Development and Analysis of a Sensitivity Matrix of a 3-phase Voltage Unbalance Factor

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Abstract—**Unbalanced phase voltages in electricity networks prevent efficient and stable delivery of power and energy services from utility companies to end-users, and vice versa. While variations in voltage magnitude and power loss have been studied extensively, little attention has been paid to the development of linearized expressions to account for the variation in voltage unbalance. Using a standard definition of the voltage unbalance factor (VUF), this study is the first, to the best knowledge of the authors, in which a VUF sensitivity matrix is mathematically developed and analyzed with respect to variations in 3-phase (3-*ph*) voltages and, consequently, 3-*ph* unbalanced load demand. To demonstrate its utility, we performed simulation case studies and compared the estimates made using the matrix to those from a power flow calculation algorithm. The VUF sensitivity matrix is expected to have a variety of practical applications for power systems, such as optimal scheduling and real-time control, which enables improvement in network power quality.**

Index Terms—**distribution network, power flow calculation, power quality, sensitivity matrix, voltage unbalance factor (VUF)**

I. INTRODUCTION

UNBALANCED voltages and, consequently, currents usually occur when single-phase (1-*ph*) loads are connected between two phases, or between one phase and the neutral. In particular, 1-*ph* building loads are randomly connected to any two points of three-phase (3-*ph*) power lines. Therefore, it frequently occurs that one or two phases are heavily charged, causing large unbalances in phase voltages. It is difficult to change load connections, as this requires retrofitting of building electricity systems. Voltage unbalance also can occur in 3-*ph* loads that are operating normally when the line impedances are asymmetrical. Voltage unbalance is considered to be one of the main causes of power quality problems. It is also detrimental to the performance of power equipment [1]. Under unbalanced voltage conditions, power networks incur more power losses and heating effects, rendering them less stable [2]. Voltage unbalance also has adverse effects on equipment such as induction motors and power electronics converters. For example, if the voltages on the stator of an induction motor are unbalanced, the winding insulation may fail prematurely or the operational efficiency can be reduced [3]. Voltage unbalances in AC-DC converters can cause low-order frequency power ripples, leading to current harmonics and poor DC-link voltage regulation [4]. The International Electrotechnical Commission (IEC) recommends that voltage unbalance in power grids should be limited to 2% [1]. Consequently, grid operators or electric utility companies need to optimize the operation of distributed energy resources (DERs) and controllable loads while taking variations in phase

voltage unbalances into consideration, so as to achieve stable and reliable operations of electric power networks in practice.

For the practical application, it is important to estimate how power inputs and outputs of the devices will affect voltage unbalance in the entire network. To make these estimates, it is necessary to undertake a sensitivity analysis of the voltage unbalance. However, while voltage magnitudes and power losses have been widely studied, little attention has been paid to developing a linearized expression of voltage unbalance. Previously, the effects of voltage unbalance on optimal power flow were analyzed using nonlinear optimization algorithms with iterative gradient methods, which are often computationally expensive. These methods also require additional effort to ensure convexity and hence avoid getting stuck in local optima.

In this paper, we propose a sensitivity matrix of the 3-*ph* voltage unbalance, based on a standard definition of the voltage unbalance factor (VUF). To the best knowledge of the authors, this study is the first in which a mathematical tool is developed to estimate a VUF sensitivity matrix that can be directly incorporated into linear programming (LP) or mixed-integer LP (MILP) algorithms. These algorithms have been widely used to comprehensively model and determine the optimal operation of complicated power systems mainly because convergence to the global optimum is guaranteed within a reasonable time using off-the-shelf software. Therefore, this incorporation enables grid operators or electric utility companies to easily determine optimal power flow or optimal operations of 1-*ph* DERs and loads considering voltage unbalance variations. Case studies are performed to validate the estimates calculated using the sensitivity matrix by comparing them to those obtained from power flow calculations, where a backward-forward sweep (BFS) algorithm is used [5]. The comparison is valid as the BFS algorithm does not require a Jacobian or sensitivity matrix.

