representational re-description (e.g. Karmiloff-Smith, 1992; Siegler & Stern, 1998). A third possibility, sometimes labelled *inactivation view* (Haapasalo & Kadijevich, 2000), is that conceptual and procedural knowledge develop independently (Resnick, 1982; Resnick & Omanson, 1987). A fourth possibility is an *iterative view*. The causal relations are said to be bi-directional, with increases in conceptual knowledge leading to subsequent increases in procedural knowledge and vice versa (Baroody, 2003; Rittle-Johnson & Siegler, 1998; Rittle-Johnson et al., 2001).

The iterative view is now the most well-accepted perspective. An iterative view accommodates gradual improvements in each type of knowledge over time. If knowledge is measured using continuous, rather than categorical, measures, it becomes clear that one type of knowledge is not well developed before the other emerges, arguing against a strict concepts- or procedures-first view. In addition, an iterative view accommodates evidence in support of concepts-first and procedures-first views, as initial knowledge can be conceptual or procedural, depending upon environmental input and relevant prior knowledge of other topics. An iterative view was not considered in early research on conceptual and procedural knowledge (see Rittle-Johnson & Siegler, 1998, for a review of this research in mathematics learning), but over the past 15 years there has been an accumulation of evidence in support of it.

First, positive correlations between the two types of knowledge have been found in a wide range of ages and domains. The domains include counting (Dowker, 2008; LeFevre et al., 2006), addition and subtraction (Canobi & Bethune, 2008; Canobi et al., 1998; Jordan et al., 2009; Patel & Canobi, 2010), fractions and decimals (Hallett, Nunes, & Bryant, 2010; Hecht, 1998; Hecht, Close, & Santisi, 2003; Reimer & Moyer, 2005), estimation (Dowker, 1998; Star & Rittle-Johnson, 2009), and equation solving (Durkin, Rittle-Johnson, & Star, 2011). In general, the strength of the relation is fairly high. For example, in a meta-analysis of a series of eight studies conducted by the first author and colleagues on equation solving and estimation, the mean effect size for the relation was 0.54 (Durkin, Rittle-Johnson, & Star, 2011). Further, longitudinal studies suggest that the strength of the relation between the two types of knowledge varies over time (Jordan et al., 2009; Schneider, Rittle-Johnson, & Star, 2011). The strength of the relation varies across studies and over time, but it is clear that the two types of knowledge are often related.

Second, evidence for predictive, bi-directional relations between conceptual and procedural knowledge has been found in mathematical domains ranging from fractions to equation solving. For example, in two samples differing in prior knowledge, middle-school students' conceptual and procedural knowledge for equation solving was measured before and after a 3-day classroom intervention in which students studied and explained worked examples with a partner (Schneider et al., 2011). Conceptual and procedural knowledge were modelled as latent variables to better account for the indirect relation between overt behaviour and the underlying knowledge structures. A cross-lagged panel design was used to directly test and compare the predictive relations from conceptual knowledge to procedural knowledge and vice versa. As expected, each type of knowledge predicted gains in the other type of knowledge, with standardized regressions coefficients of about 0.3, and the relations were symmetrical (i.e. they did not differ significantly in their strengths). Similar bi-directional relations have been found for elementary-school children learning about decimals (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson et al., 2001). Overall, knowledge of one type is a good and reliable predictor of improvements in knowledge of the other type.

The predictive relations between conceptual and procedural knowledge are even present over several years (Cowan et al., 2011). For example, elementary-school children's knowledge of fractions was assessed in the winter of Grade 4 and again in the spring of Grade 5 (Hecht & Vagi, 2010). Conceptual knowledge in Grade 4 predicted about 5% of the variance in procedural knowledge in Grade 5 after controlling for other factors, and procedural knowledge in Grade 4 predicted about 2% of the variance in conceptual knowledge in Grade 5.

