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Development and application of cartesian tensor mathematics for kinematic analysis of spatial mechanisms

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DEVELOPMENT AND APPLICATION OF CARTESIAN TENSOR MATHEMATICS

FOR KINEMATIC ANALYSIS OF SPATIAL MECHANISMS

by

ROBERT MYRL CRANE, 1941

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ABSTRACT

The complexity of spatial mechanisms in themselves and the absence of an attractive analytical tool for their study has left this field of engineering analysis largely unexplored. In recent years several analytic methods have emerged. One of the most attractive of these is the tensor method. Literature surveys reveal that the tensor method is largely unexploited in the U.S.A., with regard to spatial mechanisms as well as simpler kinematic problems.

The purpose of this work is to develop tensor mathematics for application to the kinematic analysis of spatial mechanisms. Methods are developed for position solutions and the determination of velocities and accelerations of points of interest. Included are tensor methods for obtaining angular velocities and accelerations as well as the formulae for treating moving coordinate frames. Iterative procedures are discussed for cases where a closed form solution is not possible. Sufficient applications are included to exemplify the methods developed including some which are numerically solved by computer.

It is concluded that the methods developed represent a cogent and tractable method of analysis of kinematic problems.

ACKNOWLEDGEMENT

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I

INTRODUCTION

A. Historical Background

A study of the history of man from any academic point of view
invariably marks the debut of man as a tool maker as the onset of
important changes in the evolutionary order. From these earliest
times man has recognized the efficacy of tools and their import upon
his existence. The later development of what we have come to call
machines began with the fashioning of tools. They were developed to
ease the burden of man's struggle with the environment, to save him
time, and to provide his livelihood. These are precisely the reasons,
that today man is yet building newer, more advanced, and more sophis-
ticated machines and the study of things mechanical has come to hold
a prominent position in the ranking of scientific pursuits.

It was not until relatively recent times that detailed scien-
tific analysis became necessary for the designing and building of
machines. Indeed most of the basic machines and mechanisms in use
today were first conceived of and utilized without the use of even
the most basic mathematics. Machines were fashioned in an intuitive
and empirical way to fill a need. They were revised and tested until
they satisfactorily performed the function for which they were in-
tended. Man has always had this power of intuitive reasoning, it
seems, and for thousands of years this method of design and inven-
tion was adequate. Even today this method of design is sometimes
the most practical although more scientific methods have been devel-
oped.

It is this quality more than any other, the propensity for empirical but sophisticated mechanical inventiveness, that is responsible for the slow growth of mechanism analysis as a scientific discipline. While other sciences flourished, the study of machines lanquished simply because its development was not needed. It is believed that the first published recognition of this void in the scientific disciplines was made by Ampère in 1834. [1]

> "There exist certain considerations which, if sufficiently developed, would constitute a complete science, but which have hitherto been neglected, or have formed only the subject of memoires and special essays. This science ought to include all that can be said with respect to motion in its different kinds, independently of the forces by which it is produced. ... It should treat, in the first place of spaces passed over, and of times employed in different motions, and of the determination of velocities according to the different relations which may exist between those spaces and times".

Among the many who have contributed to the field since that time are such notables as Franz Reuleaux (1829-1905) and Johann Bernoulli. Leonhard Euler (1707-1783) is generally credited with the division of the broad conception of machine analysis into the mechanical view, mechanics, and the geometrical view, kinematics. [2]

Although the science of kinematics had its beginnings over a hundred years ago, it is yet somewhat of a random art. The notion that the concept of kinematics is as old as the mechanisms it seeks to analyze is a widespread one. The fact is that kinematic analysis is relatively new as a scientific discipline. Many methods of analysis and systems of notations have been developed for the purpose of kinematic analysis. Perhaps the best known of these are vector analysis, graphics, and complex numbers. Vector analysis seems particu-

larly applicable in that almost all kinematic quantities are either vectors or magnitudes of vectors. However, vector notation can become cumbersome and difficult to work with for complex mechanisms. Graphic methods eliminate much computation and are suited for visual comprehension of a problem but are difficult to apply to three-dimensional cases. Complex mathematics can be used successfully for two-dimensional problems and has been extended to three-dimensions [3] but seems best suited for the two-dimensional case.

Until recently, the analysis of three-dimensional mechanisms has not occupied a prominent place in the work of kinematicians. Their reluctance to study these mechanisms was perhaps due to the apparently formidable and tedious task of mathematically formulating problems and obtaining solutions with existing methods of analysis. The task of modern kinematics was well stated by Uicker [2].

> "The problem is not to search for new principles, which would revolutionize the field of design: it is to try to find a better, a more extensive, a more universal method of analyzing mechanisms which have been known for ages."

B. Review of Current Literature

In an effort toward that goal, a universal method of analysis, recent publications have applied various tools to kinematic studies of mechanisms. Most notably in the field of three-dimensional mechanisms in an attempt to simplify the computational process and make the undertaking of work in this area more attractive.

Matrix methods utilizing an iterative method based on 4x4 matrixes have been developed by Hartenberg, Denavit, and Uicker [4,5, 6,7]. The method provides solutions to a large category of mechanisms but allows little interpretation of the matrix equations and is more

powerful than necessary for simpler linkages.

Vector analysis has been extended to three-dimensional mecha-
nisms by Chace [8,9] and represents a comprehensive exploitation of
the inherent applicability of the vector method to kinematic analysis.
Among others who have used the vector approach are Beyer and Harris-
berger [10,11]. Yang and Freudenstein have applied dual quaternious
[12] and kinematicians in the USSR, for example, Mangeron and Dregan
[13], and Kalitsin applied tensor analysis. There are many others
who have contributed to the study of three-dimensional mechanisms
besides those mentioned here. The authors and methods mentioned are
indicative of the many avenues of approach to the problem of three-
dimensional mechanisms and the lack of emergence of a single most
advantageous method.

C. Purpose of the Research

The authors interest in spatial mechanisms was stimulated by
exposure to the problems of teaching vector kinematics and the early
work in tensor kinematics by Professor C.Y. Ho [14]. Preliminary
investigations into current literature pointed out the diversity of
analytical methods being employed and the lack of acceptance of a
universal language for kinematic analysis. In particular, the absence
of the application of tensor mathematics in the U.S.A. indicated a
void that would merit investigation.

Tensor calculus first came into prominence as a device well
suited for dealing with the general theory of relativity. It origi-
nated as a consequence of the fact that physical laws must be inde-
pendent of any particular coordinate system used in describing them

mathematically [15]. The advantages of tensor mathematics are well recognized with regard to advanced theories but little interest has been displayed in its application to more elementary subjects.

It is perhaps the formidable character of most of the formulae of general relativity that is the source of the awe that many hold for tensor analysis, and their reluctance to study it. What is not realized is that this formidable character is removed when the formulae of tensors are referred to cartesian axes while the simplicity and conciseness of the tensor notation is retained.

Previous publications by the author and Professor C.Y. Ho have shown that tensor notation provides a convenient and compact means for expressing relationships in three-dimensional (spatial) mechanisms. Some tensor operations that have no counterpart in vector algebra are powerful aids to obtaining problem solutions. The tensor analysis contains the inherent naturalness of vector analysis but in a more tractable form for complex mechanisms. One of the newest and most important tools for use in kinematic analysis is the digital computer. A useful kinematic method of analysis must be compatible with the language of computer analysis. Vector equations do not directly lend themselves to computer programming whereas tensor equations are written in algorithm form and are readily translated into Fortran language.

The purpose of the research then was to investigate the development of tensor mathematics for clear application to kinematic analysis of spatial mechanisms.

D. Kinematic Definitions

The terms used in reference to things mechanical like mechanical things themselves are ones which are more or less intuitively understood. Terms like machine, mechanism and kinematics were used in the introduction without definition but without loss of understanding. However, from an analytical point of view, it is necessary to more rigorously define some of these terms in order to form a common basis from which to proceed.

The term kinematics is broadly used as a title to a division of the general applied science of the theory of machines. This division is the study of geometry in motion. More specifically, it is the study of position, geometry, displacement, rotation, speed, velocity, and acceleration [16]. The concept of force and motion resulting from the action of forces is not considered in kinematics.

A mechanism is a set of machine elements (links, gears, joints, pulleys, etc.) constructed so as to produce a desired output motion when it is driven by a particular input motion. That is, it transforms one kind of motion into another. One element of a mechanism is considered to be the base, or ground link, and all motions are viewed with respect to this link as a reference. Within the scope of this work, the definition of a mechanism can be further qualified in that the concepts of elasticity, bending, manufacturing tolerance, etc. are neglected. For purposes of analytical studies, a mechanism possesses perfect geometry and perfect rigidity. A useful definition of a mechanism then is: a set of kinematic links connected by kinematic pairs (joints) forming a closed kinematic chain, the whole having one degree of freedom when one link is viewed as the ground frame.

This definition of a mechanism clearly includes a broad category of devices but these can be further classified into two divisions, planar and spatial mechanisms. Planar mechanisms are characterized by motion in a single plane. All the variables and parameters necessary to mathematically describe the mechanism may be measured in a single projection, the plane of the motion. An exception to this is the method of defining angular velocities and accelerations in vector mathematics. These quantities are characteristically described as a vector having direction perpendicular to the plane of motion. A piston and crank device is an example of a planar mechanism.

The second basic category of mechanisms are those whose motions cannot be described in a single projection. These have three-dimensional motion and hence are called spatial mechanisms. The complexity of their motion has always made them difficult to analyze with existing methods. Graphical methods, at best, demand a high degree of skill in visualizing proper projections and some of the more recent mathematical methods can become quite involved in specialized mathematics and cumbersome notational difficulties. It is hoped that the compactness and brevity of the tensor method may alleviate some of these problems and make the study of spatial mechanisms more attractive.

There were two terms in the definition of a mechanism that require further definition. These were kinematic link and kinematic pair. Figure 1 is a pictorial representation of a spatial mechanism and its component parts, kinematic links and kinematic pairs. Kinematic links are the machine members comprising a mechanism whose func-

FIGURE 1

PICTORIAL REPRESENTATION OF A SPATIAL MECHANISM

AND ITS COMPONENT PARTS, KINEMATIC LINKS AND

KINEMATIC PAIRS.

Figure 1
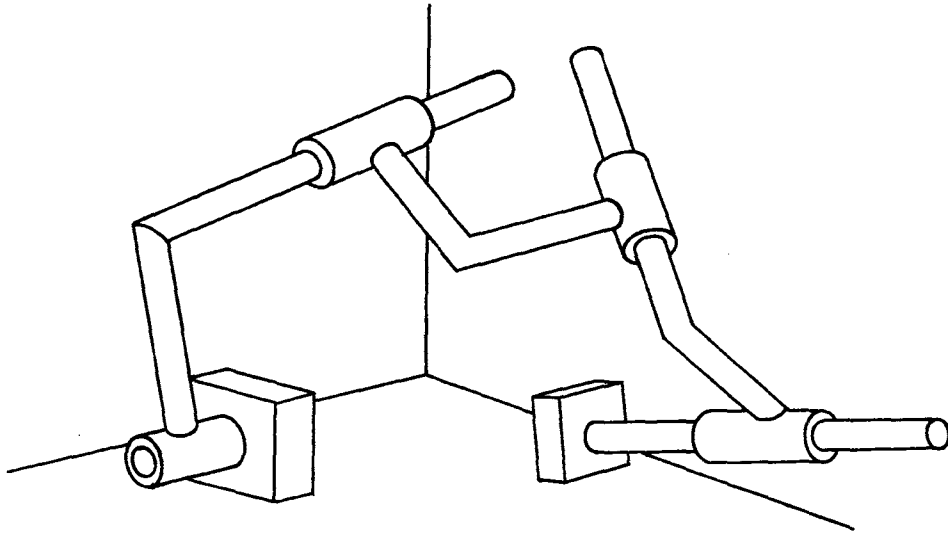
RCCC SPATIAL MECHANISM
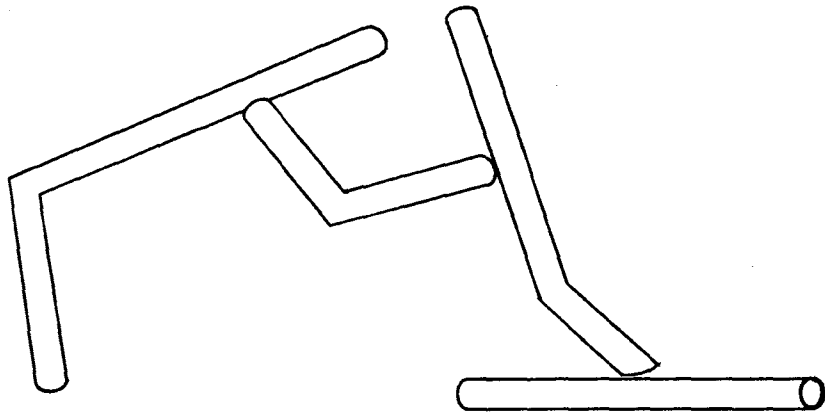
KINEMATIC LINKS
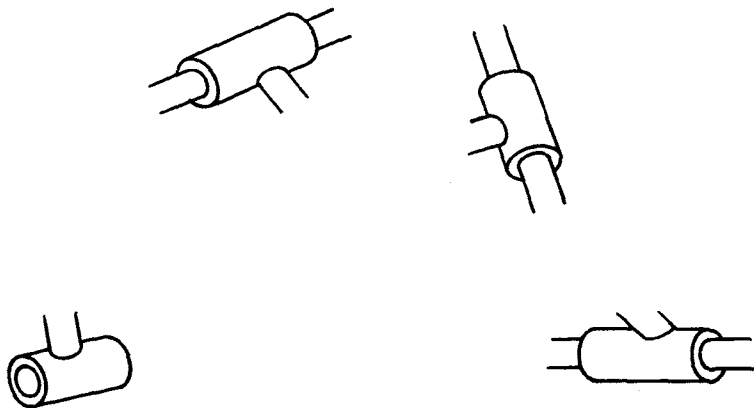
KINEMATIC PAIRS

tion it is to hold specific spatial relationships among the kinematic pairs. They are considered to be perfectly rigid and their shape is incidental to the study of the mechanism itself. In order to perform its function, a link must make contact with the elements of at least two joints although this does not preclude the possibility of contact with more than two joints. Hence, links are accordingly described as binary, ternary, quaternary, etc. Within the scope of this work, binary links will be exclusively used. A kinematic link, then, may be said to be; a rigid body containing elements of at least two kinematic pairs whose function is to maintain a specific spatial relationship between those respective pairs for the purpose of transmitting and transforming motion.

In order to perform their purpose, kinematic links must be connected by movable joints. These are traditionally called kinematic pairs. The purpose of a kinematic pair is to restrict the relative motion between connected links to a certain predetermined kind. A kinematic pair definition, then is; a movable joint whose function and purpose is to provide a connection between kinematic links which limits the relative motion between those links to a certain type. A common hinge joint, for example, limits link motion to a revolution about a common axis.

Kinematic pairs exist in diverse shapes and forms. Their physical appearance is often little clue as to their function. They can be further classified into three categories given the names lower pairs, higher pairs, and wrapping connectors. The latter are systems of belts and pulleys and are not of much interest in current thinking on spatial mechanisms. Lower pairs are the most common and most

interesting. These are six in number and were identified by Reuleaux [2]. They are shown pictorially in Figure 2.

A revolute pair (R pair) permits rotation about one axis. This pair is often called a hinge joint. To describe the relative motion between links connected by a revolute joint only one variable is needed, the angle of rotation. Hence, the revolute joint has one degree of freedom. These variables which describe relative motion permitted by the various pairs are called pair variables. It is customary to label each type of lower pair by a symbolic letter. These vary among authors on the subject and the ones to be used here are shown in Table I along with the degrees of freedom and pair variables for the lower pairs.

A prismatic pair (P pair) permits translation along a straight line. One pair variable is needed and the P pair then has one degree of freedom. A screw pair (H pair) permits helical motion involving both rotation and translation. Because the translation is related to the rotation by the pitch of the screw, the screw pair has only one degree of freedom and only one pair variable is necessary, either describing the translation or rotation but not both. A cylindrical pair (C pair) permits rotation about and translation along one axis. It, therefore, has two degrees of freedom and two pair variables are required. A spherical pair (S pair) permits three independent rotations about a point, has three degrees of freedom and three pair variables. This pair is often referred to as a ball joint and sometimes a globular joint. A planar pair (F pair) permits motion in a plane. There are two translational degrees of freedom and one rotational requiring three pair variables for specification. In usage, the most

FIGURE 2

PICTORIAL REPRESENTATION OF THE LOWER PAIRS

Figure 2
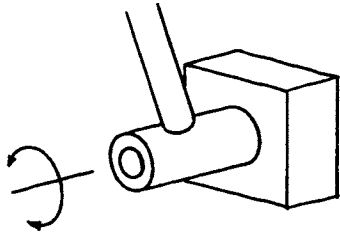
REVOLUTE PAIR
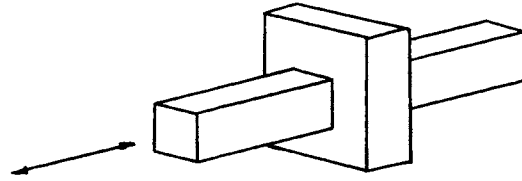
PRISMATIC PAIR
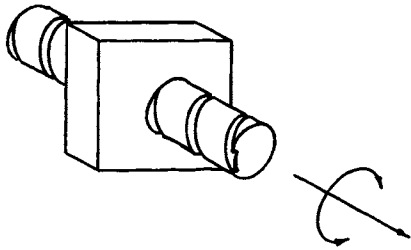
SCREW PAIR
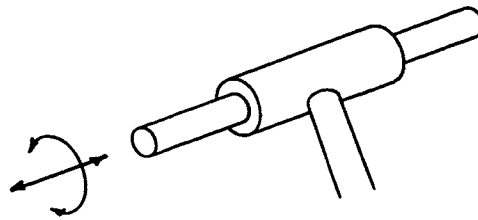
CYLINDRIC PAIR
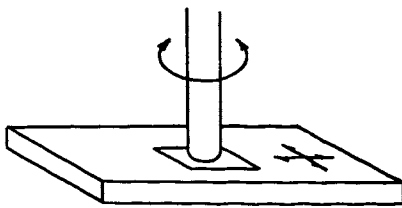
PLANAR PAIR
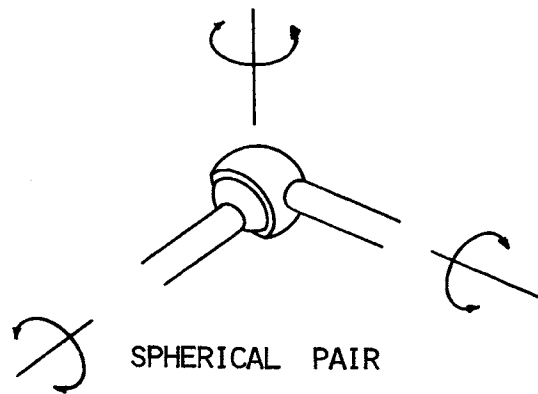
SPHERICAL PAIR

TABLE I

Classification and Nomenclature of the Lower Pairs

Pair Name	Symbol	Degrees of Freedom	Pair Variables
Revolute	R	1	θ
Prismatic	P	1	x
Helical(screw)	H	1	x or θ
Cylindrical	C	2	x, θ
Spherical	S	3	ϕ, θ, ψ
Planar	F	3	x, y, ϕ

common of these six lower pairs are the revolute, cylindrical, and spherical pairs. The prismatic and helical pairs are seen occasionally but the planar pair is rare.