II. SENSITIVITY MATRIX OF VUF

The IEC defines the VUF as the ratio of negative sequence voltage $|V_-|$ to positive sequence voltage $|V_+|$ [6]. In (1), the 3-*ph* voltages at bus k are expressed in rectangular coordinates. The derivation of the VUF at bus k is shown in (2)–(5).

$$V_a^k = E_a^k + jF_a^k, \; V_b^k = E_b^k + jF_b^k, \text{ and } V_c^k = E_c^k + jF_c^k \qquad (1)$$

$$VUF^k = f\left(V_a^k, V_b^k, V_c^k\right) = g\left(E_a^k, F_a^k, E_b^k, F_b^k, E_c^k, F_c^k\right) \qquad (2)$$

$$= \left|\frac{V_-^k}{V_+^k}\right| = \left|\frac{V_a^k + \alpha^2 V_b^k + \alpha V_c^k}{V_a^k + \alpha V_b^k + \alpha^2 V_c^k}\right| = \left|\frac{n_1^k + jn_2^k}{d_1^k + jd_2^k}\right| = \left(\frac{n^k}{d^k}\right)^{1/2},$$

where $\alpha = e^{j2\pi/3} = \alpha_{re} + j\alpha_{im}$, $\qquad\qquad (3)$

$$\begin{bmatrix} n_1^k \\ n_2^k \\ d_1^k \\ d_2^k \end{bmatrix} = \begin{bmatrix} 1 & \alpha_{re} & \alpha_{re} & 0 & \alpha_{im} & -\alpha_{im} \\ 0 & -\alpha_{im} & \alpha_{im} & 1 & \alpha_{re} & \alpha_{re} \\ 1 & \alpha_{re} & \alpha_{re} & 0 & -\alpha_{im} & \alpha_{im} \\ 0 & \alpha_{im} & -\alpha_{im} & 1 & \alpha_{re} & \alpha_{re} \end{bmatrix} \cdot \mathbf{V}^k = \begin{bmatrix} \mathbf{\alpha_{n1}} \\ \mathbf{\alpha_{n2}} \\ \mathbf{\alpha_{d1}} \\ \mathbf{\alpha_{d2}} \end{bmatrix} \cdot \mathbf{V}^k, \qquad (4)$$

and $\mathbf{V}^k = \begin{bmatrix} \mathbf{E}^k & \mathbf{F}^k \end{bmatrix}^T = \begin{bmatrix} E_a^k & E_b^k & E_c^k & F_a^k & F_b^k & F_c^k \end{bmatrix}^T. \qquad (5)$

From (2), the variation in the VUF at bus k can be estimated with respect to the change in $\mathbf{V}^k$ as

$$\Delta VUF^k = \left[\frac{\partial g}{\partial E_a^k} \quad \frac{\partial g}{\partial E_b^k} \quad \frac{\partial g}{\partial E_c^k} \quad \frac{\partial g}{\partial F_a^k} \quad \frac{\partial g}{\partial F_b^k} \quad \frac{\partial g}{\partial F_c^k} \right] \left[\Delta \mathbf{E}^k \quad \Delta \mathbf{F}^k \right]^{\mathbf{T}} \quad (6)$$

$$= \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},k} \left[\Delta \mathbf{E}^k \quad \Delta \mathbf{F}^k \right]^{\mathbf{T}} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},k} \Delta \mathbf{V}^k.$$

For $x = E_a^k, E_b^k, E_c^k, F_a^k, F_b^k,$ and F_c^k, the partial derivatives in (6) can be represented as

$$\frac{\partial g}{\partial x} = c^k \left\{ \left(n_1^k \frac{\partial n_1^k}{\partial x} + n_2^k \frac{\partial n_2^k}{\partial x} \right) d^k - \left(d_1^k \frac{\partial d_1^k}{\partial x} + d_2^k \frac{\partial d_2^k}{\partial x} \right) n^k \right\}, \quad (7)$$

where
$$c^k = \left(d^k / n^k \right)^{1/2} \cdot \left(1 / d^k \right)^2 \quad . \quad (8)$$