In addition to the predictive relations between conceptual and procedural knowledge, there is evidence that experimentally manipulating one type of knowledge can lead to increases in the other type of knowledge. First, direct instruction on one type of knowledge led to improvements in the other type of knowledge (Rittle-Johnson & Alibali, 1999). Elementary-school children were given a very brief lesson on a procedure for solving mathematical equivalence problems (e.g. $6 + 3 + 4 = 6 + \_\_$), the concept of mathematical equivalence, or were given no lesson. Children who received the procedure lesson gained a better understanding of the concept, and children who received the concept lesson generated correct procedures for solving the problems. Second, practice-solving problems can support improvements in conceptual knowledge when constructed appropriately (Canobi, 2009; McNeil et al., 2012). For example, elementary-school children solved packets of problems for 10 minutes on nine occasions during their school mathematics lessons. The problems were arithmetic problems sequenced based on conceptual principles (e.g. 6 + 3 followed by 3 + 6), the same arithmetic problems sequenced randomly, or non-mathematical problems (control group). Solving conceptually sequenced practice problems supported gains in conceptual knowledge, as well as procedural knowledge. Together, this evidence indicates that there are causal, bi-directional links between the two types of knowledge; improving procedural knowledge can lead to improved conceptual knowledge and vice versa, especially if potential links between the two are made salient (e.g. through conceptually sequencing problems).

An iterative view predicts that the bi-directional relations between conceptual and procedural knowledge persist over time, with increases in one supporting increases in the other, *which in turn supports increases in the first type of knowledge*. Indeed, prior conceptual knowledge of decimals predicted gains in procedural knowledge after a brief problem-solving intervention, which in turn predicted gains in conceptual knowledge (Rittle-Johnson et al., 2001). In addition, iterating between lessons on concepts and procedures on decimals supported greater procedural knowledge and equivalent conceptual knowledge compared to presenting concept lessons before procedure lessons (Rittle-Johnson & Koedinger, 2009). Both studies suggest that relations between the two types of knowledge are bi-directional over time (i.e. iterative).

Overall, there is extensive evidence from a variety of mathematical domains indicating that the development of conceptual and procedural knowledge of mathematics is often iterative, with one type of knowledge supporting gains in the other knowledge, which in turn supports gains in the other type of knowledge. Conceptual knowledge may help with the construction, selection, and appropriate execution of problem-solving procedures. At the same time, practice implementing procedures may help students develop and deepen understanding of concepts, especially if the practice is designed to make underlying concepts more apparent. Both kinds of knowledge are intertwined and can strengthen each other over time.

However, the relations between the two types of knowledge are not always symmetrical. In Schneider, Rittle-Johnson, and Star (2011), the relations were symmetrical—the strength of the relationship from prior conceptual knowledge to later procedural knowledge was the same as from prior procedural knowledge to later conceptual knowledge. However, in other studies, conceptual knowledge or conceptual instruction has had a stronger influence on procedural knowledge than vice versa (Hecht & Vagi, 2010; Matthews & Rittle-Johnson, 2009; Rittle-Johnson & Alibali, 1999). Furthermore, brief procedural instruction or practice solving problems does not always support growth in conceptual knowledge (Canobi, 2009; Perry, 1991; Rittle-Johnson, 2006), and increasing school experience is associated with gains in procedural knowledge for counting and arithmetic, but much less so with gains in conceptual knowledge (Canobi, 2004; LeFevre et al., 2006).

How much gains in procedural knowledge support gains in conceptual knowledge is influenced by the nature of the procedural instruction or practice. For example, in Canobi (2009) and McNeil et al. (2012), sequencing arithmetic practice problems so that conceptual relations were easier to notice supported conceptual knowledge, while random ordering of practice problems did not. In Peled and Segalis (2005), instruction that encouraged students to generalize procedural steps and connect subtraction procedures across whole numbers, decimals, and fractions led to greater conceptual knowledge than instruction on individual procedures. In general, it is best if procedural lessons are crafted to encourage noticing of underlying concepts.

The symmetry of the relations between conceptual and procedural knowledge also varies between individuals. Children in Grades 4 and 5 completed a measure of their conceptual and procedural knowledge of fractions (Hallett et al., 2010). A cluster analysis on the two measures suggested five different clusters of students, with clusters varying in the strength of conceptual and procedural knowledge. For example, one cluster had above-average conceptual knowledge and below-average procedural knowledge, another cluster was the opposite, and a third cluster was high on both measures. These cluster differences suggest that, although related in all clusters, the strength of the relations varied. Similar findings were reported for primary-school children's knowledge of addition and subtraction (Canobi, 2005), including a meta-analysis of over 14 studies (Gilmore & Papadatou-Pastou, 2009). At least in part, these individual differences may reflect different instructional histories between children.