In addition to their easily recognized pair variables, the lower pairs have other characteristics in common. In each case one element of a lower pair surrounds the other and is said to be self-connecting or have form closure and the connections between elements of lower pairs are surfaces (area contact) [16].

Connections between links which are not surfaces but lines or points are named higher pairs by Reuleaux [16]. Meshed gear teeth and ball and race contact are examples of higher pairs. Although potentially there are an infinite number of higher pairs, their appearance in the literature as related to spatial mechanisms is rare. The lower pairs only will be considered in this work.

Having formed working definitions of a mechanism and its component parts, the next step is to consider how the various elements may be combined to form mechanisms that are useful. That is, what are the possibilities for connecting binary links and lower pairs in a closed kinematic chain such that the result will satisfy the definition of a mechanism? To be useful, such a device must have one resultant degree of freedom. One input motion should be sufficient to determine the output motion.

Clearly, if one begins to arbitrarily connect various combinations of links and pairs to form closed chains, there will result a very large number of devices indeed. However, obviously not all will have the desired one degree of freedom. Some will have multiple degrees and some will be locked, not move at all.

Using what is known as the Gruebler or Kutzbach mobility criterion or Gruebler-Kutzbach criterion, depending upon the form in which it is written [11], Harrisberger has examined a large number of possibilities for spatial mechanisms. The mobility criterion determines the number of degrees of freedom of a mechanism by assuming six rigid body degrees of freedom for each link in the mechanism except the ground link which has none, and then subtracting the constraints imposed by the pairs. Harrisberger found 417 different kinds of spatial mechanisms having one degree of freedom [17]. Many of these were too mechanically complex to be of practical use. Of the many types investigated, Harrisberger found the four-link mechanisms to be of special appeal. He found 138 kinds of four-link mechanisms nine of which he deemed to be the most practical due to their desirable input-output motions. Five of these will be analyzed here as examples of the method to be developed. These five mechanisms are shown pictorially in Figure 3. It is customary to label a spatial mechanism by writing the symbols for its joints in successive order as they occur beginning at the input link. Thus, a mechanism with a revolute input connected to a spherical pair, another spherical pair and a revolute output would be labeled a RSSR mechanism.

It is a well known and curious fact that there exist several working mechanisms which do not satisfy the mobility criteria. Among the better known of these are the Bennett four-link RRRR mechanism and the Bricard six-link RRRRRR mechanism. These are shown in Figure 4. Another useful mechanism which does not satisfy the mobility criterion is the RSSR mechanism shown in Figure 5.

To gain further insight into this anomaly, it is helpful to

FIGURE 3

PICTORIAL   REPRESENTATION   OF   FOUR-BAR

SPATIAL   MECHANISMS

Figure 3

RCCC
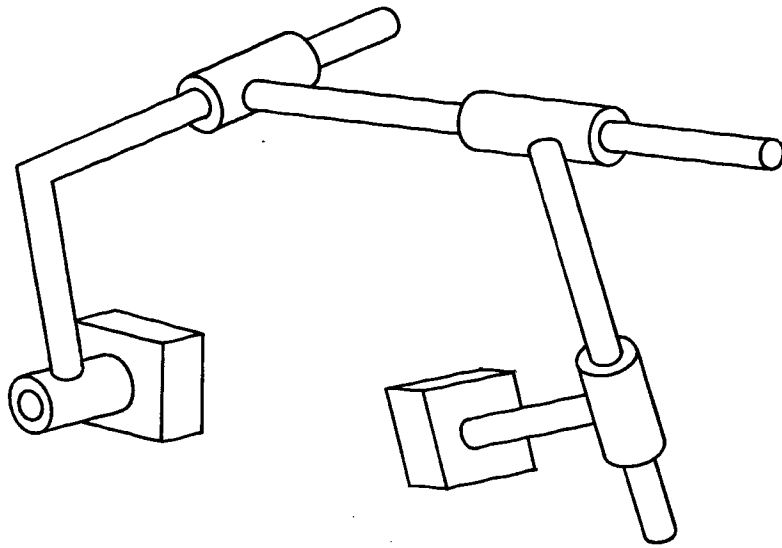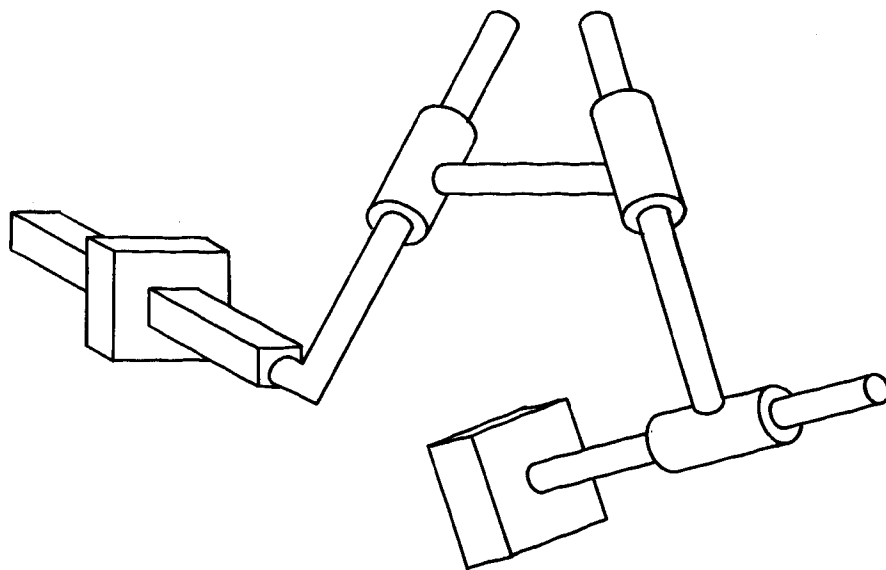
PCCC
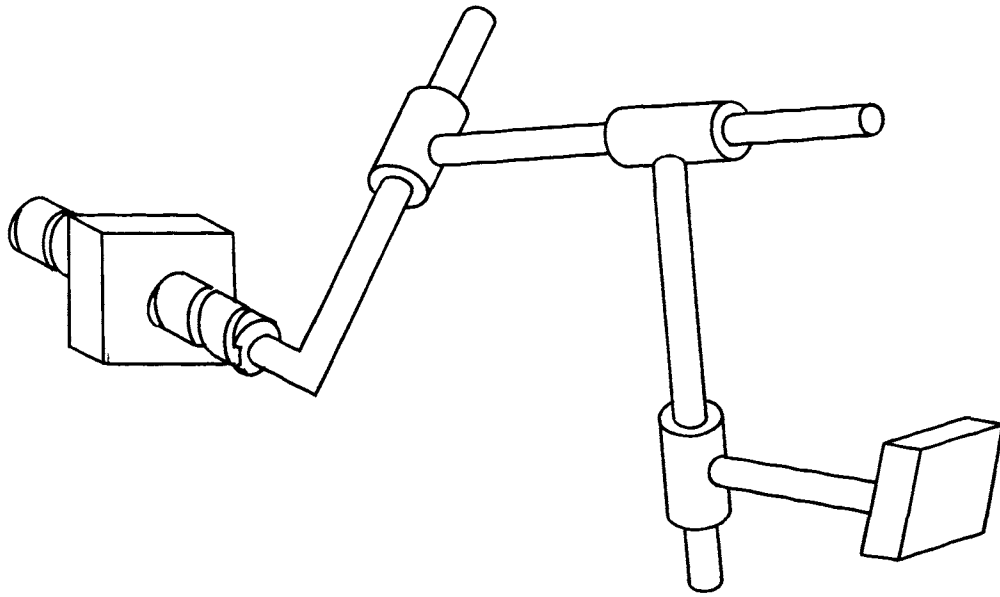
Figure 3 Cont.

HCCC
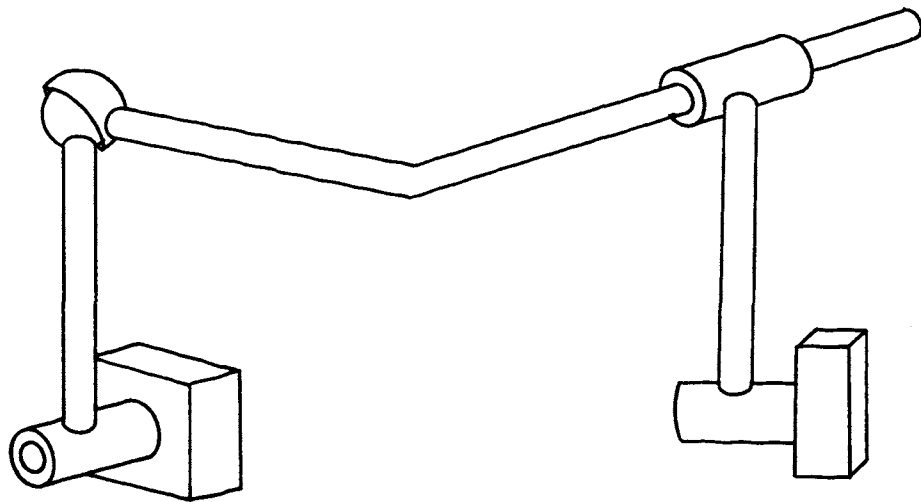
RSCR

Figure 3 Cont.
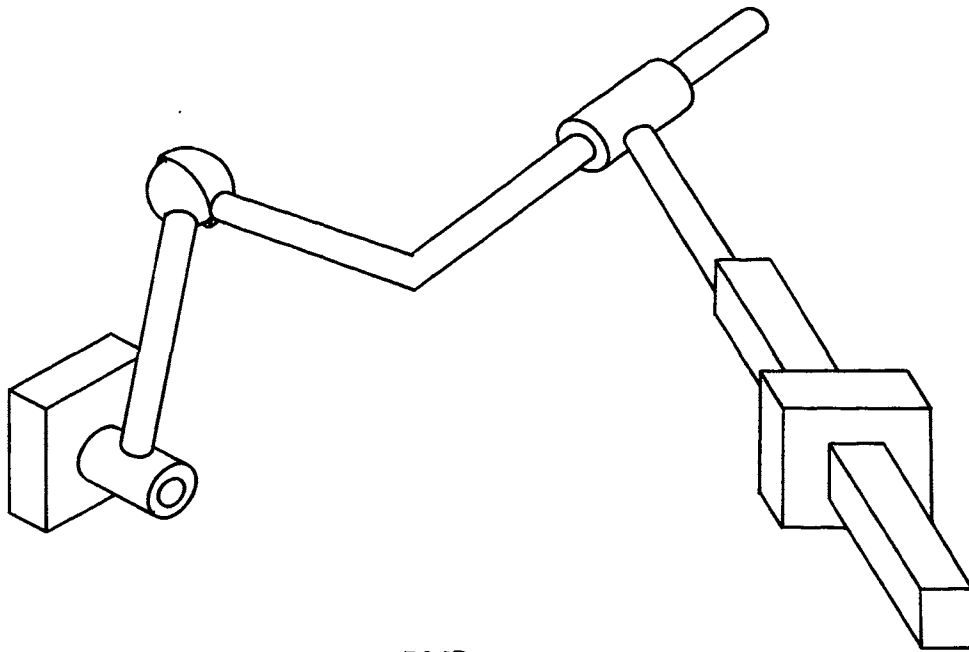
RSCP

FIGURE 4

PICTORIAL REPRESENTATION OF THE BENNETT FOUR-BAR AND

THE BRICARD SIX-BAR SPATIAL MECHANISMS

Figure 4
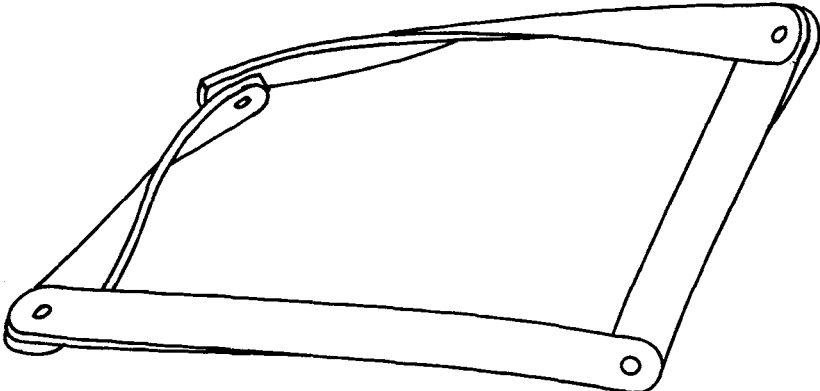
BENNETT FOUR-BAR
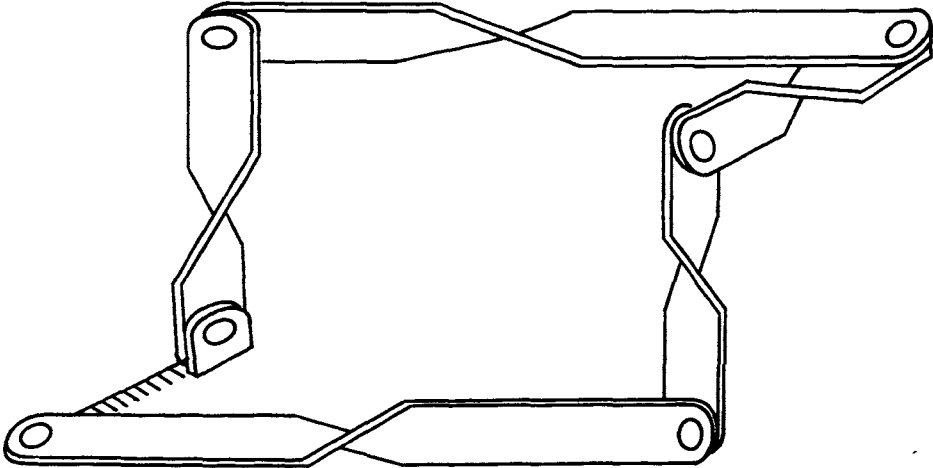
BRICARD SIX-BAR

FIGURE 5

RSSR MECHANISM - NOTE THE PASSIVE DEGREE OF

FREEDOM IN THE CONNECTING LINK

Figure 5
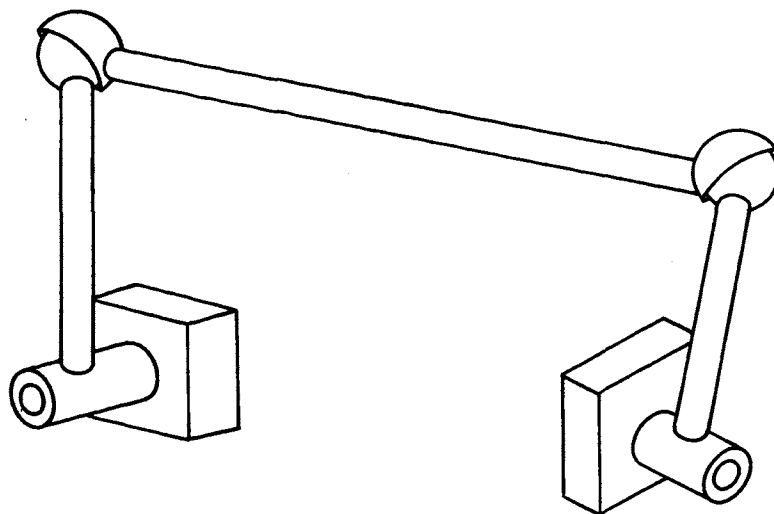
RSSR MECHANISM

mathematically demonstrate the mobility criteria. A collection of n links in space would have 6n degrees of freedom. If one link is fixed as the ground link, 6(n-1) degrees of freedom remain. Joining these links with kinematic pairs would impose additional constraints. These constraints are found by considering the number of degrees of freedom a pair would have if it were a rigid body in space, 6, and subtracting the number of degrees of freedom, it possesses due to its own peculiar motion when viewed as a joint. The relation may be written:

$$f = 6(n-1) - \sum_{i=1}^{m} C_i \qquad\qquad (I-1)$$

where f is the resultant number of degrees of freedom, n is the number of links, m is the number of pairs, and C_i is the number of constraints imposed by the ith pair. For example, an R pair has one degree of freedom so that C_i for an R pair is 5.

Application of the criterion to a mechanism which results in an f of 0 or a negative number would seem to indicate that the mechanism was over constrained and would not move at all. However, applying the criterion to the Bennett mechanism results in f = -2 and to the Bricard, f = 0. An f of greater than one would seem to indicate that the mechanism was under constrained and more than one input motion would be necessary to effect a single output motion. However, the RSSR mechanism of Figure 5 yields f = 2. It may be surmised that these anomalous mechanisms possess redundant constraints and passive degrees of freedom. Referring to Figure 5, a passive degree of freedom may be observed in the link connecting the two spherical pairs in that the link may rotate about its longitudinal axis without affecting the motion of the mechanism.

II

TENSOR MATHEMATICS FOR KINEMATICS

A. Definitions and Terminology

The use of cartesian axes results in two great simplifications
in the powerful but formidable tensor calculus. The distinction be-
tween covariant and contravariant vectors disappears and the terms
arising from the curvature of surfaces of reference are no longer
present. Absolute differentiation of cartesian tensors is equivalent
to ordinary differentiation. Some tensor properties that have no
counterpart in vector mathematics are powerful aids to problem solv-
ing. Ordinarily the use of subscripts and superscripts in tensor no-
tation is to distinguish the covariant and contravariant qualities of
tensors. Since that distinction is no longer necessary, when tensors
are referred to cartesian axes, we are free to retain the subscripts
for indication of tensor character and utilize superscripts for label-
ing purposes.

It is difficult to succinctly define what tensors are, as they
are rigorously defined only by their properties. A loose definition
might be that a tensor is an abstraction that contains an ordered set
of elements or components, the properties of which, taken together
are independent of the coordinate frame used to describe them. It is
this quality that makes tensors an ideal instrument for the study of
physical laws and when the simplifications for cartesian axes are
introduced, they become an exceptionally useful tool for the study
of spatial relationships. One great advantage to the use of the ten-
sor method to be presented is that it is not a new notation but a con-
cise way of writing the ordinary vector notation with the additional

benefits of the brevity of tensor operations.

1. Cartesian Tensors

A point in three-dimensional space located with respect to a cartesian coordinate frame (X_1, X_2, X_3) by a set of three coordinates may be also located with respect to another cartesian coordinate frame (X_1', X_2', X_3') having the same origin by another set of three coordinates. The coordinates of the point p in the unprimed system may be called (p_1, p_2, p_3), and the coordinates of p in the primed system (p_1', p_2', p_3'). The coordinates of p in the prime frame may be written in terms of the coordinates in the unprimed frame by the relations

$$p_1' = A_{11} p_1 + A_{12} p_2 + A_{13} p_3$$

$$p_2' = A_{21} p_1 + A_{22} p_2 + A_{23} p_3 \qquad \text{(II-1)}.$$

$$p_3' = A_{31} p_1 + A_{32} p_2 + A_{33} p_3$$

The quantities $(A_{11}, A_{12}, \cdots, A_{33})$ are the cosines of the angles between the various axes; for example, A_{11} is the cosine of the angle between X_1' and X_1, A_{23} is the cosine of the angle between X_2' and X_3, and so on. Also, the coordinates of p in the unprimed frame may be expressed in terms of those in the primed frame by the relations

$$p_1 = A_{11} p_1' + A_{21} p_2' + A_{31} p_3'$$

$$p_2 = A_{12} p_1' + A_{22} p_2' + A_{32} p_3'$$

$$p_3 = A_{13}p_1' + A_{23}p_3' + A_{33}p_3' \qquad \text{(II-2).}$$

The writing of equations (II-1) and (II-2) can be considerably shortened by a change in notation. We introduce the range and summation conventions.

a. Range Convention

For the coordinates of the point p we write p_i and p_i' where the index i is understood to take, in turn, each value in the range of that index. In three-dimensional space the range is 3 so that an unrepeated index will always be 1, 2, 3. Thus, the term p_i represents the three coordinates (p_1, p_2, p_3) and p_i' represents (p_1', p_2', p_3'). Equations (II-1) and (II-2) may then be written

$$p_i' = A_{i1}p_1 + A_{i2}p_1 + A_{i3}p_3 \qquad \text{(II-3)}$$

$$p_i = A_{1i}p_1' + A_{2i}p_2' + A_{3i}p_3' \qquad \text{(II-4).}$$

Each of these equations can be expressed as a summation.

$$p_i' = \sum_{j=1}^{3} A_{ij}p_j \qquad \text{(II-5)}$$

$$p_i = \sum_{j=1}^{3} A_{ji}p_j' \qquad \text{(II-6).}$$

b. Summation Convention

If we adopt the convention that when an index is repeated in a term, as is j in equation (II-5) and (II-6), that a summation over the range of that index is implied, we may write

$$p_i^{'} = A_{ij} p_j \tag{II-7}$$

$$\text{and} \quad p_i = A_{ji} p_j^{'} \tag{II-8}.$$

Thus, equations (II-7) and (II-8) each represent three equations and are completely equivalent to equations (II-1) and (II-2) but the twelve terms in each are now compactly expressed in the tensor notation. The range and summation convention will be implicitly present in the notation used henceforth.