Therefore, $\mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},k}$ in (6) is expressed in compact form as

$$\mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},k} = c^k \left\{ \left(n_1^k \boldsymbol{\alpha_{n1}} + n_2^k \boldsymbol{\alpha_{n2}} \right) d^k - \left(d_1^k \boldsymbol{\alpha_{d1}} + d_2^k \boldsymbol{\alpha_{d2}} \right) n^k \right\} \quad , \quad (9)$$

where $\boldsymbol{\alpha_{n1}}$, $\boldsymbol{\alpha_{n2}}$, $\boldsymbol{\alpha_{d1}}$, and $\boldsymbol{\alpha_{d2}}$ are the constant row vectors in (4), respectively. Since ΔVUF^k only depends on the phase voltages at bus k, the sensitivity matrix of the VUFs for all 3-*ph* buses in a power network can be established as shown in (10), which can be simplified to (11). In (10), $\mathbf{O_{1 \times 6}}$ is a zero row vector, and $N_{3\phi}$ and N_T are the number of 3-*ph* buses and the total number of buses in the power network, respectively: i.e., $N_T \geq N_{3\phi}$. For convenience, Bus 1 is assumed to be a 3-*ph* bus in (10).

$$\begin{bmatrix} \Delta VUF^1 \\ \vdots \\ \Delta VUF^k \\ \vdots \\ \Delta VUF^{N_{3\phi}} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},1} & \cdots & \mathbf{O}_{1\times6} & \cdots & \mathbf{O}_{1\times6} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{O}_{1\times6} & \cdots & \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},k} & \cdots & \mathbf{O}_{1\times6} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{O}_{1\times6} & \cdots & \mathbf{O}_{1\times6} & \cdots & \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF},N_{3\phi}} & \cdots \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{V}^1 \\ \Delta \mathbf{V}^2 \\ \vdots \\ \Delta \mathbf{V}^k \\ \vdots \\ \Delta \mathbf{V}^{N_{3\phi}} \\ \vdots \\ \Delta \mathbf{V}^{N_T} \end{bmatrix} \quad (10)$$

$$\Delta \mathbf{VUF} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \Delta \mathbf{V} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \begin{bmatrix} \Delta \mathbf{E} \\ \Delta \mathbf{F} \end{bmatrix} \quad (11)$$

Furthermore, (12) can be derived from (11) by using the sensitivity matrix $\mathbf{J}_{\mathbf{PQ}}^{\mathbf{EF}}$ of complex power injections at all buses with respect to the bus voltages. This enables the estimation of variations in the VUFs with respect to changes in the active and reactive power injections. In (12), $\mathbf{J}_{\mathbf{PQ}}^{\mathbf{EF}}$ can be obtained from general power flow equations with or without consideration of nonlinear equipment, such as voltage regulators or switched capacitors [7], [8]. Note that the sensitivity matrix $\mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}}$ is independent of nonlinear control actions or equipment operating characteristics, because it is derived from the standard definition of the VUF (i.e., $|V_-|/|V_+|$).