Overall, the relations between conceptual and procedural knowledge are bi-directional, but sometimes they are not symmetrical. At times, conceptual knowledge more consistently and strongly supports procedural knowledge than the reverse. Crafting procedural lessons to encourage noticing of underlying concepts can promote a stronger link from improved procedural knowledge to gains in conceptual knowledge.

Promising Methods for Improving Both Types of Knowledge

Given the importance of developing both conceptual and procedural knowledge, instructional techniques that support both types of knowledge are critical. Here, we highlight examples of general instructional methods that are promising.

Promoting comparison of alternative solution procedures is one effective instructional approach. In a series of studies, students studied pairs of worked examples illustrating two different, correct procedures for solving the same problem and were prompted to compare them or studied the same examples one at a time and were prompted to reflect on them individually. For students who knew one of the solution procedures at pre-test, comparing procedures supported greater procedural knowledge (Rittle-Johnson & Star, 2007; Rittle-Johnson, Star, & Durkin, 2009) or greater conceptual knowledge (Rittle-Johnson & Star, 2009; Rittle-Johnson et al., 2009; Star & Rittle-Johnson, 2009). For novices, who did not know one of the solution procedures at pre-test, no benefits were found for conceptual or procedural knowledge (although comparison did improve procedural flexibility; see Rittle-Johnson et al., 2009; Rittle-Johnson, Star, & Durkin, 2011). In addition, having students compare incorrect procedures to correct ones aided conceptual and procedural knowledge and reduced misconceptions (Durkin & Rittle-Johnson, 2012). Overall, comparing procedures can help students gain conceptual and procedural knowledge, but its advantages are more substantial if students have sufficient prior knowledge.

A second approach is to encourage self-explanation when studying solution procedures. For example, prompting primary-school children to explain why solutions to mathematical equivalence problems were correct or incorrect supported greater procedural transfer (Rittle-Johnson, 2006). Similarly, prompting high-school students to self-explain when studying worked examples of probability problems supported greater conceptual knowledge of probability (although it seemed to hamper procedural knowledge; Berthold & Renkl, 2009).

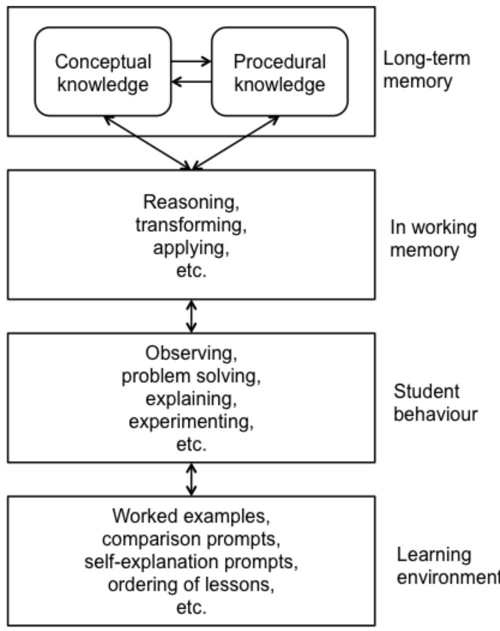
A third approach is to offer opportunities for problem exploration *before* instruction (Schwartz, Chase, Chin, & Oppezzo, 2011). For example, primary-school children solved a set of unfamiliar mathematics problems and received a lesson on the concept of equivalence, and the order of problem solving and the lesson was manipulated (DeCaro & Rittle-Johnson, 2011). Children who solved the unfamiliar problems before the lesson made greater gains in conceptual knowledge, and comparable gains in procedural knowledge, compared to children who solved the problems after the lesson. Similarly, middle-school students who explored problems and invented their own formula for calculating density before instruction on density gained deeper conceptual and procedural knowledge of density than students who received the lessons first (Schwartz et al., 2011). Initial problem exploration fits with the recommendation from the mathematics education literature that students have opportunities to struggle—to figure out something that is not immediately apparent (Hiebert & Grouws, 2009).

Comparison, self-explanation, and exploration are all promising instructional methods for promoting conceptual and procedural knowledge, as are sequencing problems so that conceptual relations are more apparent (Canobi, 2009) and iterating between lessons on concepts and procedures (Rittle-Johnson & Koedinger, 2009). These are just some examples of effective methods; certainly there are numerous others (e.g. McNeil & Alibali, 2000) and more need to be identified.