Sets of three quantities such as p_i and $p_i^{'}$ which satisfy equations (II-7) and (II-8) are called tensors of the first order, or vectors. The individual p_1, p_2, p_3 are called the components of the tensors. It can be seen that a first order cartesian tensor is equivalent to a cartesian vector.

There are, of course, tensors of other orders than one. A tensor of order zero is a scalar and has the same value for all sets of axes. A tensor of second order can be constructed by the product of two vectors.

$$r_i^{'} s_j^{'} = t_{ij}^{'} \quad \text{or} \quad r_i s_j = t_{ij} \tag{II-9}$$

and from equation (II-7)

$$t_{ij}^{'} = (A_{ij} r_k)(A_{jm} s_m) \tag{II-10},$$

$$\text{or} \quad t_{ij}^{'} = A_{ik} A_{jm} r_k s_m \tag{II-10},$$

then from equation (II-9)

$$t'_{ij} = A_{ij}A_{jm}t_{km} \qquad\qquad (II-11).$$

The tensor t'_{ij} is of second order, formed by the product of two first order tensors, and transforms according to the rule (II-11). In general, a set of nine quantities ω_{ij} referred to a coordinate system and transforming to another system by the rule (II-11) is a second order tensor.* There are many quantities that satisfy these conditions besides the product of two vectors. Tensors of higher order can be constructed and defined. In general, a tensor of order n has n indices, $t_{ijk \ldots n}$, and transforms according to the rule

$$t'_{ijk \ldots n} = A_{ia}A_{jb}A_{ki} \cdots A_{ne}t_{ab \ldots e} \qquad (II-12).$$

c. Symmetric and Skew-Symmetric Tensors

A tensor t_{ij} is said to be symmetric in the indices ij if upon interchange of these indices

$$t_{ij} = t_{ji} \qquad\qquad (II-13).$$

A tensor s_{ij} is said to be skew-symmetric in the indices ij if

$$s_{ij} = -s_{ji} \qquad\qquad (II-14).$$

If s_{ij} is a skew-symmetric tensor then

$$s_{11} = 0, \; s_{22} = 0, \; s_{33} = 0 \qquad (II-15).$$

*Much of the development in this section is adapted from references [15, 18, and 19]. The reader is referred to these for more detailed study.

Clearly, the product of a symmetric tensor and a skew-symmetric tensor is zero

$$t_{ij}s_{ij} \equiv 0 \qquad \text{(II-16)}.$$

The product of first order tensors

$$r_i r_j = p_{ij} \qquad \text{(II-17)}$$

is a symmetric second order tensor, for

$$r_i r_j = r_j r_i \qquad \text{(II-18)}.$$

Equation (II-17) is called the symmetric product, has no counterpart in vector mathematics, and is of great importance in tensor kinematics.

d. The Kronecker Delta

Consideration of the gradient operator in tensor form leads to some important relations. The gradient operator is written $\dfrac{\partial}{\partial X_i}$ where X_i are the cartesian axis considered as vectors. The gradient of a scalar results in a vector for

$$\partial \phi / \partial X_i' = (\partial X_k / \partial X_i') \ (\partial \phi / \partial X_k) = A_{ik} \ (\partial \phi / \partial X_k) \qquad \text{(II-19)}.$$

which is a transformation according to the vector rule, hence the gradient of a scalar is a vector. In tensor mathematics, the gradient of a vector has meaning and is a second order tensor. There is no counterpart in vector mathematics.

$$\partial r_i'/\partial X_k' = (\partial X_m/\partial X_k')(\partial r_i'/\partial X_m) = A_{km}(\partial/\partial X_m)A_{ij}r_j \qquad \text{(II-20)},$$

then $\qquad \partial r_i'/\partial X_k' = A_{km}A_{ij}(\partial r_j/\partial X_m) \qquad \text{(II-21)}.$

Equation (II-21) satisfies the transformation rule for second order tensors, hence the gradient of a vector is a second order tensor.

Since the axes X_i may be considered as vectors, then $\partial X_i/\partial X_k$ is a tensor of the second order. But clearly

$$\partial X_i/\partial X_k = \delta_{ki} \qquad \text{(II-22)}$$

where δ_{ki} is the Kronecker delta, and

$$\delta_{ki} = \begin{array}{l} 0 \text{ if } i \neq j \\[6pt] 1 \text{ if } i = j \end{array} \qquad \text{(II-23)}.$$

Hence, the Kronecker delta is a tensor of second order. Therefore, it transforms according to

$$\delta_{ij}' = A_{ik}A_{jn}\delta_{kn} \qquad \text{(II-24)}.$$

Setting k = n we have

$$\delta_{ij}' = A_{ik}A_{jk} \qquad \text{(II-25)},$$

which confirms the orthogonality of the coordinate axis. It can be shown that

$$A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij} \qquad \text{(II-26)}.$$

Clearly, the Kronecker delta is a symmetric tensor. It can be shown that it is also isotropic. That is, its components retain the same values in any coordinate system.

e. The Permutational Tensor

The permutational tensor, ε_{ijk}, is defined by the conditions on its components

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ if any two indices have the same value} \\[2mm] 1 \text{ if the values of the indices ijk represent} \\ \text{an even permutation of the sequences } 1,2,3. \\[2mm] -1 \text{ if the values of ijk represent an odd permutation of the sequence } 1,2,3. \end{cases}$$

A permutation of the sequence 1,2, ..., n is even if an even number of interchanges of adjacent integers is required to attain the permutation. Similarly, a permutation is odd if an odd number of interchanges is required. Thus,

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1, \text{ and } \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1.$$

It can be shown that the permutation tensor is a third order, completely skew-symmetric, isotropic tensor.

An important relation between δ_{ij} and ε_{ijk} is given by

$$\varepsilon_{ijk} \, \varepsilon_{mpj} = \delta_{im} \, \delta_{kp} - \delta_{ip} \, \delta_{km} \qquad \text{(II-27).}$$

f. The Duality Property

It can be shown that for any vector there is associated a skew-symmetric tensor of second order. The converse is also true. Multiply the vector r_n by ε_{ijk}. This is a tensor of fourth order. We can form a second order tensor by contracting, let $n = k$.

$$\varepsilon_{ijk} \, r_k = w_{ij} \qquad \text{(II-28)}$$

Now form the vector u_k by multiplying the second order tensor w_{mn} by ε_{ijk} and contracting twice.

$$\varepsilon_{ijk} \, w_{ij} = u_k \qquad \text{(II-29)}.$$

If w_{ij} is a symmetric tensor, the left side of equation (II-29) is zero. If w_{ij} is skew-symmetric the left side is non-zero and expansion of equation (II-29) leads to the conclusion that the components of u_k are numerically twice those of w_{ij}. Therefore, we may write the relations

$$w_k = \tfrac{1}{2}\varepsilon_{ijk} w_{ij} \qquad \text{(II-30)},$$

and $$w_{ij} = \varepsilon_{ijk} w_k \qquad \text{(II-31)},$$

where w_k is the vector uniquely associated with the skew-symmetric second order tensor w_{ij}. The consequences of the duality property of vectors with skew-symmetric, second order, tensors in three-dimensional space will have important applications in the development of angular velocity relations.

2. Tensor-Vector Correspondence

The various formulae of vector analyses are easily reconstructed using the tensor notation. The following is a condensed but fairly complete account of the vector analysis by the tensor method. The method clarifys some of the ambiguities of the vector relations. For example, the emergence of vector and scalar products.

a. Vector

As previously defined, a vector in tensor notation is written as a letter with a single subscript.

$$\vec{t} = t_i \tag{II-32}$$

b. Multiplication of a vector $\vec{t}$ by a scalar ϕ.

$$\phi \vec{t} = \phi t_i \tag{II-33}$$

c. Addition and subtraction of vectors $\vec{t}$ and $\vec{s}$

$$\vec{t} + \vec{s} = t_i + s_i \tag{II-34}$$

$$\vec{t} - \vec{s} = t_i - s_i \tag{II-35}$$

d. Scalar product of vectors $\vec{t}$ and $\vec{s}$

$$\vec{t} \cdot \vec{s} = t_i s_i \tag{II-36}$$

e. Vector product of vectors $\vec{t}$ and $\vec{s}$

$$\vec{t} \times \vec{s} = \varepsilon_{ijk} t_j s_k \tag{II-37}$$

Note that the ordering of the indices in equation (II-37) must be preserved in order to conform to the right hand rule (sign convention) when forming cross products.

f. <u>Second order tensor product of two vectors $\vec{t}$ and $\vec{s}$</u>

$$t_i s_j = p_{ij} \tag{II-38}$$

This product becomes a second order tensor; vector notation fails to define this quantity.

g. <u>Triple scalar product</u>

$$\vec{p} \cdot (\vec{q} \times \vec{r}) = \varepsilon_{ijk} p_i q_j r_k \tag{II-39}$$

h. <u>Triple vector product</u>

$$\vec{p} \times (\vec{q} \times \vec{r}) = \varepsilon_{ijk} p_j \varepsilon_{klm} q_l r_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} p_j q_l r_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) p_j q_l r_m$$

$$= p_m r_m q_i - p_l q_l r_i \tag{II-40}$$

The correspondence between vector and tensor notations are summarized in Table II.

TABLE II

Common Formulae of Vector Analysis in Tensor Notation

Operation	Vector Notation	Tensor Notation
Denotation	$\vec{t}$	t_i
Addition and	$\vec{p} = \vec{t} + \vec{s}$	$p_i = t_i + s_i$
Subtraction	$\vec{q} = \vec{t} - \vec{s}$	$q_i = t_i - s_i$
Multiplication by a Scalar	$\vec{r} = \phi\vec{t}$	$r_i = \phi t_i$
Scalar Product	$\phi = \vec{t} \cdot \vec{s}$	$\phi = t_i s_i$
Vector Product	$\vec{m} = \vec{t} \times \vec{s}$	$m_i = \varepsilon_{ijk} t_j s_k$
Tensor Product		$n_{ij} = t_i s_j$
Triple Scalar Product	$\phi = \vec{p} \cdot \vec{m}$ $= \vec{p} \cdot (\vec{t} \times \vec{s})$	$\phi = p_i m_i$ $= p_i \varepsilon_{ijk} t_j s_k$
Triple Vector Product	$\vec{l} = \vec{p} \times \vec{m}$ $= \vec{p} \times (\vec{t} \times \vec{s})$	$l_i = \varepsilon_{ijk} p_j m_k$ $= p_j t_i s_j - p_j t_j s_i$

III

POSITION ANALYSIS

A. Problem Formulation

The most difficult problem in the study of spatial mechanisms is the position solution. Position solutions invariably result in non-linear transcendental equations. Velocity and acceleration problems are characteristically linear and present no difficulties beyond their often cumbersome length. The use of the computer alleviates this difficulty to a great degree as lengthy equations can be defined in pieces using implicit notation and later pieced together explicitly by the computer.

Sets of simultaneous, non-linear, transcendental equations resulting from position analysis yield readily to simpler iterative techniques on the computer. Much work has been done resolving such equations into polynomials for which solution techniques are well known [9]. This approach leads to a polynomial in one unknown but one of higher degree than the number of unknowns in the problem. Some complex problems do not allow the reduction of equations to a single polynomial but rather simultaneous polynomials with additional computational difficulties. The present approach is to simultaneously solve the system of non-linear equations by well known iterative techniques for a particular set of system parameters. This results in one set of solutions for a particular case of interest. The particulars of this statement will become clear in the example solutions given.

1. Notation and Terminology

The notation problem that arises when physical problems require
the concept of three-dimensional space is at best difficult. Various
schemes have been devised which have particular advantages and dis-
advantages. It is felt that the flexibility of the tensor notation
provides a solution which alleviates many of the difficulties.

It will be found convenient to express vectors as a product of
magnitude and a unit vector defining the direction of the vector.
For example,

$$R_i = R r_i \tag{III-1}$$

where R is the magnitude of the vector R_i and r_i is a unit vector
describing the direction of the R_i vector in a particular coordinate
frame. The three axes of a coordinate frame are labeled X_1, X_2, X_3.
In general, when a vector is not implied, we may speak of a frame as
simply the X_i frame. The notation X_i (no superscript) is reserved
for the ground frame, the frame used as a reference for all other
considerations. If another frame is required, different from the
ground frame, a superscript is used to distinguish it from the ground
frame. For example,

$$X_i^r = X_1^r, \; X_2^r, \; X_3^r \tag{III-2}$$

identifies a frame, called the r frame, which has in general a dif-
ferent orientation than the ground frame. When it becomes necessary
to treat an axis of a coordinate frame as a vector, it is understood
to be a unit vector and a subscript must be added to indicate the
vector character.

$$(X_1^r)_i = (X_1^r)_1, \ (X_1^r)_2, \ (X_1^r)_3 \qquad \text{(III-3)}$$

Superscripts are also used with vectors to indicate in which coordinate frame the components are expressed. The equation

$$r_i^t = r_1^t, \ r_2^t, \ r_3^t \qquad \text{(III-4)},$$

indicates the r vector expressed in the t frame. That is, the components of r_i^t are perpendicular projections on the axes of the t frame. The absence of a superscript on a vector indicates that that vector is expressed in the ground frame. The components of the $(X_1^r)_i$ vector in equation (III-3), as given are the components of the X_1^r axis treated as a unit vector and expressed in the ground frame. The expression $(X_1^r)_i^r$ is the same vector but expressed in the r frame and has components in that frame.

$$(X_1^r)_i^r = 1,0,0 \qquad \text{(III-5)}.$$

When more than a single coordinate frame transformation is involved in a problem, the transformation coefficients must be labeled. Superscripts are again used. The transformation

$$r_i^t = A_{ij}^{tx} r_j \qquad \text{(III-6)},$$

indicates that the vector r_j (expressed in the ground frame) has been transformed to the t frame by the transformation relation for vectors. The coefficients A_{ij}^{tx} are the transformation coefficients between the x frame and the t frame. The resultant vector r_i^t is expressed in the t frame. Note that the ordering of indices in equation (III-6)

is established by convention and must be preserved. The inverse

transformation is

$$r_i = A_{ji}^{tx} r_j^t \qquad\qquad \text{(III-7)}.$$

Spherical polar coordinates are used throughout to define the

cartesian components of vectors. Two angles and a magnitude are re-

quired to define a vector with respect to a particular coordinate

frame. The symbols ϕ and θ will be used for the polar and azimuthal

angles throughout. Figure 6 shows the conventional polar coordinate

used. Superscripts are again used to define which vector these angles

describe and which coordinate frame they are measured in. That is,

ϕ^{rt} and θ^{rt} are the polar and azimuthal angles of the r vector and

are measured from the X_3^t and X_1^t axes respectively.

It is important to note that r_i and r_i^t are the same vector in

space (a fact denoted by the letter r) but are expressed in different

coordinate frames, the X_i and X_i^t frames respectively. In vector anal-

ysis this is accomplished by writing each component as a product of

its coordinate magnitude and a unit vector along its axis. In tensor

notation a single superscript serves the same purpose. Equation

(III-7) does not imply a relation between two vectors, but a relation

between the components of the same vector as they appear in two

different coordinate frames.

The nomenclature is summarized in Table III. Many mechanism

problems allow expression and solution of all vectors in a single

frame, the ground frame with perhaps one coordinate transformation.

For these cases, the notation is more detailed than necessary but

allows for expansion to more complex cases.

FIGURE 6

NOMENCLATURE FOR POLAR COORDINATE VECTOR REPRESENTATION

COORDINATES OF THE VECTOR ARE SHOWN IN THE X_i FRAME AS r_i. THE SAME VECTOR IS SHOWN IN THE X_i^t FRAME AS r_i^t.

IF THE TRANSFORMATION IS DEFINED, THE RELATION BETWEEN COMPONENTS OF r_i AND r_i^t MAY BE EXPRESSED AS

$$r_i = A_{ji}^{tx} r_j^t$$

OR

$$r_i^t = A_{ij}^{tx} r_j$$

Figure 6
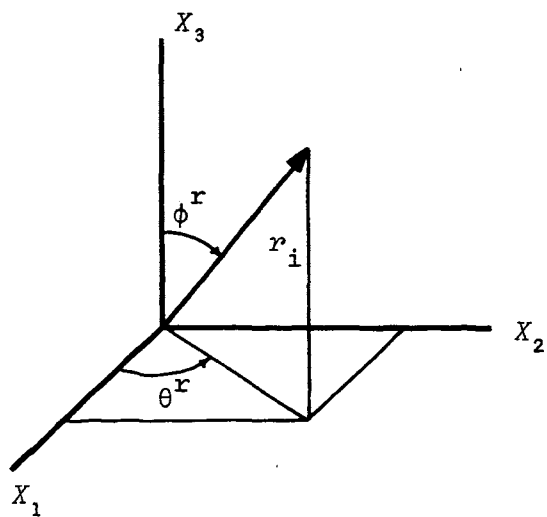
$$r_i = \sin\phi^r \cos\theta^r, \ \sin\phi^r \sin\theta^r, \ \cos\phi^r$$
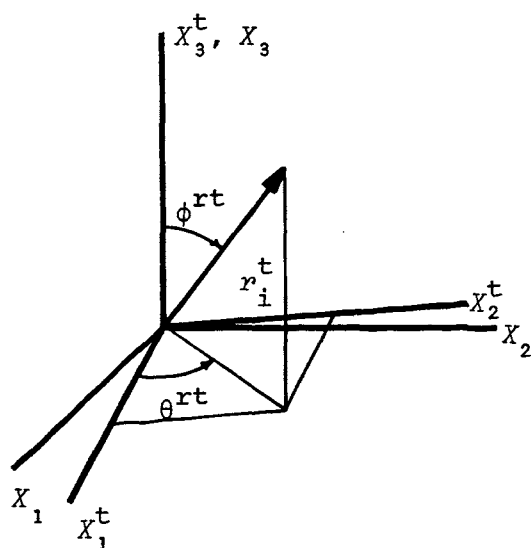
$$r_i^t = \sin\phi^{rt} \cos\theta^{rt}, \ \sin\phi^{rt} \sin\theta^{rt}, \ \cos\phi^{rt}$$

TABLE III

Summary of Tensor Nomenclature

1. Unit vector r_i

Unit vectors are indicated by lower case letters with a subscript. Absence of a superscript indicates the vector is expressed in the ground frame. A vector of unit magnitude $r_i r_i = 1$

2. Magnitude R

Magnitudes are indicated by upper case letters. A scalar quantity - no super or subscripts. Magnitude of a vector is always positive; if a solution for a magnitude yields a negative number, the associated unit vector is reversed.