$$\Delta \mathbf{VUF} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \begin{bmatrix} \Delta \mathbf{E} \\ \Delta \mathbf{F} \end{bmatrix} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \begin{bmatrix} \begin{bmatrix} \Delta \mathbf{E}^1 \\ \Delta \mathbf{F}^1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Delta \mathbf{E}^k \\ \Delta \mathbf{F}^k \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Delta \mathbf{E}^{N_T} \\ \Delta \mathbf{F}^{N_T} \end{bmatrix} \end{bmatrix} = \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \begin{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{E}}{\partial \mathbf{P}} \end{bmatrix} & \begin{bmatrix} \frac{\partial \mathbf{E}}{\partial \mathbf{Q}} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{P}} \end{bmatrix} & \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{Q}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \Delta \mathbf{P}^1 \\ \Delta \mathbf{Q}^1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Delta \mathbf{P}^k \\ \Delta \mathbf{Q}^k \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Delta \mathbf{P}^{N_T} \\ \Delta \mathbf{Q}^{N_T} \end{bmatrix} \end{bmatrix} \quad (12)$$

$$= \mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}} \, \mathbf{J}_{\mathbf{PQ}}^{\mathbf{EF}} \begin{bmatrix} \Delta \mathbf{P} \\ \Delta \mathbf{Q} \end{bmatrix} = \mathbf{J}_{\mathbf{PQ}}^{\mathbf{VUF}} \begin{bmatrix} \Delta \mathbf{P} \\ \Delta \mathbf{Q} \end{bmatrix}$$

The sensitivity matrices $\mathbf{J}_{\mathbf{EF}}^{\mathbf{VUF}}$ or $\mathbf{J}_{\mathbf{PQ}}^{\mathbf{VUF}}$ in (12) can then be

effectively used to develop various optimization problems for optimal operations of DERs and controllable loads in distribution networks, taking into account changes in the network power quality (e.g., [9]). The optimization problems can be easily solved with LP, rather than more computationally expensive iterative approaches. Note that in practice, distribution and transmission system operators use LP or MILP algorithms for optimal operation of their power networks. Iterative, brute force approaches such as the BFS algorithm also can be used to estimate $\Delta \mathbf{VUF}$; however, iterative approaches cannot be incorporated into LP or MILP.

III. CASE STUDIES AND SIMULATION RESULTS

The IEEE 123 Test Feeder was used as a test bed for the simulation case studies, with the slight modifications discussed in [9] incorporated. Fig. 1 shows that the test bed includes 1-*ph* and 3-*ph* buses and, consequently, contains not only 3-*ph* balanced loads but also 3-*ph* unbalanced loads and 1-*ph* loads. The load demand at each bus was scaled so that the total load demand was set to $P_L + jQ_L = 7.04$ MW + $j3.88$ MVar (i.e., $pf = 0.87$).
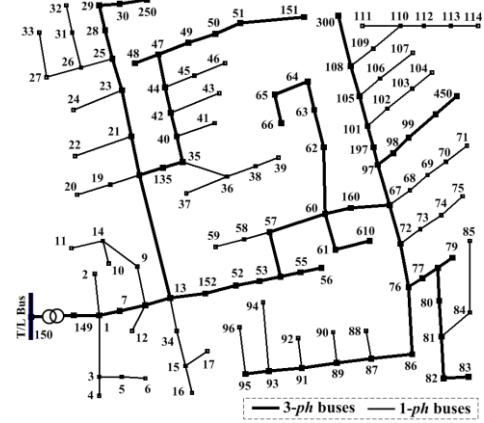

Fig. 1. Test distribution network used in the case studies
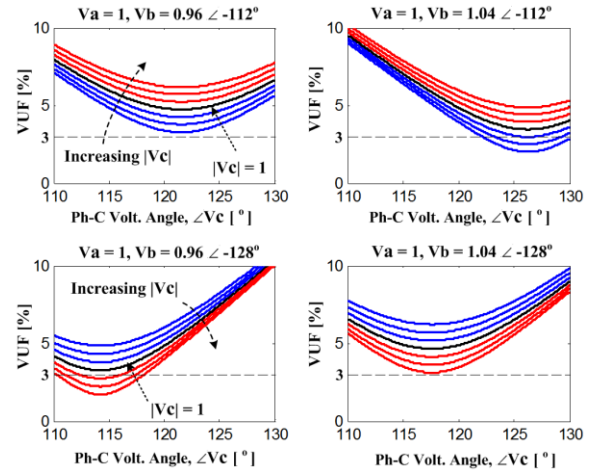