Future Directions

Considerable progress has been made in understanding the development of conceptual and procedural knowledge of mathematics over the past 15 years. An important next step is to develop a more comprehensive model of the relations between conceptual and procedural knowledge. Some components that need to be considered in such a model are shown in Figure 1. To flesh out such a model, we will need a better understanding of numerous components. For example, are conceptual and procedural knowledge stored independently in long-term memory and does this change with expertise? How do age and individual differences impact the relations between conceptual and procedural knowledge and the effectiveness of different instructional methods? What additional instructional methods can be integrated into learning environments and what student behaviours and mental activities do they support? How do differences across topics impact the model (e.g. learning about counting vs. algebra)? What are alternative models for understanding the relations between conceptual and procedural knowledge?

Figure 1: Potential components of an information-processing model for the relations between conceptual and procedural knowledge.



However, before more progress can be made in understanding the relations between conceptual and procedural knowledge, we must pay more attention to the validity of measures of conceptual and procedural knowledge. Currently, no standardized approaches for assessing conceptual and procedural knowledge with proven validity, reliability, and objectivity have been developed. This is deeply problematic because knowledge is stored in memory and has to be inferred from overt behaviour. However, human behaviour arises from a complex interplay of a multitude of cognitive processes and usually does not reflect memory content in a pure and direct form. This makes it difficult to attribute learners' answers exclusively to one type of knowledge.

Each potential measure of conceptual or procedural knowledge has at least four different variance components (Schneider & Stern, 2010). First, if the measure has been developed carefully, it can be assumed to reflect the amount of the kind of knowledge it is supposed to assess. Second, each assessment task also requires task-specific knowledge. For example, when children answer interview questions, their answers reflect not only their conceptual knowledge, but also their vocabulary in the respective domain and more general verbal abilities. A diagram task designed to assess procedural knowledge about fractions reflects not only knowledge about fractions but also knowledge of and experience with the specific diagrams used in that task.

Third, under many circumstances, learners can derive new procedures from their conceptual knowledge (Gelman & Williams, 1998) and they can abstract new concepts from their procedural experience (Karmiloff-Smith, 1992). Thus, measures of conceptual knowledge often reflect some procedural knowledge and measures of procedural knowledge might also reflect conceptual knowledge to some degree.

Finally, random measurement error is present in virtually all psychological measures. This makes it hard to interpret findings about conceptual and procedural knowledge. For example, when a measure of conceptual knowledge and a measure of procedural knowledge show a low inter-correlation, is this due to a dissociation of conceptual and procedural knowledge, due to task-specific knowledge, or due to high measurement error?

A confirmatory factor analysis (Schneider & Stern, 2010) demonstrated that this problem is not just theoretical. Four commonly used hypothetical measures of conceptual knowledge and four commonly used hypothetical measures of procedural knowledge were completed by fifth and sixth graders. Conceptual and procedural knowledge were modelled as latent factors underlying these eight measures. However, each latent factor explained less than 50% of the variance of the measured variables, indicating that the measures reflected measure-specific variance components and random measurement error to a higher degree than the kind of knowledge they were supposed to assess.

Very little attention has been given to measurement validity in the literature on conceptual and procedural knowledge. Clearly, attention to validity is greatly needed. Future studies will have to validate tasks and measures to ensure that we are using good measures of conceptual and procedural knowledge. As noted by Hill and Shih (2009):

> *'Without conducting and reporting validation work on key independent and dependent variables, we cannot know the extent to which our instruments tap what they claim to. And without this knowledge, we cannot assess the validity of inferences drawn from studies' (p. 248).*

Likely progress will require some mixture of traditional psychometric approaches, newer approaches based on item-response theory, and perhaps innovations in alternative ways to validate measures, especially of conceptual knowledge.

Conclusion

Mathematical competence rests on developing both conceptual and procedural knowledge. Although there is some variability in how these constructs are defined and measured, there is general consensus that the relations between conceptual and procedural knowledge are often bi-directional and iterative. Instructional methods for supporting both types of knowledge have emerged, such as promoting comparison of alternative solution methods, prompting for self-explanation, and providing opportunities for exploration before instruction. Future research needs to focus on more rigorous measurement of conceptual and procedural knowledge, providing evidence for the validity of the measures, and specify more comprehensive models for understanding how conceptual and procedural knowledge develop.

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