3. Vector $R_i = Rr_i$

Vectors are written as a product of magnitude and a unit vector defining its direction.

4. Vector expressed in other than ground frame Rr_i^t

This is the same vector as Rr_i but is now expressed in the t^1 frame. May be obtained by transformation or empirically written.

5. Transformation Coefficients A_{ij}^{rx}

A set of nine quantities (direction cosines) relating the two coordinate frames indicated by the superscripts.

6. Coordinate Frame X_i^r

Three mutually perpendicular axes having a common origin. Right handed coordinate frames are used exclusively. Superscripts distinguish frames other than the ground frame.

TABLE III (cont.)

7. Coordinate frame
   axis

   $(X_2^r)_i^r$

   Parenthesis are used when
   it is necessary to treat an
   axis as a unit vector. As
   written this vector has com-
   ponents 0, 1, 0.

   $(X_2^r)_i$

   The same vector but now ex-
   pressed in the ground frame.

8. Spherical
   Coordinates

   R, ϕ^r, θ^r

   Polar and azimuthal angles
   used to define orientation
   of a vector in a frame and
   a magnitude to define its
   length. As shown, these
   angles describe the r_i vec-
   tor and are measured from
   the ground frame axes.

   R, ϕ^{rt}, θ^{rt}

   Describe the r_i vector but
   are measured from the axes
   of the t frame.

9. Components in
   Spherical
   Coordinates

   $r_1 = \sin\phi^r \cos\theta^r$

   $r_2 = \sin\phi^r \sin\theta^r$

   $r_3 = \cos\phi^r$

   Perpendicular projections
   of the vector r_i on the axes
   of the ground frame.

   $r_1^t = \sin\phi^{rt} \cos\theta^{rt}$

   $r_2^t = \sin\phi^{rt} \sin\theta^{rt}$

   $r_3^t = \cos\phi^{rt}$

   Perpendicular projections
   of the vector r_i on the
   axes of the t frame.

2. The Tetrahedron Problem

The position of a spatial mechanism is given by a vector loop equation.

$$Cc_i + Rr_i + Ss_i + Tt_i + \ldots + Nn_i = 0 \qquad \text{(III-8)}$$

that is, its position at any instant of time can be represented by equation (III-8) where the $C, R, S, T, \ldots, N$ are the lengths of the various links and the c_i, r_i, s_i, etc. are unit vectors defining the orientation of the links. The vector tetrahedron problem is formed when all vectors except three are completely known and are summed into a single constant vector Cc_i. Equation (III-8) then becomes

$$Cc_i + Rr_i + Ss_i + Tt_i = 0 \qquad \text{(III-9)}.$$

Figure 7 is a pictorial representation of a vector loop defining the position of a space mechanism. In three-dimensional space, equation (III-9) represents three equations and can be solved for three scalar unknowns.

These unknowns may be randomly distributed throughout the three vectors Rr_i, Ss_i, and Tt_i. Thus, the possible unknowns are any three of the nine quantities R, ϕ^r, Θ^r, S, ϕ^s, Θ^s, T, ϕ^t, Θ^t. Examination of the possibilities shows that there are just nine combinations of unknowns that result in different solutions. Chace [9] has classified the nine cases and solved them by vector methods. He classified them by the distribution of the unknowns, whether they occurred in one, two or three vectors. His solutions were accomplished by reduction to a polynomial in cases where explicit solutions were not pos-

FIGURE 7

A RSSP SPATIAL MECHANISM AND THE VECTOR LOOP

EQUIVALENT DEFINING THE POSITION OF THE MECHANISM

Figure 7
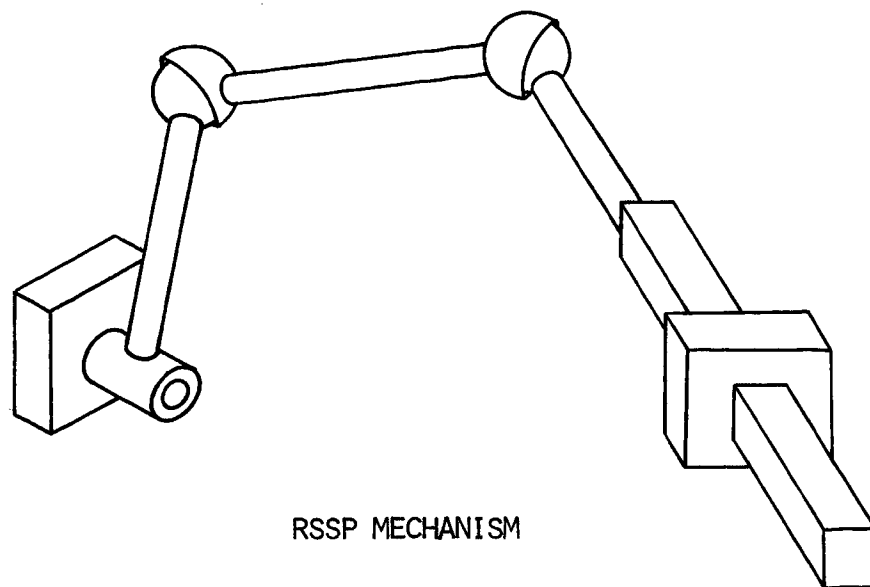
RSSP MECHANISM
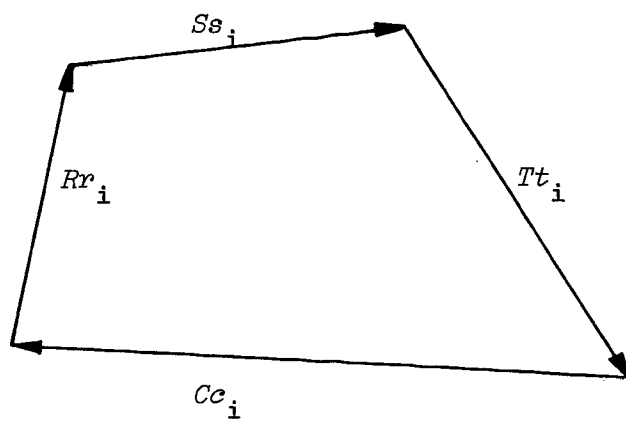
VECTOR LOOP EQUIVALENT

$$Cc_i + Rr_i + Ss_i + Tt_i = 0$$

sible by vector algebra. Classification of the nine tetrahedron problems is shown in Table IV after the system used by Chace. The four solutions of case 3 have been presented by tensor methods by Ho [14]. The complete set of all nine solutions will be presented here to provide insight into the tensor method and to accomplish the completion of the vector tetrahedron solution in tensor form.

Case 1. unknown R, θ^r, ϕ^r

known Cc_i

The unknowns occur in a single vector Rr_i. The other vectors are therefore known and are summed into the single known vector Cc_i. Equation (III-9) becomes

$$Cc_i + Rr_i = 0 \qquad\qquad (III-10).$$

The solution is trivial $\quad Rr_i = -Cc_i \qquad\qquad (III-11).$

Cases 2a-2b.

The unknowns are contained in the vectors Rr_i and Ss_i. The vector Tt_i is known and is summed into the Cc_i vector. Equation (III-9) for cases 2a-2b becomes

$$Cc_i + Rr_i + Ss_i = 0 \qquad\qquad (III-12)$$

Case 2a. unknown R, θ^r, S

known Cc_i, s_i, ϕ^r

Vector loop equation $Cc_i + Rr_i + Ss_i = 0 \qquad\qquad (III-13)$

TABLE IV

Classification of the Solutions to the Vector

Tetrahedron Equation (after Chace)

Case Number	Unknown	Known	Possible Solutions
1	R, θ^r, ϕ^r	Cc_i	1 (trivial)
2a	R, θ^r, S	Cc_i, s_i, ϕ^r	2
2b	R, θ^r, θ^s	Cc_i, ϕ^r, S, ϕ^s	4
2c	θ^r, ϕ^r, S	Cc_i, s_i, R	2
2d	$\theta^r, \phi^r, \theta^s$	Cc_i, R, S, ϕ^s	2
3a	R, S, T	Cc_i, r_i, s_i, t_i	1
3b	R, S, θ^t	$Cc_i, r_i, s_i, t, \phi^t$	2
3c	R, θ^s, θ^t	$Cc_i, r_i, \phi^s, T, \phi^t$	4
3d	$\theta^r, \theta^s, \theta^t$	$Cc_i, R, \phi^r, S, \phi^s, T, \phi^t$	8

Solution: Multiply through equation (III-13) by the vector product

$$\varepsilon_{ijk} c_j s_k$$

$$C\varepsilon_{ijk} c_i c_j s_k + R\varepsilon_{ijk} r_i c_j s_k + S\varepsilon_{ijk} s_i c_j s_k = 0 \qquad \text{(III-14)}$$

Recall that the second order products $c_i c_j$ and $s_i s_k$ form symmetric tensors and that ε_{ijk} is a completely skew-symmetric tensor. Hence, the products in the first and third terms of equation (III-14) are identically zero.

$$R\varepsilon_{ijk} r_i c_j s_k = 0 \qquad \text{(III-15)}$$

Assuming $R \neq 0$ we have

$$\varepsilon_{ijk} r_i c_j s_k = 0 \qquad \text{(III-16)}.$$

Equation (III-16) is a scalar equation involving one unknown, Θ^r, and is of the form

$$a \cos\Theta^r + b \sin\Theta^r + c = 0 \qquad \text{(III-17)},$$

where a, b and c are known scalar constants.

Equation (III-17) may be solved for Θ^r by iteration techniques. There are two solutions but the choice of starting values (determined by a visual inspection of the values of the known parameters) is sufficient to insure convergence to the proper solution for the case of interest.

Equation (III-17) may be simplified by transforming the problem to a coordinate frame which has particular characteristics. Define the vector

$$u_i = \varepsilon_{ijk} c_j s_k \qquad\qquad \text{(III-18)}$$

The vector u_i is known being the vector product of the two known vectors c_i and s_i. Equation (III-16) may be written

$$u_i r_i = 0 \qquad\qquad \text{(III-19)}.$$

Define a coordinate frame, X_i^r, such that

$$(X_1^r)_i = \frac{\varepsilon_{ijk} u_j (X_3)_k}{\left| \varepsilon_{ijk} u_j (X_3)_k \right|} \qquad\qquad \text{(III-20)},$$

$$(X_2^r)_i = \frac{\varepsilon_{ijk} (X_3)_j (X_1^r)_k}{\left| \varepsilon_{ijk} (X_3)_j (X_1^r)_k \right|} \qquad\qquad \text{(III-21)}.$$

$$(X_3^r)_i = (X_3)_i \qquad\qquad \text{(III-22)}.$$

Equations (III-20,21,22) form the transformation coefficients, A_{ij}^{rx}. The transformation relations for the vectors r_i and u_i may be written

$$u_i = A_{ji}^{rx} u_j^r \qquad\qquad \text{(III-23)}$$

$$r_i = A_{ji}^{rx} r_j^r \qquad\qquad \text{(III-24)}$$

Substitute equations (III-22) and (III-23) into equation (III-18)

$$A_{ji}^{rx} A_{mi}^{rx} u_j^r r_m^r = 0 \qquad\qquad \text{(III-25)}.$$

Using the orthogonality property

$$\delta_{jm} u_j^r r_m^r = 0 \qquad\qquad \text{(III-26)},$$

$$u^r_j r^r_j = 0 \qquad\qquad \text{(III-27)}.$$

Since the $(X^r_1)_i$ axis is formed by the vector product of u_j and $(X_3)_k$ it follows that u_j is perpendicular to the $(X^r_1)_k$ axis. Hence, the first component of u_i in the X^r_i frame is zero.

$$u^r_1 = 0 \qquad\qquad \text{(III-28)}$$

Expand equation (III-27)

$$u^r_2 r^r_2 + u^r_3 r^r_3 = 0 \qquad\qquad \text{(III-29)}$$

Now the components of r^r_i are

$$r^r_i = \sin\phi^{rr}\cos\theta^{rr},\ \sin\phi^{rr}\sin\theta^{rr},\ \cos\phi^{rr} \qquad\qquad \text{(III-30)},$$

but the $(X^r_3)_i = (X_3)_i$, therefore $\phi^r = \phi^{rr}$.

then, $\qquad r^r_i = \sin\phi^r\cos\theta^{rr},\ \sin\phi^r\sin\theta^{rr},\ \cos\phi^r \qquad\qquad \text{(III-31)}.$

Expand equation (III-29) in terms of the unknown θ^{rr},

$$u^r_2 \sin\phi^r \sin\theta^{rr} + u^r_3\, r^r_3 = 0 \qquad\qquad \text{(III-32)}.$$

then, solving for θ^{rr}

$$\sin\theta^{rr} = \frac{-u^r_3\, r_3}{u^r_2 \sin\phi^r} \qquad\qquad \text{(III-33)}.$$

The vector r^r_i as defined by equation (III-30) is now known. The vector r_i may be obtained from equation (III-24).

Explicit expressions for the remaining unknowns R and S can now

be obtained. Multiply equation (III-13) by $\varepsilon_{ijk} s_j$

$$C\varepsilon_{ijk} c_i s_j + R\varepsilon_{ijk} r_i s_j = 0 \qquad \text{(III-34)}.$$

Solving for R,

$$R = \left| \frac{-C\varepsilon_{ijk} c_i s_j}{\varepsilon_{ijk} r_i s_j} \right| \qquad \text{(III-35)}.$$

Expand equation (III-13) in the first component

$$Cc_1 + Rr_1 + Ss_1 = 0 \qquad \text{(III-36)}.$$

Solving for S

$$S = \left| \frac{-(Cc_1 - Rr_1)}{s_1} \right| \qquad \text{(III-37)}.$$

Case 2b. Unknown: R, θ^r, θ^s

known: Cc_i, ϕ^r, ϕ^s, S

Vector loop equation: $Cc_i + Rr_i + Ss_i = 0 \qquad \text{(III-38)}$

Solution: Multiply through equation (III-38) by $\varepsilon_{ijk} r_j$ to eliminate the unknown R

$$C\varepsilon_{ijk} c_i r_j + S\varepsilon_{ijk} s_i r_j = 0 \qquad \text{(III-39)}$$

Expand equation (III-39) in the first and second indices,

$$\varepsilon_{ij1}(Cc_i r_j + Ss_i r_j) = 0 \qquad\qquad \text{(III-40)}.$$

$$\varepsilon_{ij2}(Cc_i r_j + Ss_i r_j) = 0$$

Equations (III-40) are of the form

$$a \sin\theta^s + b \sin\theta^r + c = 0 \qquad\qquad \text{(III-41)}.$$

$$a \cos\theta^s + b \cos\theta^r + d = 0$$

where

$$a = Sr_3 \qquad\qquad\qquad \text{(III-42)}$$

$$b = -(Cc_3 + Ss_3)$$

$$c = Cc_2 r_3$$

$$d = Cc_1 r_3$$

Equations (III-41) may be solved for the unknowns θ^r, and θ^s. The vectors r_i and s_i are then known. The remaining unknown, R, may be obtained by expansion of equation (III-38).

$$Cc_1 + Rr_1 + Ss_1 = 0 \qquad\qquad \text{(III-42)}$$

$$R = \left| \frac{-(Ss_1 + Cc_1)}{r_1} \right| \qquad\qquad \text{(III-43)}$$

<u>Case 2c.</u>   Unknown: θ^r, ϕ^r, S

known: Cc_i, R, s_i

Vector loop equation: $Cc_i + Rr_i + Ss_i = 0 \qquad\qquad \text{(III-44)}$

In this case the vector r_i is completely unknown. Isolate this vector on the left side of equation (III-44).

$$Rr_i = -Cc_i - Ss_i \tag{III-45}$$

Take the scalar product of each side of equation (III-45) by itself.

$$R^2 = S^2 + C^2 + 2SCs_ic_i \tag{III-46}$$

Equation (III-46) is quadratic in S with known coefficients.

Therefore,

$$S = -Cs_ic_i \pm \left[C^2 (s_ic_i)^2 - (C^2 - R^2) \right]^{1/2} \tag{III-47}.$$

The unknown vector r_i may be obtained from equation (III-44).

$$r_i = - \frac{Ss_i + Cc_i}{R} \tag{III-48}.$$

Case 2d.   Unknown: $\theta^r, \phi^r, \theta^s$

known: ϕ^s, Cc_i, R, S

Vector loop equation: $Cc_i + Rr_i + Ss_i = 0 \tag{III-49}$

Expand equation (III-49) in the third index

$$Cc_3 + Rr_3 + Ss_3 = 0 \tag{III-50}.$$

Equation (III-50) contains only one unknown, contained in the term r_3

$$r_3 = \frac{-Cc_3 - Ss_3}{R} = \cos\phi^r \tag{III-51}.$$

Isolate the vector Rr_i in the manner of case 2c.

$$Rr_i = -Ss_i - Cc_i \qquad \text{(III-52)}$$

Take the scalar product of each side of equation (III-52) with itself.

$$R^2 = S^2 + C^2 + 2SCs_i c_i \qquad \text{(III-53)}$$

then

$$s_i c_i = \frac{R^2 - S^2 - C^2}{2SC} \qquad \text{(III-54)}$$

Expanding and rearranging equation (III-54), we have

$$c_1 \sin\phi^S \cos\theta^S + c_2 \sin\phi^S \sin\theta^S = \frac{R^2 - S^2 - C^2}{2SC} - s_3 c_3 \qquad \text{(III-55).}$$

Equation (III-55) may be simplified in the manner of case 2a. Define the auxiliary coordinate frame X_i^S in the following manner

$$(X_1^S)_i = \frac{\varepsilon_{ijk} c_j (X_3)_k}{|\varepsilon_{ijk} c_j (X_3)_k|} \qquad \text{(III-56)}$$

$$(X_2^S)_i = \frac{\varepsilon_{ijk} (X_3)_j (X_1^S)_k}{|\varepsilon_{ijk} c_j (X_3)_k|} \qquad \text{(III-57)}$$

$$(X_3^S)_i = (X_3)_i \qquad \text{(III-58)}$$

Equations (III-56,57,58) form the transformation coefficient A_{ij}^{sx}. The vectors s_i and c_i may be written

$$s_i = A_{ji}^{sx} s_j^S \qquad \text{(III-59),}$$

$$c_i = A_{ji}^{sx} c_j^S \qquad \text{(III-60).}$$

Equation (III-54) may be written

$$s_i c_i = K \qquad\qquad\qquad \text{(III-61)},$$

where K is the known expression on the right side of equation (III-54). Substitute equations (III-59 and 60) into equation (III-61).

$$A^{sx}_{ji} s^S_j A^{sx}_{mi} c^S_m = K \qquad\qquad\qquad \text{(III-62)},$$

or $\qquad s^S_j c^S_j = K \qquad\qquad\qquad \text{(III-63)}.$

Expand equation (III-63)

$$s^S_1 c^S_1 + s^S_2 c^S_2 + s^S_3 c^S_3 = K \qquad\qquad\qquad \text{(III-64)}$$

Since the X^S_1 axis is formed by the vector product of c_i and $(X_3)_i$, and $(X_3)_i = (X^S_3)_i$, it follows that the c_i vector and the X^S_1 axis are perpendicular. Therefore, the first component of the c_i vector is zero in the X^S_i frame.

$$c^S_1 = 0 \qquad\qquad\qquad \text{(III-65)}$$

Equation (III-64) becomes

$$s^S_2 c^S_2 + s^S_3 c^S_3 = K \qquad\qquad\qquad \text{(III-66)}.$$

Expand equation (III-66) in terms of the unknowns

$$c^S_2 \sin\phi^{SS} \cos\theta^{SS} + s^S_3 c^S_3 = K \qquad\qquad\qquad \text{(III-67)}.$$

Since X^S_3 is defined to be equal to X_3 we have that $\phi^{SS} = \phi^S$. Therefore, the only unknown in equation (III-67) is θ^{SS}.

$$\cos\theta^{ss} = \frac{K - s_3^s c_3^s}{c_2^s \sin\phi^{ss}}$$ (III-68).