Fig. 2. VUF variations with respect to the voltage magnitudes and angles

Fig. 2 shows example variations in the VUF as the angles and magnitudes of the 3-*ph* voltages V_a, V_b, and V_c are varied. It can be seen that the VUF is a convex function of the voltage magnitudes and angles and, consequently, of the active and reactive power drawn from the network. Fig. 3(a) shows that the maximum and average values of the VUFs at the 3-*ph* buses increased gradually as the total load demand increased. This is because the unbalanced current flows increased with the

unbalanced load, leading to larger unbalanced voltage drops. Note that the *x*-axis in Fig. 3 is represented using the ratio r of ΔP_L and ΔQ_L to P_L and Q_L, respectively. In addition, Figs. 3(b) and (c) show VUF^{max} and VUF^{avg} for the load variations in phases B and C, respectively, at 10 arbitrarily selected buses, while keeping the load demands in the other phases fixed. This implies that the VUFs in the test network can be minimized in practice by optimizing the unbalanced power inputs of the 1-*ph* loads, rather than by simply reducing all their power inputs.
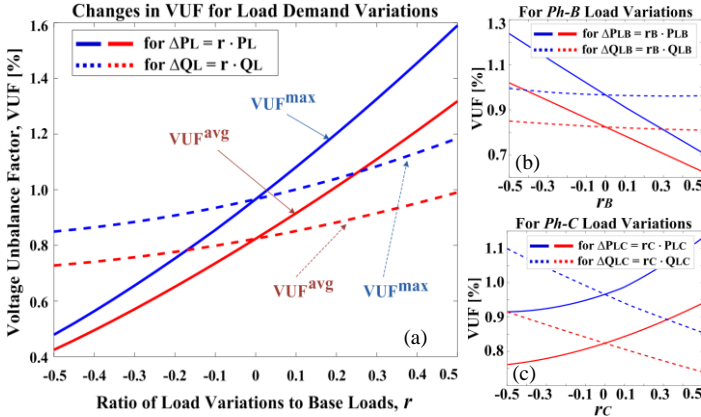

Fig. 3. Variations in the VUF with respect to the changes in active and reactive load demands: (lines) ΔP_L and (dotted lines) ΔQ_L.

Fig. 4 shows a comparison of the VUFs estimated using (12) and a BFS power flow calculation algorithm [5] (i.e., $\mathbf{VUF}_{(12)}$ and $\mathbf{VUF}_{BFS}$, respectively) for the case of a 10% increase in the total load demand. Figs. 4(a) and (b) show the VUFs at all the 3-*ph* buses for ΔP_L and ΔQ_L, respectively. It can be seen that the $\Delta\mathbf{VUF}$ estimates calculated using (12) are reasonably accurate.
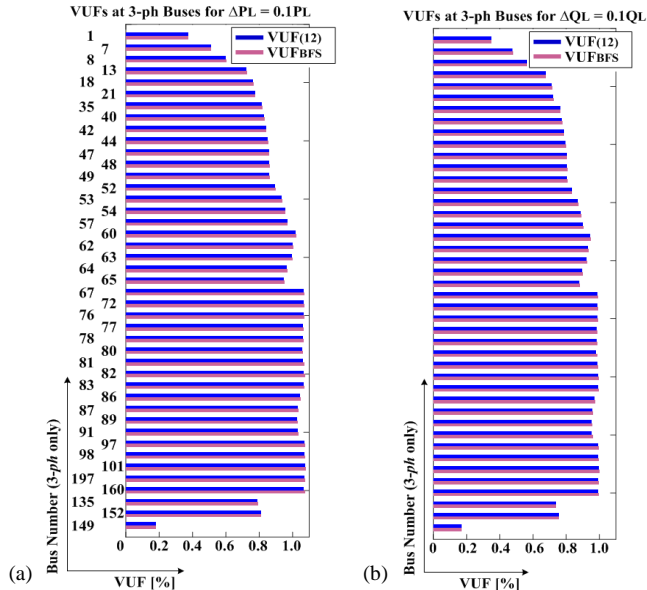