The vector s_i^s is now known. Equation (III-59) may be used to obtain the vector s_i. The remaining unknown, θ^r, may be obtained from the expansion of equation (III-49)

$$r_1 = \frac{-(Cc_1 + Ss_1)}{R}$$ (III-69),

then

$$\cos\theta^r = \frac{-(Cc_1 + Ss_1)}{R \sin\phi^r}$$ (III-70).

Cases 3a-3d   These cases are categorized in that the unknowns are distributed through all three vectors Rr_i, Ss_i, and Tt_i.

Case 3a.   Unknown: R, S, T

known: Cc_i, r_i, s_i, t_i

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-71)

Multiply equation (III-71) in turn by the vector products $\varepsilon_{ijk} s_j t_k$, $\varepsilon_{ijk} r_j t_k$, $\varepsilon_{ijk} r_j s_k$. There results

$$R = \left|\frac{Cc_{ijk} c_i s_j t_k}{\varepsilon_{ijk} r_i s_j t_k}\right|$$ (III-72)

$$S = \left|\frac{C\varepsilon_{ijk} c_i r_j t_k}{\varepsilon_{ijk} s_i r_j t_k}\right|$$ (III-73)

$$T = \left| \frac{C\varepsilon_{ijk}c_i r_j s_k}{\varepsilon_{ijk}t_i r_j s_k} \right| \qquad \text{(III-74)}$$

Case 3b. Unknown: R, S, θ^t

known: Cc_i, r_i, s_i, ϕ^t

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-75)

The unknowns R and S may be eliminated by multiplying equation (III-75) by the vector product $\varepsilon_{ijk}r_j s_k$. There results

$$C\varepsilon_{ijk}c_i r_j s_k + T\varepsilon_{ijk}t_i r_j s_k = 0 \qquad \text{(III-76)}.$$

Equation (III-76) contains only one unknown θ^t in the form of $\sin\theta^t$ and $\cos\theta^t$. This equation may be simplified by a coordinate transformation. Define the known vector W_i.

$$W_i = \varepsilon_{ijk}r_j s_k \qquad \text{(III-77)}.$$

Equation (III-76) may then be written

$$CW_i c_i + TW_i t_i = 0 \qquad \text{(III-78)}.$$

Now define the X_i^t coordinate frame such that

$$(X_1^t)_i = \frac{\varepsilon_{ijk}W_j (X_3)_k}{\left| \varepsilon_{ijk}W_j (X_3)_k \right|} \qquad \text{(III-79)}$$

$$(X_2^t)_i = \frac{\varepsilon_{ijk}(X_3)_j (X_1^t)_k}{\left| \varepsilon_{ijk}(X_3)_j (X_1^t)_k \right|} \qquad \text{(III-80)}$$

$$(X_3^t)_i = (X_3)_i \qquad \text{(III-81)}$$

Equations (III-79, 80, 81) form the transformation coefficients A_{ij}^{tx}. The vectors W_i and t_i may be written

$$W_i = A_{ji}^{tx} W_j^t \qquad \text{(III-82)}$$

$$t_i = A_{ji}^{tx} t_j^t \qquad \text{(III-83)}$$

Substitute equations (III-82) and (III-83) into the second term of equation (III-78)

$$CW_i c_i + TA_{ji}^{tx} A_{mi}^{tx} W_j^t t_m^t = 0 \qquad \text{(III-84)},$$

or $$CW_i c_i + TW_m^t t_m^t = 0 \qquad \text{(III-85)}.$$

Since the X_1^t axis is defined by the vector product of the vectors W_j and $(X_3)_k$ it follows that W_j is perpendicular to X_1^t and hence the first component of W_j^t is zero in the X_i^t frame,

$$W_1^t = 0 \qquad \text{(III-86)}.$$

Expand equation (III-85) in terms of the unknowns.

$$CW_i c_i + T(W_2^t \sin\phi^{tt} \sin\theta^{tt} + W_3^t t_3^t) = 0 \qquad \text{(III-87)}$$

Since $\phi^{tt} = \phi^t$ the only unknown in equation (III-87) is θ^{tt}.

$$\sin\theta^{tt} = \frac{-(CW_i c_i + TW_3^t t_3^t)}{TW_2^t \sin\phi^{tt}} \qquad \text{(III-88)}$$

The vector t_i^t is now known and t_i may be obtained from equation (III-83). Obtain the remaining unknowns R and S in the following manner. Multiply equation (III-75) by $\varepsilon_{ijk} s_j t_k$ and solve for R.

$$R = \left| \frac{-C\varepsilon_{ijk} c_i s_j t_k}{\varepsilon_{ijk} r_i s_j t_k} \right| \qquad \text{(III-89)}.$$

Expand equation (III-75) and solve for S

$$S = \left| \frac{-(Cc_1 + Rr_1 + Tt_1)}{s_1} \right| \qquad \text{(III-90)}.$$

Case 3c. Unknown: R, θ^s, θ^t

known: Cc_i, S, T, r_i, ϕ^s, ϕ^t

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-91)

Expanding equation (III-91) in the third component results in

$$R = \left| \frac{-(Ss_3 + Tt_3 + Cc_3)}{r_3} \right| \qquad \text{(III-92)}$$

Expand equation (III-91)

$$Cc_1 + Rr_1 + S\sin\phi^s \cos\theta^s + T\sin\phi^t \cos\theta^t = 0 \qquad \text{(III-93)},$$

$$Cc_2 + Rr_2 + S\sin\phi^s \sin\theta^s + T\sin\phi^t \sin\theta^t = 0 \qquad \text{(III-94)}.$$

Equations (III-93, 94) contain the unknowns θ^s and θ^t and may be solved by iteration.

Case 3d. Unknown: θ^r, θ^s, θ^t

known: Cc_i, R, S, T, ϕ^r, ϕ^s, ϕ^t

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-95)

The distribution of the unknowns in this problem is such that a solution cannot be obtained directly from equation (III-95). The expansion of the loop equation in the third index contains only a relation among the known parameters leaving the problem of two equations with three unknowns. However, a conceptual change in the formulation of the problem alleviates the difficulty. If we regard the problem as one with the azimuthal angles of three vectors unknown while the polar angles are known and allow that the direction (axis) from which these polar angles are measured may be different but known we can formulate the problem in the following manner.

Unknown: θ^{rr}, θ^{ss}, θ^{tt}

known: Cc_i, R, S, T, ϕ^{rr}, ϕ^{ss}, ϕ^{tt}

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-96)

Restated the problem is mathematically the same but now we are allowing the vectors r_i, s_i, t_i to be measured in a coordinate frame other than X_i, and whose third axis is known in the X_i frame. Thus, three auxiliary coordinate frames may be defined using the known axis vectors $(X_3^r)_i$, $(X_3^s)_i$, $(X_3^t)_i$ in the vector product manner of previous cases. Therefore, the A_{ij}^{rx}, A_{ij}^{sx}, A_{ij}^{tx} are known.

Restate equation (III-96) replacing r_i, s_i, and t_i with $A_{mi}^{rx}r_m^r$, $A_{mi}^{sx}s_m^r$, and $A_{mi}^{tx}t_m^t$ respectively.

$$Cc_i + RA_{mi}^{rx}r_m^r + SA_{mi}^{sx}s_m^s + TA_{mi}^{tx}t_m^t = 0 \qquad \text{(III-97)}$$

Multiply equation (III-97) by A_{3i}^{rx}, A_{3i}^{sx}, A_{3i}^{tx} and recall that

$$A_{3i}A_{mi} = \delta_{3m}$$

$$CA_{3i}^{rx}c_i + Rr_3^r + SA_{3i}^{rx}A_{mi}^{sx}s_m^s + TA_{3i}^{rx}A_{mi}^{tx}t_m^t = 0 \qquad \text{(III-98)}$$

$$CA_{3i}^{sx}c_i + RA_{3i}^{sx}A_{mi}^{rx}r_m^r + Ss_3^s + TA_{3i}^{sx}A_{mi}^{tx}t_m^t = 0 \qquad \text{(III-99)}$$

$$CA_{3i}^{tx}c_i + RA_{3i}^{tx}A_{mi}^{rx}r_m^r + SA_{3i}^{tx}A_{mi}^{sx}s_m^s + Tt_3^t = 0 \qquad \text{(III-100)}$$

Expanding and rearranging equations (III-98, 99, 100)

$$CA_{3i}^{rx}c_i + Rr_3^r + SA_{3i}^{rx}A_{3i}^{sx}s_3^s + TA_{3i}^{rx}A_{3i}^{tx}t_3^t$$

$$+ SA_{3i}^{rx}\sin\phi^{ss}(A_{1i}\cos\theta^{ss} + A_{2i}\sin\theta^{ss})$$

$$+ TA_{3i}^{rx}\sin\phi^{tt}(A_{1i}^{tx}\cos\theta^{tt} + A_{2i}^{tx}\sin\theta^{tt}) = 0 \qquad \text{(III-101)},$$

$$CA_{3i}^{sx} + Ss_3^s + RA_{3i}^{sx}A_{3i}^{rx}r_3^r + TA_{3i}^{sx}A_{3i}^{tx}t_3^t$$

$$+ RA_{3i}^{sx}\sin\phi^{rr}(A_{1i}^{rx}\cos\theta^{rr} + A_{2i}^{rx}\sin\theta^{rr})$$

$$+ TA_{3i}^{sx}\sin{}^{tt}(A_{1i}^{tx}\cos{}^{tt} + A_{2i}^{tx}\sin{}^{tt}) = 0 \qquad \text{(III-102)},$$

$$CA_{3i}^{tx}c_i + Tt_3^t + RA_{3i}^{tx}A_{3i}^{rx}r_3^r + SA_{3i}^{tx}A_{3i}^{sx}s_3^s$$

$$+ RA_{3i}^{tx}\sin\phi^{rr}(A_{1i}^{rx}\cos\theta^{rr} + A_{2i}^{rx}\sin\theta^{rr})$$

$$+SA_{3i}^{tx} \sin\phi^{ss}(A_{1i}^{sx} \cos\theta^{ss} + A_{2i}^{sx} \sin\theta^{ss}) = 0 \qquad \text{(III-103)}.$$

Equations (III-101, 102, 103) are of the form

$$a \cos\theta^{ss} + b \sin\theta^{ss} + c \cos\theta^{tt} + \sin\theta^{tt} = e \qquad \text{(III-104)},$$

$$a' \cos\theta^{rr} + b' \sin\theta^{rr} + c' \cos\theta^{tt} + d' \sin\theta^{tt} = e' \qquad \text{(III-105)},$$

$$a'' \cos\theta^{rr} + b'' \sin\theta^{rr} + c'' \cos\theta^{ss} + d'' \sin\theta^{ss} = e' \text{(III-106)}.$$

The invariants a,b,c,d,e and their primed and double-primed counter-parts may be determined from equations (III-101, 102, 103). Equations (III-104, 105, 106) may be solved for θ^{ss}, θ^{tt}, θ^{rr}, by numerical analysis. The vectors r_i^r, s_i^s, t_i^t are then known and the vectors r_i, s_i, t_i may be obtained from the transformation relations.

As has been demonstrated the solution of difficult vector equations often require simultaneous solution of sets of non-linear transcendental equations. Appendix A is a description of Newton or Newton-Raphson iterative procedures that have been successfully employed in the solving of such equations. The simplicity and adaptability of this procedure was the original stimulus for the present approach of solving the simultaneous set of equations as opposed to reducing them to a polynomial and applying iterative techniques for polynomials.

B. Application to Mechanism Solutions

The vector tetrahedron problems are somewhat abstract in nature as they can be formulated without the concept of a mechanism. Now from a different view-point the methods developed in the solution

of the tetrahedron cases will be applied to the solution of partic-
ular mechanisms. That is, a mechanism is conceived of, its link
and joint geometry selected, the problem of knowns and unknowns
defined and a solution then sought. The following are ten examples
of the application of tensor methods to the solution of spatial
mechanisms. The symmetry and constraints present in such problems
will be seen to have three effects. The constraints present due
to the pairs selected allows coordinate frames to be defined to
advantage, thereby eliminating some terms in the solution equations.
Constraints are often present in the form of scalar products which
tends to increase the number of simultaneous equations which must
be solved numerically. The independent constraints allow a solu-
tion to be obtained for more than three unknowns.

Case 1:   RSSP Mechanism

Referring to Figure 8, an RSSP mechanism is constructed as
shown. The problem is defined as follows.

Input: r_i

Known: $R,\ S,\ C,\ c_i,\ t_i$

Unknown: $T,\ s_i$

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ (III-107)

Since the vectors Cc_i and Rr_i are completely known, let

$$K_i = Cc_i + Rr_i$$ (III-108),

FIGURE 8

CASE 1   RSSP MECHANISM

INPUT: r_i

KNOWN: R, S, C, c_i, t_i

UNKNOWN: T, s_i

$$Cc_i + Rr_i + Ss_i + Tt_i = 0$$

Figure 8
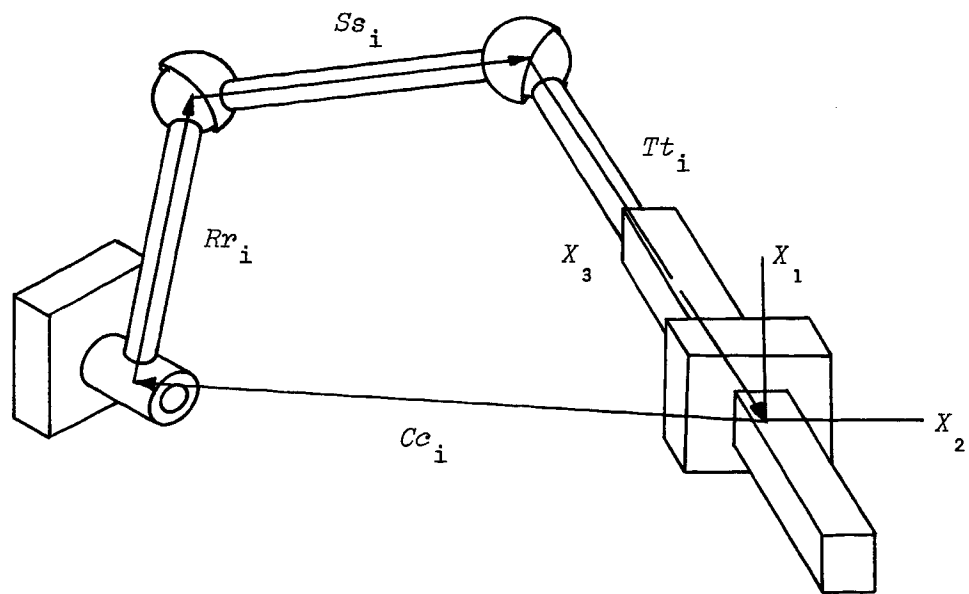
RSSP   MECHANISM

then $\qquad K_i + Ss_i + Tt_i = 0$ $\hspace{3cm}$ (III-109),

The problem as now stated is analogous to Case 2c of the tetrahedron problems. However, the choice of the ground frame leads to a simpler solution than was obtained in Case 2c. Choose the ground frame as shown in Figure 8 so that $t_1 = t_2 = 0$, $t_3 = -1$. Multiply equation (III-109) by $\varepsilon_{ijk} t_j$ eliminating the unknown T.

$$\varepsilon_{ijk}(K_i t_j + Ss_i t_j) = 0 \hspace{3cm} \text{(III-110)}.$$

Expand equation (III-110)

$$K_1 + Ss_1 = 0 \hspace{3cm} \text{(III-111)},$$

$$K_2 + Ss_2 = 0 \hspace{3cm} \text{(III-112)}.$$

then

$$s_1 = \frac{-K_1}{S} = \sin\phi^S \cos\theta^S \hspace{3cm} \text{(III-113)},$$

$$s_2 = \frac{-K_2}{S} = \sin\phi^S \sin\theta^S \hspace{3cm} \text{(III-114)}.$$

If desired, the polar and azimuthal angles, ϕ^S and θ^S are directly obtainable from equations (III-113 and 114). The remaining unknown T is obtained from the expansion of equation (III-109)

$$T = \left| K_3 + Ss_3 \right| \hspace{3cm} \text{(III-115)}.$$

Case 2:   RSCC Mechanism

Referring to Figure 9, a RSCC mechanism is constructed and the problem is defined as follows:

Input: r_i

Known: R, C, c_i, t_i

Unknown: S, s_i, T

Constraint: $s_i \, t_i = P$ where P is a known scalar constant

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$ $\qquad$ (III-116)

Sum the known vectors into a single vector

$$K_i = Cc_i + Rr_i \qquad \text{(III-117)}.$$

then $K_i + Ss_i + Tt_i = 0$ $\qquad$ (III-118)

Choose the ground frame so that $t_1 = t_2 = 0$, $t_3 = -1$ then expanding the constraint equation yields a solution for ϕ^S

$$s_3 = -P = \cos\phi^S \qquad \text{(III-119)}.$$

Multiply equation (III-117) by $\varepsilon_{ijk} t_j s_k$ to eliminate the unknowns S and T.

$$\varepsilon_{ijk}(K_i t_j s_k) = 0 \qquad \text{(III-120)}$$

Expanding, we have

FIGURE 9

CASE 2   RSCC MECHANISM

INPUT: r_i

KNOWN: $R,\ C,\ c_i,\ t_i$

UNKNOWN: $S,\ s_i,\ T$

CONSTRAINT: $s_i t_i = P$

$$Cc_i + Rr_i + Ss_i + Tt_i = 0$$

Figure 9
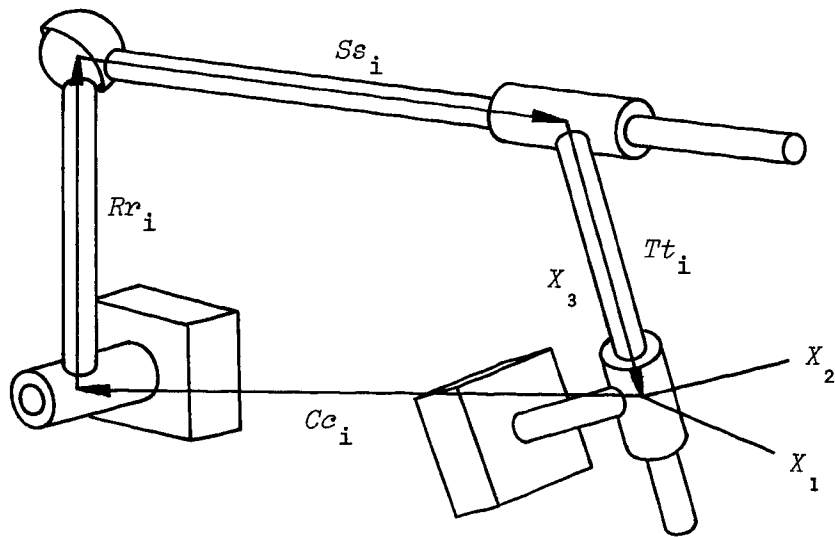
RSCC MECHANISM

$$K_1 s_2 - K_2 s_1 = 0 \qquad\qquad \text{(III-121)}$$

or
$$K_1 \sin\theta^S - K_2 \cos\theta^S = 0 \qquad\qquad \text{(III-122)}$$

The form of equation (III-122) allows a direct solution for θ^S.
The vector s_i is now known; the remaining unknowns are S and T.
Multiply equation (III-118) by $\varepsilon_{ijk} t_j$

$$\varepsilon_{ijk}(K_i t_j + S s_i t_j) = 0 \qquad\qquad \text{(III-123).}$$

Expanding (III-123) we have

$$K_1 + S s_1 = 0 \qquad\qquad \text{(III-124)}$$

$$K_2 + S s_2 = 0 \qquad\qquad \text{(III-125)}$$

Either equation (III-123 or 124) yields a solution for S

$$S = \frac{-K_1}{s_1} = \frac{-K_2}{s_2} \qquad\qquad \text{(III-126).}$$

The remaining unknown T may be obtained by expanding equation (III-118)

$$T = \left| K_3 + S s_3 \right| \qquad\qquad \text{(III-127)}$$

Case 3: RSCP Mechanism

Referring to Figure 10, an RSCP mechanism is constructed and the problem defined as follows:

Input: r_i

Known: C, R, S, c_i, t_i, d_i

FIGURE 10

CASE 3   RSCP MECHANISM

INPUT: r_i

KNOWN: C, R, S, c_i, t_i, d_i

UNKNOWN: T, D, s_i

CONSTRAINT: $s_i t_i = P$

$$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$$

Figure 10
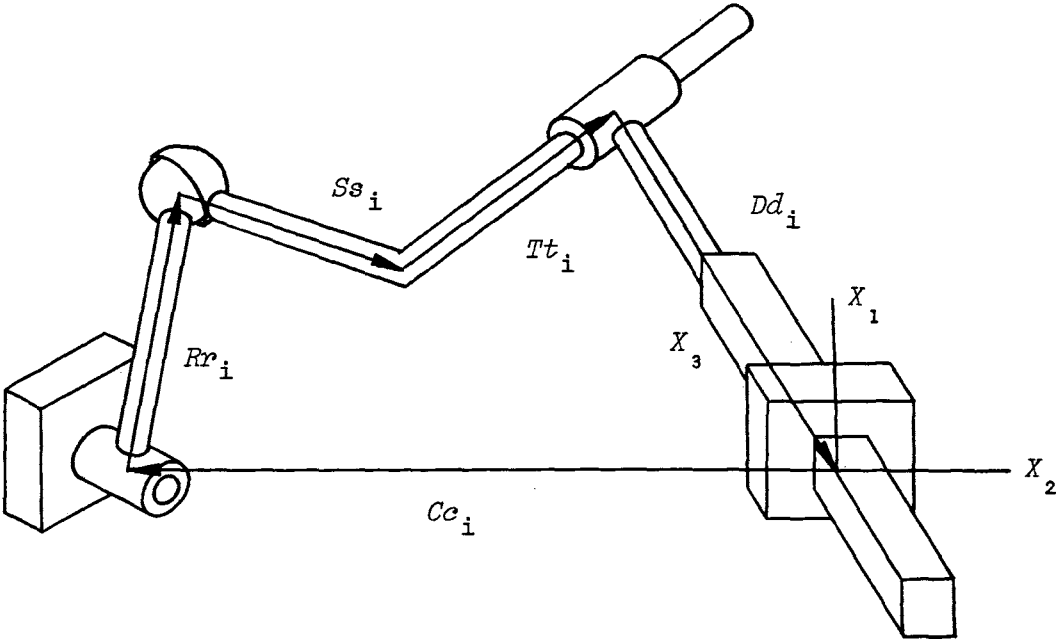
RSCP MECHANISM

Unknown: T, D, s_i

Constraint: $s_i t_i = P$ where P is a known scalar constant

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$ (III-128)

Sum the known vectors into a single vector,

let $K_i = Cc_i + Rr_i$ (III-129),

then $K_i + Ss_i + Tt_i + Dd_i = 0$ (III-130).

Choose the ground frame so that $d_1 = d_2 = 0$, $d_3 = -1$. Multiply equation (III-130) by $\varepsilon_{ijk} t_j d_k$ to eliminate the unknowns T and D.

$$\varepsilon_{ijk}(K_i t_j d_k + Ss_i t_j d_k) = 0 \qquad \text{(III-131)},$$

or $\varepsilon_{ij3}(K_i t_j + Ss_i t_j) = 0$ (III-132).

Expanding equation (III-131) we have

$$\text{(III-133).}$$
$$(K_1 t_2 - K_2 t_1) + St_2 \sin\phi^S \cos\theta^S - St_1 \sin\phi^S \sin\theta^S = 0$$

Expand the constraint equation

$$t_1 \sin\phi^S \cos\theta^S + t_2 \sin\phi^S \sin\theta^S + t_3 \cos\phi^S = 0 \qquad \text{(III-134).}$$

Equations (III-132 and 133) may be solved numerically for θ^S and ϕ^S. The vector s_i is then known. Multiply equation (III-130) by $\varepsilon_{ijk} s_j d_k$ to isolate the unknown T.

$$\varepsilon_{ij3}(K_i s_j + Tt_i s_j) = 0 \qquad\qquad \text{(III-135),}$$

then $T = \dfrac{|\varepsilon_{ij3} K_i s_j|}{|\varepsilon_{ij3} t_i s_j|}$ (III-136).

The remaining unknown D is obtained from the expansion of equation (III-130).

$$D = |K_3 + Ss_3 + Tt_3|$$ (III-137).

Case 4: HCCC Mechanism

Referring to Figure 11, an HCCC mechanism is constructed and the problem defined as follows:

Input: s_i

Known: C, c_i, r_i, d_i, h (pitch of screw)

Unknown: R, S, T, D, t_i

Constraint: $t_i d_i = Q$ $s_i t_i = P$ where Q and P are known scalar constants

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$ (III-138)

The unknown R can be expressed as a function of the screw pitch and the azimuthal angle θ^{rr}

$$R = \theta^{rr} h$$ (III-139).

Let $Cc_i + Rr_i = K_i$, then

$$K_i + Ss_i + Tt_i + Dd_i = 0$$ (III-140).

Choose the ground frame so that $d_1 = d_2 = 0$, $d_3 = -1$.
Expanding the constraint $t_i d_i = Q$ we have

FIGURE 11

CASE 4   HCCC MECHANISM

INPUT: s_i

KNOWN: C, c_i, r_i, d_i, h (pitch of screw)

UNKNOWN: R, S, T, D, t_i

CONSTRAINTS: $t_i d_i = Q$, $s_i t_i = P$

$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$

Figure 11

Tt_i

Ss_i

Dd_i

X_3

Rr_i

Cc_i

X_2

X_1
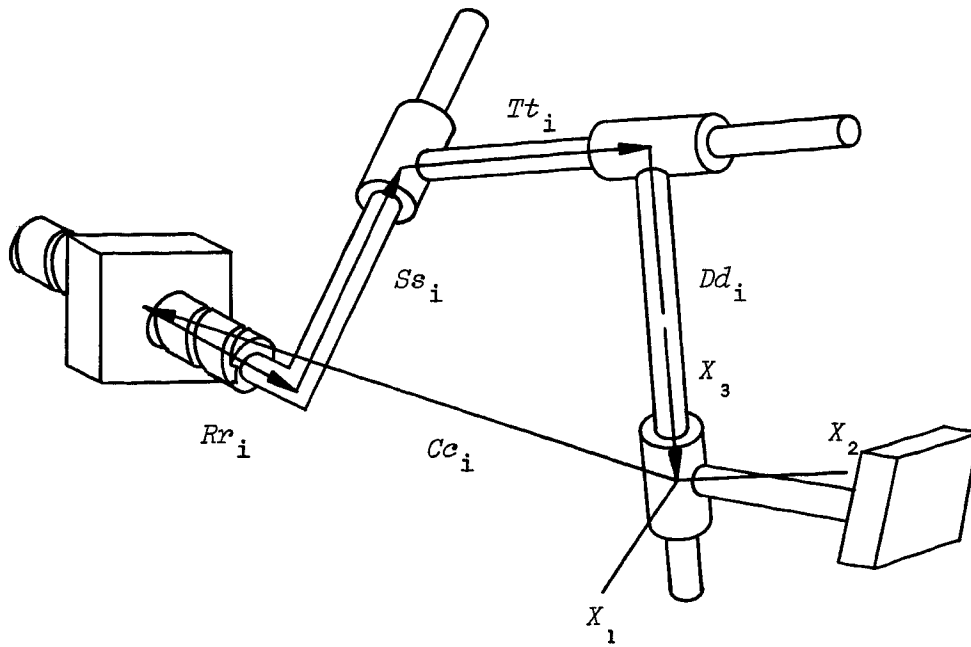
HCCC MECHANISM

$$t_3 = -Q = \cos\phi^t \qquad\qquad \text{(III-141)}$$

Equation (III-140) may be solved for ϕ^t. Expanding the constraint $s_i t_i = P$ we have

$$s_1 \sin\phi^t \cos\theta^t + s_2 \sin\phi^t \sin\theta^t + s_3 t_3 = 0 \qquad \text{(III-142)}$$

Equation (III-142) may be solved for θ^t and the vector t_i is known. Multiply equation (III-140) by $\varepsilon_{ijk} t_j d_k$

$$\varepsilon_{ijk}(K_i t_j d_k + S s_i t_j d_k) = 0 \qquad\qquad \text{(III-143)},$$

or $\qquad\qquad \varepsilon_{ij3}(K_i t_j + S s_i t_j) = 0 \qquad\qquad \text{(III-144)}.$

Equation (III-144) may be solved for S. Multiplying equation (III-140) by $\varepsilon_{ijk} s_j d_k$ yields

$$\varepsilon_{ij3}(K_i s_j + T t_i s_j) = 0 \qquad\qquad \text{(III-145)},$$

which may be solved for T. The remaining unknown D may be obtained from the expansion of equation (III-140)

$$D = \left| K_3 + S s_3 + T t_3 \right| \qquad\qquad \text{(III-146)}.$$

Case 5: PCSC Mechanism

Referring to Figure 12, a PCSC Mechanism is constructed and the problem defined as follows:

Input: R

Known: C, T, c_i, r_i, s_i, d_i

FIGURE 12

CASE 5   PCSC MECHANISM

INPUT: R

KNOWN: C, T, c_i, r_i, s_i, d_i

UNKNOWN: S, D, t_i

CONSTRAINT: $t_i d_i = Q$

$$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$$

Figure 12
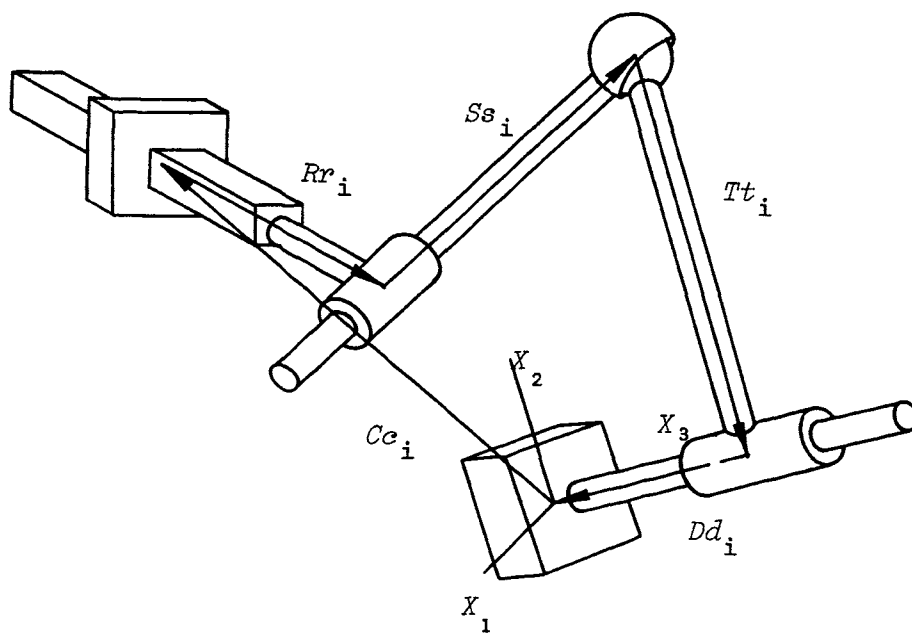
PCSC MECHANISM

Unknown: S, D, t_i

Constraint: $t_i d_i = Q$ where Q is a known scalar constant

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$ (III-147)

Let $K_i = Cc_i + Rr_i$, then

$$K_i + Ss_i + Tt_i + Dd_i = 0 \qquad (III\text{-}148).$$

Choose the ground frame so that $d_1 = d_2 = 0$, $d_3 = -1$.
From the constraint we have

$$t_3 = -Q = \cos\phi^t \qquad (III\text{-}149)$$

Multiply equation (III-148) by $\varepsilon_{ijk}s_j d_k$ and expand

$$(K_1 s_2 - K_2 s_1) + T(s_2 \sin\phi^t \cos\theta^t - s_1 \sin\phi^t \sin\theta^t = 0 \quad (III\text{-}150)$$

Equation (III-149) may be solved for θ^t. Multiply equation (III-148) by $\varepsilon_{ijk}d_j t_k$ to isolate the unknown S.

$$\varepsilon_{i3k}(K_i t_k + Ss_i t_k) = 0 \qquad (III\text{-}151)$$

Solve equation (III-151) for S. The unknown D may be obtained from the loop expansion (III-148).

$$D = \left| K_3 + Ss_3 + Tt_3 \right| \qquad (III\text{-}152)$$

Case 6: RCSC Mechanism

Referring to Figure 13, a RCSC mechanism is constructed and the

FIGURE 13

CASE 6   RCSC MECHANISM

INPUT: r_i

KNOWN: C, R, T, c_i, d_i

UNKNOWN: S, D, t_i

CONSTRAINT: $s_i t_i = P$

$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$

Figure 13
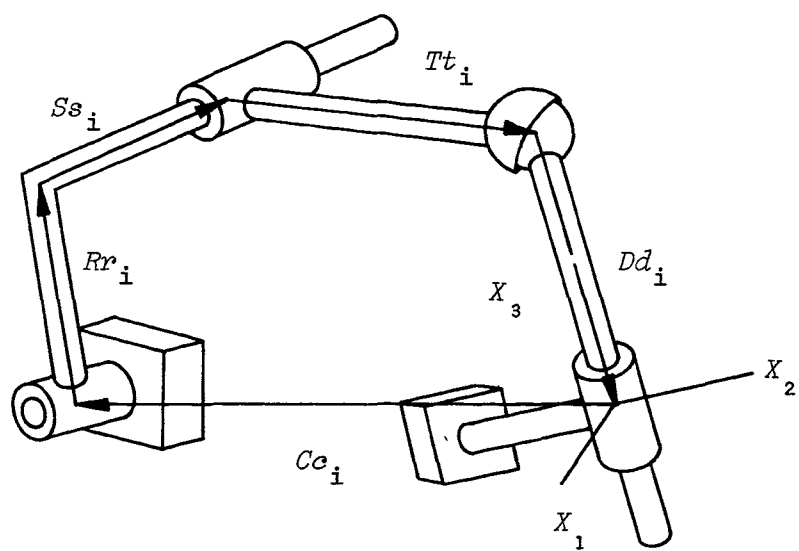
RCSC MECHANISM

problem is defined as follows:

Input: r_i

Known: C, R, T, c_i, d_i

Unknown: S, D, t_i

Constraint: $s_i t_i = P$ where P is a known scalar constant

Vector loop equation: $Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$ (III-153)

Let $K_i = Cc_i + Rr_i$, then

$$K_i + Ss_i + Tt_i + Dd_i = 0 \qquad\qquad \text{(III-154)}$$

Choose the ground frame so that $d_1 = d_2 = 0$, $d_3 = -1$. Multiply equation (III-154) by $\varepsilon_{ijk} s_j d_k$ to eliminate the unknowns S and D.

$$\varepsilon_{ij3}(K_i s_j + Tt_i s_j) = 0 \qquad\qquad \text{(III-155)}$$

Expanding equation (III-155)

$$(K_1 s_2 - K_2 s_1) + T \sin\phi^t (s_2 \cos\theta^t - s_1 \sin\theta^t) = 0 \qquad \text{(III-156)},$$

and from the constraint

$$\sin\phi^t(s_1 \cos\theta^t + s_2 \sin\theta^t) + s_3 \cos\phi^t = 0 \qquad\qquad \text{(III-157)}.$$

Equations (III-156 and 157) may be simultaneously solved for θ^t and ϕ^t. Multiply equation (III-153) by $\varepsilon_{ijk} t_j d_k$ yields

$$\varepsilon_{ij3}(K_i t_j + Ss_i t_j) = 0 \qquad \text{(III-158)}.$$

Equation (III-158) may be solved for S. The remaining unknown D may be obtained from the expansion of equation (III-153)

$$D = \left| K_3 + Ss_3 + Tt_3 \right| \qquad \text{(III-159)}.$$

Case 7: RSCR Mechanism

Referring to Figure 14, an RSCR mechanism is constructed and the problem defined as follows:

Input: r_i

Known: $R, S, D, C, c_i, \phi^t, \phi^d$

Unknown: $T, s_i, \theta^d, \theta^t$

Constraints: $s_i t_i = Q$, $t_i d_i = P$, where Q and P are known scalar constants

Vector loop equation: $K_i + Ss_i + Tt_i + Dd_i = 0 \qquad \text{(III-160)}$

where $K_i = Cc_i + Rr_i$. Eliminate the unknown T by multiplying equation (III-160) by $\varepsilon_{ijk} t_j$

$$\varepsilon_{ijk}(K_i t_j + Ss_i t_j + Dd_i t_j) = 0 \qquad \text{(III-161)}.$$

Expand equation (III-161)

$$(K_2 + Ss_2 + Dd_2) t_3 - (K_3 + Ss_3 + Dd_3) t_2 = 0 \qquad \text{(III-162)},$$

FIGURE 14

CASE 7   RSCR MECHANISM

INPUT: r_i

KNOWN: $R,\ S,\ D,\ C,\ c_i,\ \phi^t,\ \phi^d$

UNKNOWN: $T,\ s_i,\ \theta^d,\ \theta^t$

CONSTRAINT: $s_i t_i = Q$, $t_i d_i = P$

$$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$$

Figure 14
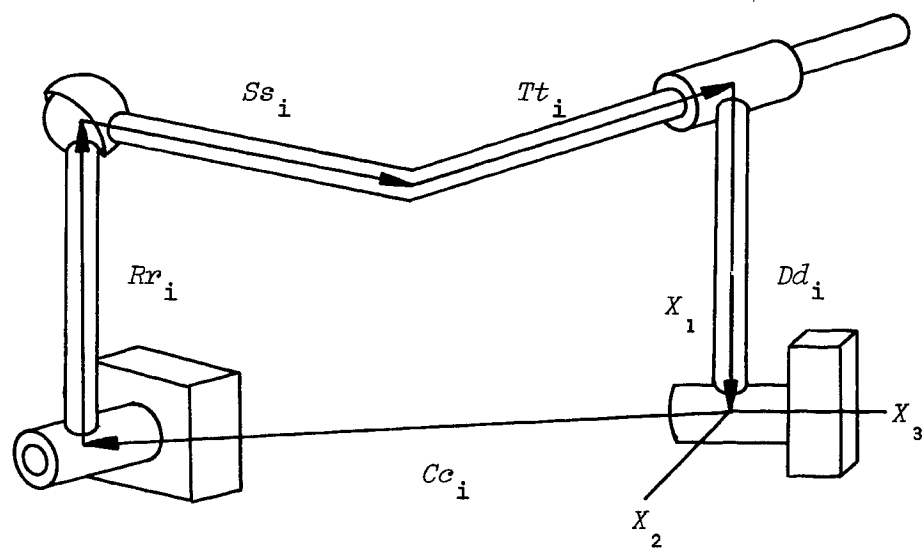
RSCR   MECHANISM

$$(K_3 + Ss_3 + Dd_3)\, t_1 - (K_1 + Ss_1 + Dd_1)\, t_3 = 0 \qquad \text{(III-163)}.$$

Expand the constraint equations

$$s_1 t_1 + s_2 t_2 + s_3 t_3 = Q \qquad \text{(III-164)},$$

$$t_1 d_1 + t_2 d_2 + t_3 d_3 = P \qquad \text{(III-165)}.$$

Equations (III-162, 163, 164, 165) contain the four unknowns θ^s, ϕ^s, θ^d, θ^t and may be solved numerically. The remaining unknown T is obtained by expanding equation (III-160).

$$T = \left| \frac{K_1 + Ss_1 + Dd_1}{t_1} \right| \qquad \text{(III-166)}.$$

Case 8: RCCC Mechanism

Referring to Figure 15 an RCCC mechanism is constructed and the problem defined as follows:

Input: r_i

Known: $R,\ C,\ c_i,\ d_i$

Unknown: $S,\ T,\ D,\ t_i$

Constraint: $s_i t_i = P \qquad d_i t_i = Q$ where P and Q are known scalar constants

Vector loop equation: $K_i + Ss_i + Tt_i + Dd_i = 0$ \qquad (III-167)

where $K_i = Rr_i + Ss_i$.

FIGURE 15

CASE 8   RCCC MECHANISM

INPUT: r_i

KNOWN: R, C, c_i, d_i

UNKNOWN: S, T, D, t_i

CONSTRAINTS: $s_i t_i = P$, $d_i t_i = Q$

$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$

Figure 15
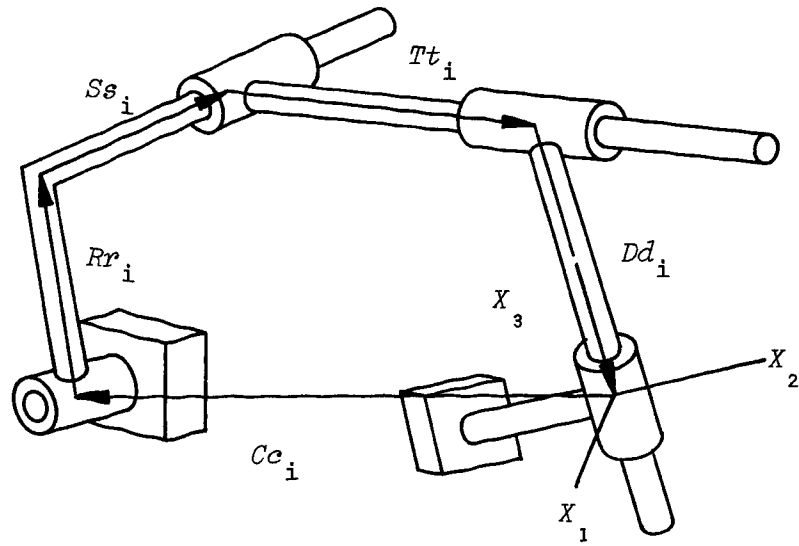
RCCC   MECHANISM

Choose the ground frame so that $d_1 = d_2 = 0$, $d_3 = -1$. Then from the constraint $d_i t_i = Q$, we have

$$t_3 = -Q = \cos\phi^t \qquad \text{(III-168)}$$

From the constraint $s_i t_i = P$, we have

$$s_1 \sin\phi^t \cos\theta^t + s_2 \sin\phi^t \sin\theta^t + s_3 t_3 = P \qquad \text{(III-169)}$$

Equation (III-169) may be solved for θ^t. The remaining unknowns are S, T and D. The problem is then analogous to Case 3a of the tetrahedron solutions and may be solved in the same manner.

Case 9:   PCCC Mechanism

Referring to Figure 16, a PCCC linkage is constructed and the problem defined as follows:

Input: R

Known: C, T, c_i, r_i, s_i, e_i, ϕ^d

Unknown: S, D, E, t_i, θ^d

Constraints: $s_i t_i = M$ $t_i d_i = N$ $s_i d_i = P$ where M, N, P are

known scalar constants

Vector loop equation: $K_i + S s_i + T t_i + D d_i + E e_i = 0$ \qquad (III-170),

where $K_i = C c_i + R r_i$. Choose the ground frame so that $e_1 = e_2 = 0$, $e_3 = -1$. Expand the three constraint equations

FIGURE 16

CASE 9   PCCC MECHANISM

INPUT: R

KNOWN: $C,\ T,\ c_i,\ r_i,\ s_i,\ e_i,\ \phi^d$

UNKNOWN: $S,\ D,\ E,\ t_i,\ \theta^d$

CONSTRAINTS: $s_i t_i = M$, $t_i d_i = N$, $s_i d_i = P$

$$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i + Ee_i = 0$$

Figure 16
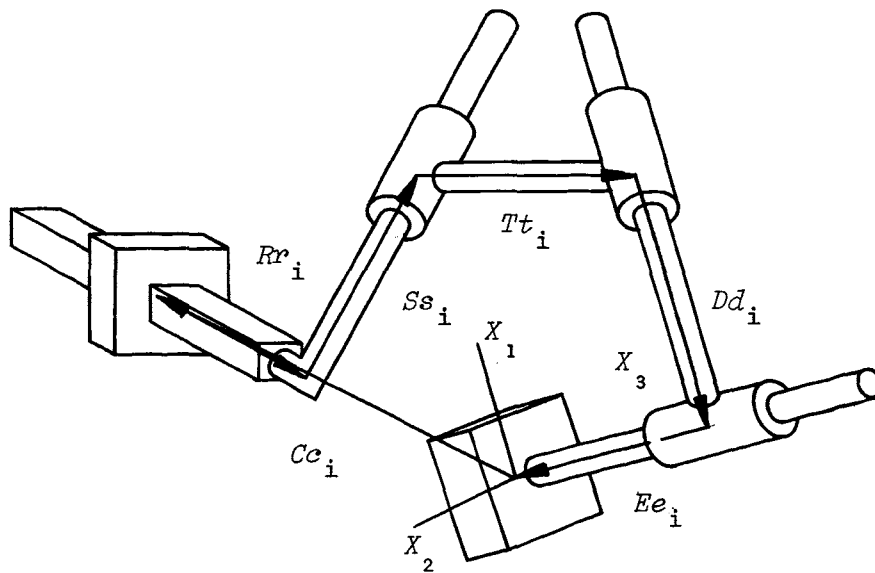
PCCC   MECHANISM

$$s_{1}t_{1} + s_{2}t_{2} + s_{3}t_{3} = M \qquad\qquad \text{(III-171)}$$

$$t_{1}d_{1} + t_{2}d_{2} + t_{3}d_{3} = N \qquad\qquad \text{(III-172)}$$

$$s_{1}d_{1} + s_{2}d_{2} + s_{3}d_{3} = P \qquad\qquad \text{(III-173)}$$

Equation (III-173) may be solved for θ^{d}. Equations (III-171 and 172) may then be solved for θ^{t}, ϕ^{t}. The remaining unknowns then are S, D, E and the problem is now analogous to Case 3a of the tetrahedron solutions and may be solved in the same manner.

Case 10: PCCR Mechanism

Referring to Figure 17, a PCCR mechanism is constructed and the problem defined as follows:

Input: R

Known: C, D, c_{i}, r_{i}, s_{i}, ϕ^{d}

Unknown: S, T, t_{i}, θ^{d}

Constraint: $s_{i}t_{i} = P$ $\quad d_{i}t_{i} = Q$ where P and Q are known scalar constants.

Vector loop equation: $K_{i} + Ss_{i} + Tt_{i} + Dd_{i} = 0 \qquad\qquad \text{(III-174)}$,

where $K_{i} = Cc_{i} + Rr_{i}$. Multiply equation (III-174) by $\varepsilon_{ijk}s_{j}t_{k}$ to eliminate S and T

$$\varepsilon_{ijk}(K_{i}s_{j}t_{k} + Dd_{i}s_{j}t_{k}) = 0 \qquad\qquad \text{(III-175)}.$$

FIGURE 17

CASE 10   PCCR MECHANISM

INPUT: R

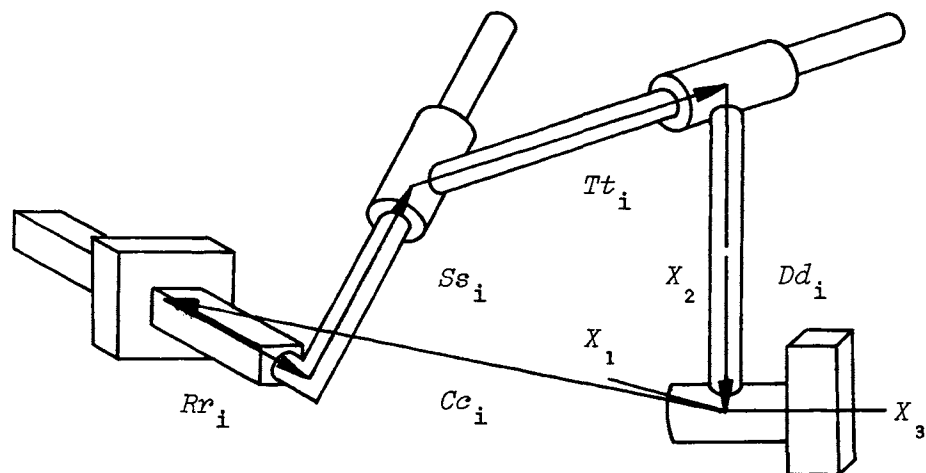KNOWN: C, D, c_i, r_i, s_i, ϕ^d

UNKNOWN: S, T, t_i, θ^d

CONSTRAINTS: $s_i t_i = P$, $d_i t_i = Q$

$$Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$$

Figure 17

PCCR   MECHANISM

Expand the constraint equations

$$s_1 t_1 + s_2 t_2 + s_3 t_3 = P \qquad \text{(III-176)},$$

$$d_1 t_1 + d_2 t_2 + d_3 t_3 = Q \qquad \text{(III-177)}.$$

Equations (III-175, 176, and 177) contain the unknowns θ^t, ϕ^t, and θ^d and may be numerically solved. Multiplying equation (III-174) by $\varepsilon_{ijk} t_j d_k$ and $\varepsilon_{ijk} s_j d_k$ yields the unknowns S and T respectively.

1. Canard Deployment Mechanism

While the foregoing ten cases are interesting and provide insight into the method of tensor analysis it is much more interesting to apply the method to a problem of practical nature from an engineering point of view. The following problem represents the application of tensor kinematics to an existing engineering problem of current interest.

Modern interest in the short-field manuverability and low level mission roles of supersonic military aircraft has led to much investigation of variable geometry configurations. One result has been the development of the retractable Canard surface system. This configuration produces a nose-up moment at low speeds and significantly improves the aircrafts takeoff and landing performance and its subsonic manuverability. Design considerations include the ability to retract and deploy the surfaces through a single drive, thus eliminating the possibility of asymmetrical deployment.

Figure 18 is a pictorial representation of the use of spatial mechanisms to provide the required deployment system. Two four-bar
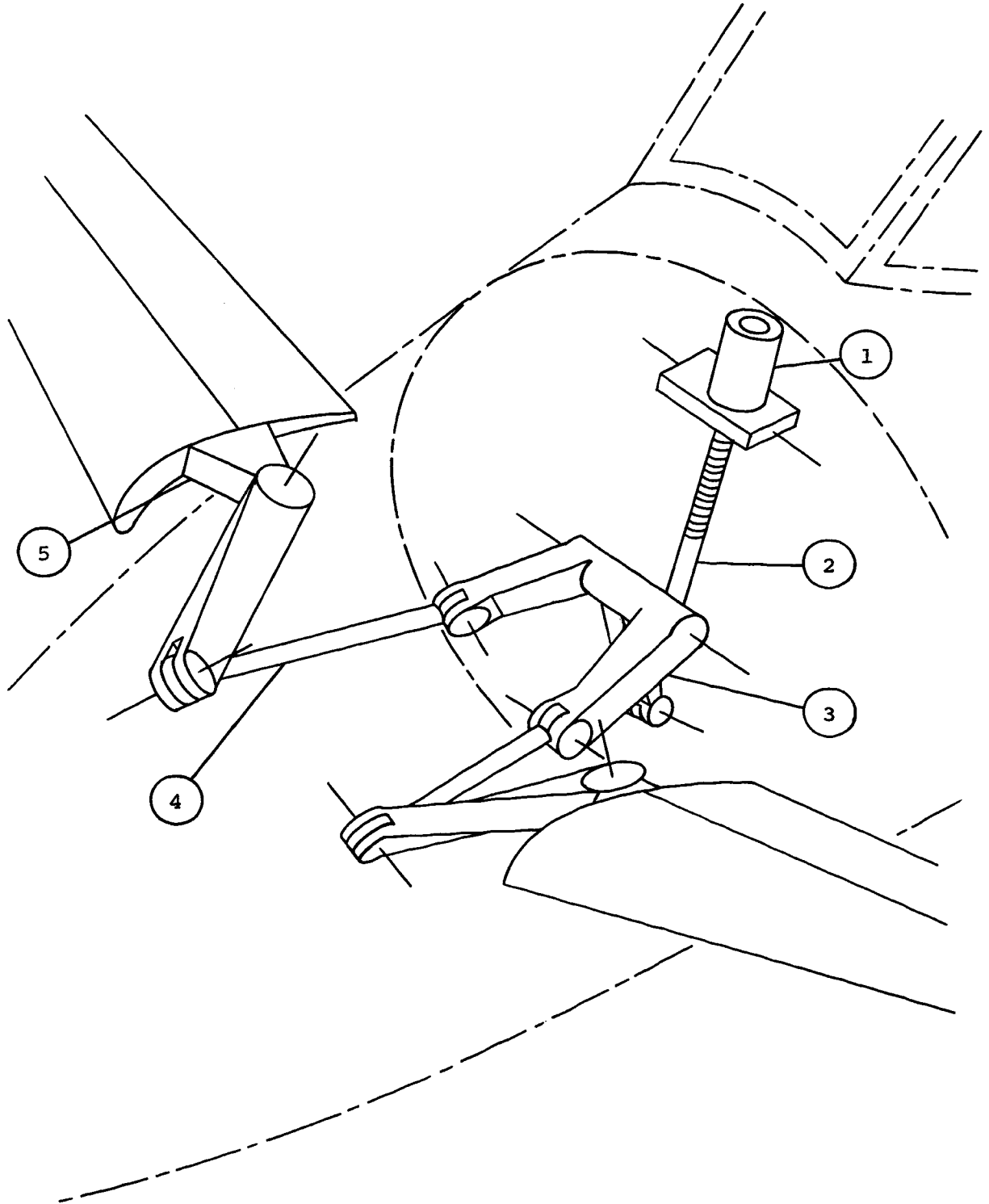
FIGURE 18

CANARD DEPLOYMENT MECHANISM

1. ELECTRIC MOTOR

2. WORM AND GEAR

3. ACTUATING BELLCRANK

4. ACTING ROD

5. CANARD SURFACE SPAR MEMBER

TWO FOUR-BAR BENNETT MECHANISMS DRIVEN BY A SINGLE JACK SCREW. THE
ARRANGEMENT ALLOWS THE USE OF ALL REVOLUTE JOINTS AND PREVENTS ASYM-
METRICAL DEPLOYMENT.

Figure 18

mechanisms with revolute joints are shown connected in parallel and each driving one of the canard surfaces. The parallel arrangement allows the use of a single jack screw to drive both canard surfaces. The four-bar linkages may be recognized as the Bennett RRRR mechanism mentioned in section I-D and shown in Figure 4. It will be recalled that the mobility criterion when applied to the Bennett mechanism yielded -2 for the number of degrees of freedom, yet it is known that the Bennett mechanism exists as a single degree of freedom linkage. This leads to the supposition that the Bennett linkage possesses three redundant constraints.

The peculiar geometric qualities that allow the Bennett mechanism to exist are well known; namely, that opposite links must have the same lengths and the same degree of skew or twist. These constraints are most often demonstrated in an after-the-fact fashion through the use of descriptive geometry. The present approach is to assume only the existence of a four-bar RRRR mechanism and establish the geometric criteria for its existence through the use of tensor analysis.

a. Existence Criteria for the Bennett Mechanism

Figure 19 is a vector loop representation of a Bennett mechanism where C, R, S, T are the link lengths and u_i, v_i, w_i, p_i are the directions of the axes of revolution of the R joints. To mathematically describe the RRRR mechanism, nine separate constraint equations must be written. Closure must exist hence the vector loop equation
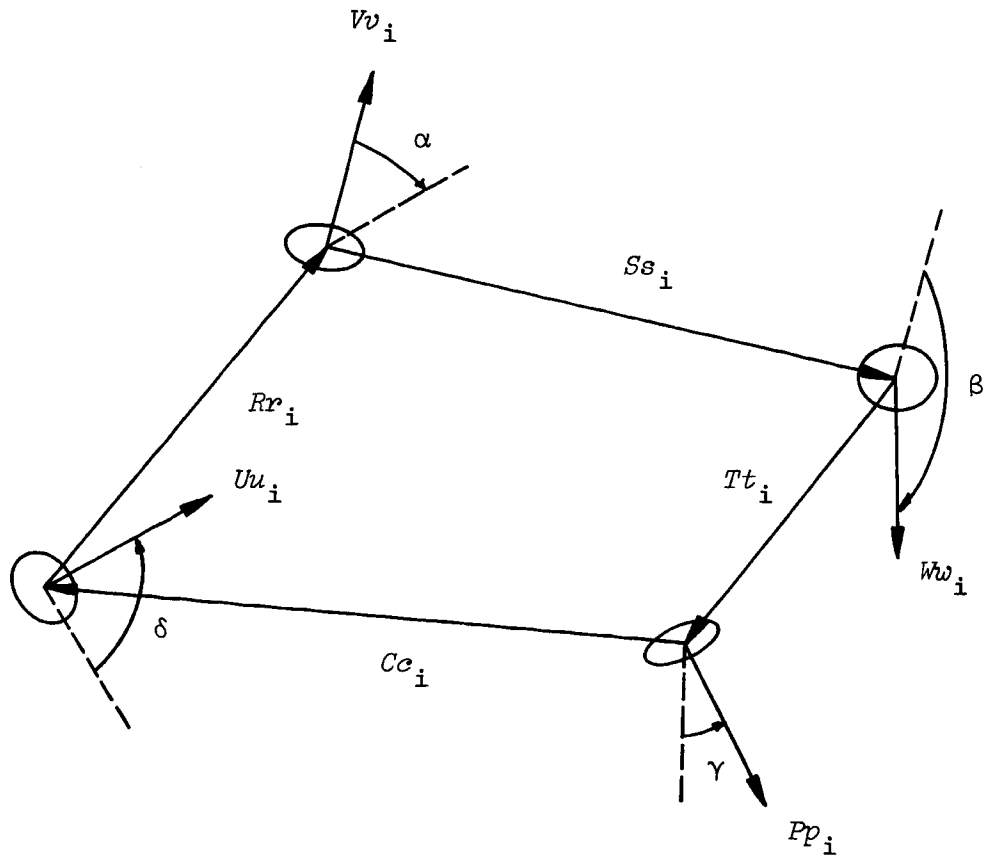
FIGURE 19

VECTOR   REPRESENTATION   OF   A   FOUR-BAR   RRRR   MECHANISM

THE REVOLUTE AXES OF THE JOINTS ARE LABELED Uu_i, Vv_i, Ww_i AND Pp_i.
THE SKEW ANGLES δ, α, β, γ REPRESENT THE TWIST IN THE C, R, S, T
LINKS RESPECTIVELY.