Fig. 4. Comparisons of VUFs at the 3-*ph* buses obtained using (12) and a BFS power flow algorithm for load variations: (a) $\Delta P_L=0.1P_L$ and (b) $\Delta Q_L = 0.1Q_L$

In addition, Fig. 5(a) shows the maximum and average errors in the VUF estimates (i.e., ε^{max} and ε^{avg}, respectively) at the 3-*ph* buses for various values of ΔP_L and ΔQ_L. The errors were calculated as shown in (13) and (14). In Fig. 5, ε^{max} remains less than 2% when P_L and Q_L decrease by approximately 33.4% and 39.8%, respectively, and when P_L and Q_L increase by approximately 22.5% and 24.4%, respectively. Moreover, Figs.

5(b) and (c) represent ε^{max} and ε^{avg} in the VUF estimates for the load variations shown in Figs. 3(b) and (c), respectively. In both cases, ε^{max} remains less than 3% for 30% load variations, verifying the accuracy of the VUF sensitivity matrix.

As the main aim of this letter is to develop and validate $\mathbf{J}_{EF}^{VUF}$ as presented in (12), only constant power loads were considered in the case studies for simplicity. The proposed VUF sensitivity matrix also can be applied to ZIP coefficients models of load devices, resulting in steady-state VUF variations and estimation errors similar to those shown in Figs. 3 and 5, respectively.

$$\varepsilon^{max} = \left| \mathbf{VUF}_{(12)}^{max} - \mathbf{VUF}_{BFS}^{max} \right| \Big/ \mathbf{VUF}_{BFS}^{max} \quad , \quad (13)$$

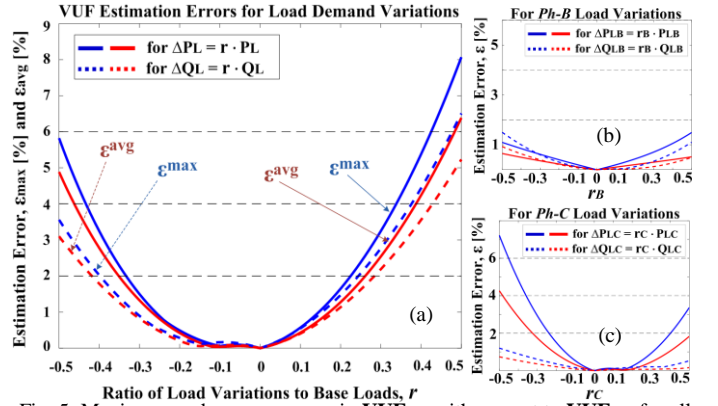$$\varepsilon^{avg} = \left| \mathbf{VUF}_{(12)}^{avg} - \mathbf{VUF}_{BFS}^{avg} \right| \Big/ \mathbf{VUF}_{BFS}^{avg} \quad . \quad (14)$$

Fig. 5. Maximum and average errors in $\mathbf{VUF}_{(12)}$ with respect to $\mathbf{VUF}_{BFS}$ for all 3-*ph* buses as a function of ΔP_L (lines) and ΔQ_L (dotted lines)

IV. CONCLUSIONS

In this paper, a sensitivity matrix for VUFs was developed with respect to variations in the unbalanced power inputs of 1-*ph* and 3-*ph* loads. Case studies demonstrated the accuracy of the VUF estimation over a wide range of power variations. This shows that the VUF sensitivity matrix can be incorporated into linear optimization problems for optimal operation of DERs and loads in unbalanced distribution networks, considering the improvement in the network power quality. The simulations were run using MATLAB on a four-core, 3.5 GHz CPU, and took insignificant lengths of time to run.

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