Figure 19

$$Cc_i + Rr_i + Ss_i + Tt_i = 0 \qquad\qquad \text{(III-178)}.$$

The condition that the axes of revolution of each joint must remain mutually perpendicular to the links which it joins leads to the constraint equations

$$\varepsilon_{ijk}r_j c_k = Uu_i \qquad\qquad \text{(III-179)},$$

$$\varepsilon_{ijk}s_j r_k = Vv_i \qquad\qquad \text{(III-180)},$$

$$\varepsilon_{ijk}t_j s_k = Ww_i \qquad\qquad \text{(III-181)},$$

$$\varepsilon_{ijk}c_j t_k = Pp_i \qquad\qquad \text{(III-182)}.$$

Definition of the skew or twist in each link leads to the equations

$$\varepsilon_{ijk}u_j p_k = c_i \, \sin(U,P) = c_i \, \sin\delta \qquad\qquad \text{(III-183)},$$

$$\varepsilon_{ijk}v_j u_k = r_i \, \sin(V,U) = r_i \, \sin\alpha \qquad\qquad \text{(III-184)},$$

$$\varepsilon_{ijk}w_j v_k = s_i \, \sin(W,V) = s_i \, \sin\beta \qquad\qquad \text{(III-185)},$$

$$\varepsilon_{ijk}p_j w_k = t_i \, \sin(P,W) = t_i \, \sin\gamma \qquad\qquad \text{(III-186)}.$$

The angles δ, α, β, γ are the skew or twist angles of the C,R,S,T links respectively.

Multiply equation (III-178) by $\varepsilon_{ijk}c_j r_k$, $\varepsilon_{ijk}c_j s_k$, $\varepsilon_{ijk}c_j t_k$, $\varepsilon_{ijk}r_j s_k$, $\varepsilon_{ijk}r_j t_k$, $\varepsilon_{ijk}s_j t_k$ to form six equations involving the possible permutations of vector products. There results

$$\varepsilon_{ijk}(Ss_i c_j r_k + Tt_i c_j r_k) = 0 \qquad \text{(III-187)},$$

$$\varepsilon_{ijk}(Rr_i c_j s_k + Tt_i c_j s_k) = 0 \qquad \text{(III-188)},$$

$$\varepsilon_{ijk}(Rr_i c_j t_k + Ss_i c_j t_k) = 0 \qquad \text{(III-189)},$$

$$\varepsilon_{ijk}(Cc_i r_j s_k + Tt_i r_j s_k) = 0 \qquad \text{(III-190)},$$

$$\varepsilon_{ijk}(Cc_i r_j t_k + Ss_i r_j t_k) = 0 \qquad \text{(III-191)},$$

$$\varepsilon_{ijk}(Cc_i s_j t_k + Rr_i s_j t_k) = 0 \qquad \text{(III-192)}.$$

After some manipulation the various terms in equations (III-187 through 192) may be recognized in the constraint equations (III-179 through 186). Substitute equations (III-179 through 182) into equations (III-183 through 186). These eight equations are reduced to

$$\varepsilon_{kmn} r_k c_m t_n = PU \sin\delta \qquad \text{(III-193)}$$

$$\varepsilon_{kmn} s_k r_m c_n = UV \sin\alpha \qquad \text{(III-194)}$$

$$\varepsilon_{kmn} t_k s_m r_n = WV \sin\beta \qquad \text{(III-195)}$$

$$\varepsilon_{kmn} c_k t_m s_n = PW \sin\gamma \qquad \text{(III-196)}$$

Substitute equations (III-193 through 196) into equations (III-187 through 192). We have

$$SV\sin\alpha - TP\sin\delta = 0 \qquad \text{(III-197)}$$

$$RUV \sin\alpha - TPW \sin\gamma = 0 \qquad \text{(III-198)}$$

$$RU \sin\delta + SW \sin\gamma = 0 \qquad \text{(III-199)}$$

$$CU \sin\alpha + TW \sin\beta = 0 \qquad \text{(III-200)}$$

$$-CPU \sin\delta + SWV \sin\beta = 0 \qquad \text{(III-201)}$$

$$CP \sin\gamma + RV \sin\beta = 0 \qquad \text{(III-202)}$$

The six equations (III-197 through 202) may be used to algebraically eliminate the magnitudes of the revolute axes P, V, W, U. Equations (III-197 through 202) can be arranged so that these unknowns occur only in the ratios P/V and W/U, therefore, two equations are required to eliminate the four parameters P, V, W, U. Of the remaining four equations one is an identity and the remaining three are triply redundant, all being the same equation.

$$CS \sin\alpha \sin\gamma = RT \sin\delta \sin\beta \qquad \text{(III-203)}.$$

Equation (III-203) represents a single constraint equation that an RRRR mechanism as defined must satisfy in order to exist. Of the nine original constraint equations, we have one remaining which is triply redundant indicating that the RRRR mechanism possesses three redundant constraints. It may be concluded that any set of parameters that will satisfy equation (III-203) will result in a workable RRRR mechanism. One such set is

$$C = S = A \qquad \sin\alpha = \sin\gamma \qquad \text{(III-204)},$$

$$R = T = B \qquad \sin\beta = \sin\delta \qquad \text{(III-205)}.$$

Then equation (III-203) may be written

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} \tag{III-206}$$

The conditions (III-204, 205) indicate that opposite links are equal and have the same skew angle and must satisfy equation (III-206).

b. Position Solution

Having established the geometric constraint criteria for the Bennett mechanism, the position solution of the canard deployment mechanism may proceed. Figure 20 depicts one half of the mechanism in its physical configuration as well as the vector loop which represents the mathematical counterpart. Physically, links Rr_i and Ss_i are offset from the vector loop. This is possible if they are constructed so as to retain the spatial relationship among the joints and links as determined by the constraint equation (III-205). That is, it is not necessary for the actual links to follow the path of the vector loop provided the direction of the revolute joint axes as shown in Figure 20 are preserved.

Referring to Figure 20, the ground frame is placed at the bell crank pivot with the X_2 axis colinear with the c_i vector and the X_3 axis along the revolute axis of the bell crank. The driving screw link Ee_i, link Ff_i, and the input link to the spatial mechanism Br_i are all coplanar. The vector r_i may be determined as a function of the length of link E. The spatial mechanism problem may then be defined as

Input: r_i
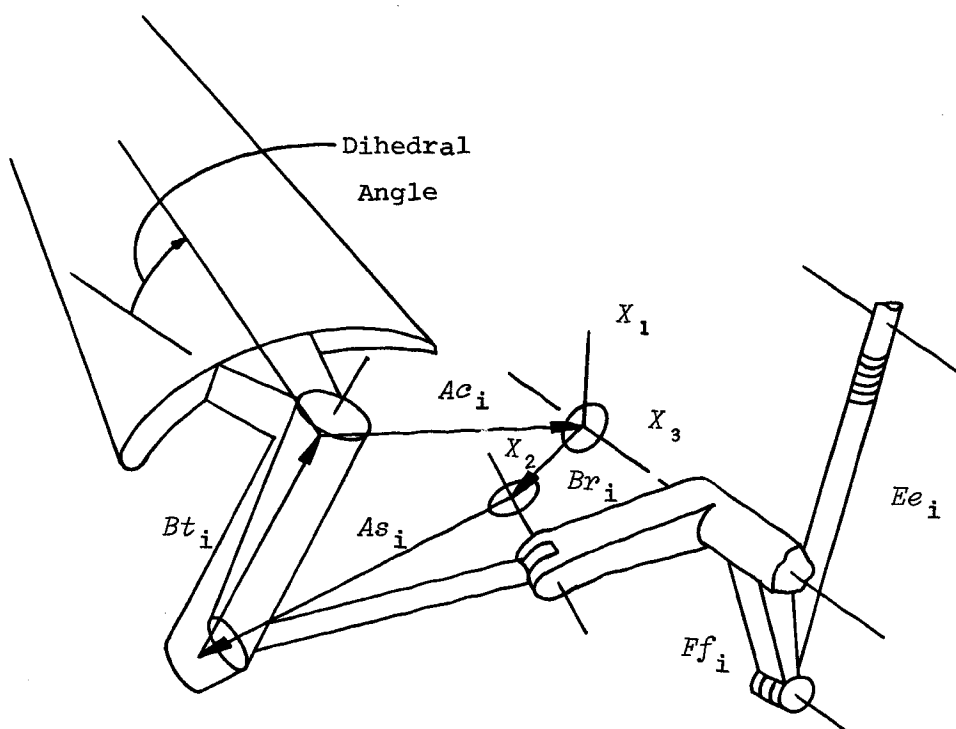
FIGURE 20

CANARD DEPLOYMENT MECHANISM

THE PHYSICAL CONFIGURATION OF THE MECHANISM AND THE VECTOR LOOP COUNTERPART. LINKS Br_i AND As_i ARE OFFSET FROM THE VECTOR PATHS BUT THE REVOLUTE AXIS RELATIONSHIPS ARE MAINTAINED.

THE DESIGN DIHEDRAL ANGLE DETERMINES THE SKEW IN THE Ac_i GROUND LINK.

Figure 20

Known: A, B, c_i

Unknown: s_i, t_i

Constraint: $A\sin\beta = B\sin\alpha$

Vector loop equation: $Ac_i + Br_i + As_i + Bt_i = 0$ (III-207)

An important design parameter is the dihedral angle of the canard surface. Flight characteristics would determine the best dihedral angle which in turn determines the skew angle of the ground link Ac_i. The length of the links A and B are specified from considerations of available space, clearances, etc. and are considered as variable parameters. The skew angle β of the links Br_i and Bt_i is then determined from the constraint equation.

We are interested in the position of the spar vector w_i as a function of E, the length of the jack screw link. It is advantageous to define an auxiliary frame X_i^t as shown in Figure 21. The X_3^t axis is aligned along the revolute axis of the joint at the intersection of the c_i and t_i vectors. This can be accomplished by a single rotation about the X_2 axis through the angle α, the skew angle of the Ac_i link. The w_i and t_i vectors then remain in the X_2^t, X_1^t plane; and the t_i^t vector is a function of the single unknown θ^{tt}.

$$t_i^t = \cos\theta^{tt},\ \sin\theta^{tt},\ 0 \qquad \text{(III-208)}$$

The transformation coefficients are seen to be

$$A_{ij}^{tx} = \begin{matrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{matrix} \qquad \text{(III-209)}$$

FIGURE 21

DEFINITION OF THE AUXILIARY FRAME x_i^t

THE x_i^t FRAME IS DEFINED BY THE ROTATION ABOUT THE x_2 AXIS THROUGH
THE ANGLE α, THE SKEW ANGLE OF THE GROUND LINK. THE x_3^t AXIS IS THEN
THE REVOLUTE AXIS OF THE SPAR VECTOR w_i. THE VECTORS w_i^t AND t_i^t
REMAIN ALWAYS IN THE x_2^t, x_1^t PLANE AND ARE FUNCTIONS OF A SINGLE
AZIMUTHAL ANGLE θ^{tt}.

Figure 21
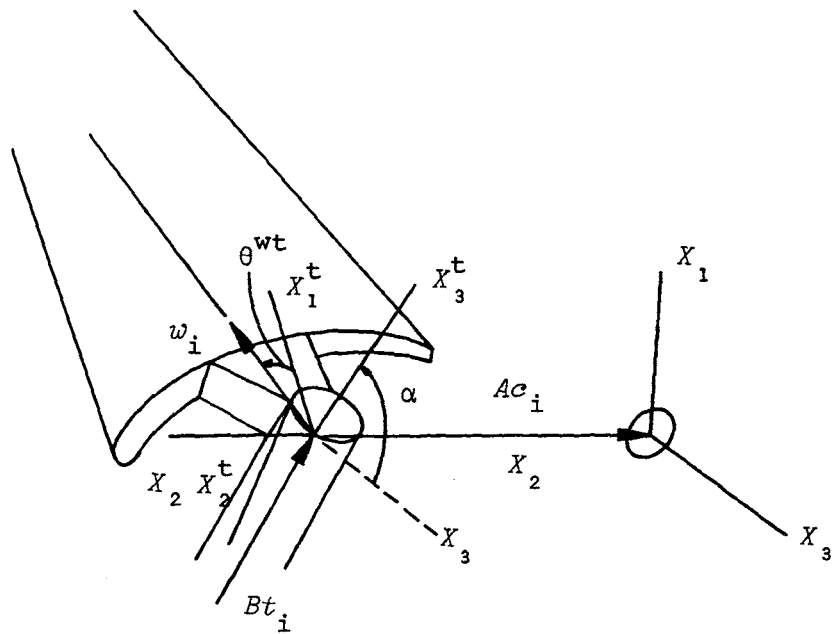
LOCATION OF THE X_i^t FRAME BY THE ROTATION α
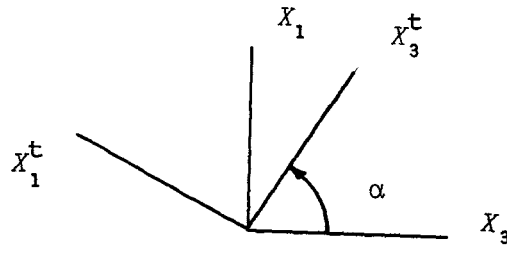
VIEW FROM THE $X_1\,X_3$ PLANE

Let $K_i = Ac_i + Br_i$ then the loop equation becomes

$$K_i + As_i + Bt_i = 0 \tag{III-210}.$$

Multiply equation (III-210) by A_{mi}^{cx}, transforming to the X_i^t frame

$$K_m^t + As_m^t + Bt_m^t = 0 \tag{III-211}.$$

Expand equation (III-211)

$$K_1^t + A \sin\phi^{st} \cos\theta^{st} + B \cos\theta^{tt} = 0 \tag{III-212},$$

$$K_2^t + A \sin\phi^{st} \sin\theta^{st} + B \sin\theta^{tt} = 0 \tag{III-213},$$

$$K_3^t + A \cos\phi^{st} = 0 \tag{III-214}.$$

Equation (III-214) yields a solution for ϕ^{st}

$$\cos\phi^{st} = -K_3^t/A$$

Equations (III-212 and 213) may be solved iteratively for θ^{st} and θ^{tt}. Having a solution for θ^{tt} enables the w_i vector to be written as a function of θ^{tt}

$$w_i = \cos\theta^{wt}, \sin\theta^{wt}, 0 \tag{III-215},$$

where $\theta^{wt} = \theta^{tt} + \xi$ ξ a constant and may be considered a parameter. For any value of E, through its range, the value of θ^{wt} may be obtained. Figure 22 is a plot of θ^{wt} for values of E from the initial configuration (canard surface fully deployed) to the final configu-

FIGURE 22

PLOT OF POSITION AZIMUTHAL ANGLE θ^{wt} VERSUS

THE LENGTH OF THE JACK SCREW E.

Figure 22
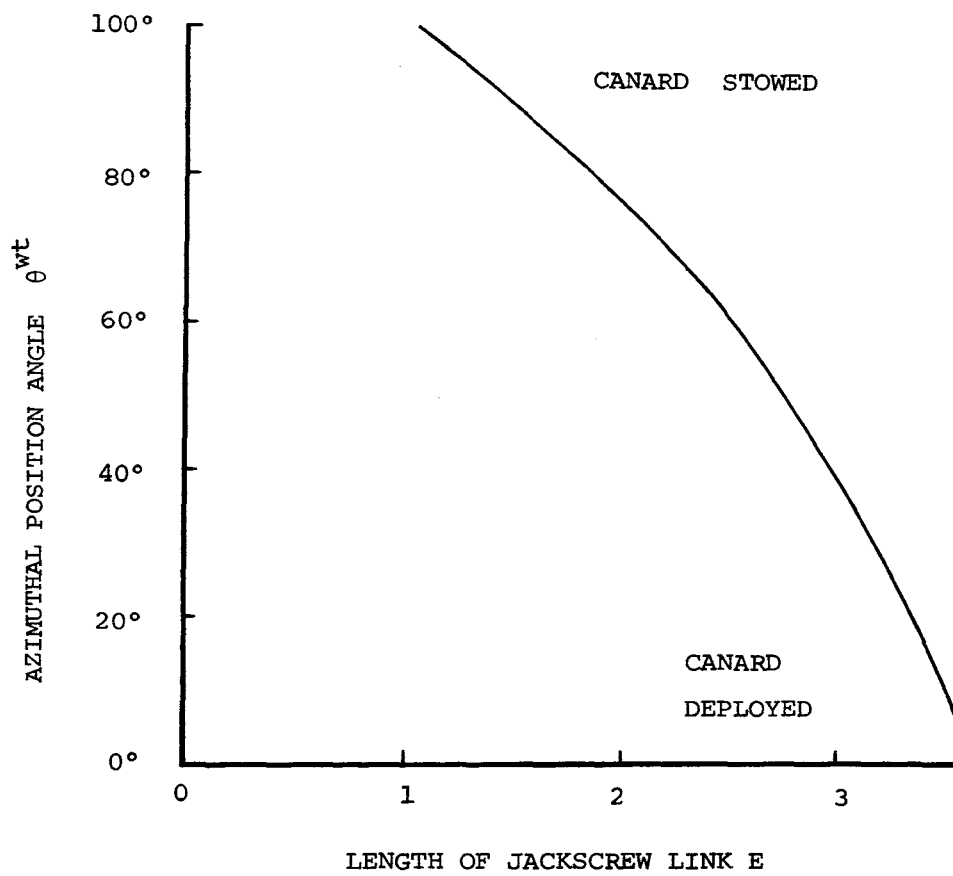
ration (fully stowed). The parameters chosen for the problem are as follows:

A = 4 units

B = 3 units

α = 60°

D = 2 units

θ^d= 330°

E (initial value) = $D\sqrt{3}$

F = 2 units

θ^c = 270°

starting values used $\theta^{tt} \simeq$ 300°, $\theta^{st} \simeq$ 105°

The problem was programmed for digital computation and the solutions obtained. The procedure used was to solve the two equations (III-212 and 213) via the Newton method for systems of nonlinear equations for the initial configuration using the estimated starting values shown. Values were calculated to four-place accuracy. The link E was then incremented and a new solution sought using the last solutions for starting values. At no step in the process was more than four iterations required to provide the accuracy specified. Less than 25 seconds of computer time were required to effect the entire solution.

IV

VELOCITY AND ACCELERATION ANALYSIS OF SPATIAL MECHANISMS

As has been demonstrated, position solutions are non-linear
and usually somewhat difficult to obtain. Motion solutions, on the
other hand, are always linear. Differentiation of a position solu-
tion will never introduce unknown vectors and the motion quantity
unknowns that are introduced are of the same or lower order and occur
in additive terms, not in products with each other. Thus, the prob-
lem of obtaining solutions for motion quantities is minimal and may
be accomplished by linear algebra. The difficulty lies instead in
obtaining the motion equations from the position solution.

A direct approach for obtaining motion equations is differen-
tiation of the position equations. This approach is relatively
tractable and straight forward when all quantities to be differenti-
ated are known with respect to the ground frame but becomes increas-
ingly difficult when one or more moving reference frames are included
in a problem. The combination of the tensor methods and the Newton
iterative process allows a large class of spatial problems to be
solved in the ground frame making the method of direct differentia-
tion a most useful tool.

Relative velocities between various points of a mechanism may
be obtained via the same procedures as well as relative angular
velocities between links.

A. Method of Direct Differentiation

If a position solution for a particular point in space can be
obtained, the velocity and acceleration of that point may always be

obtained by direct differentiation of the position vector. The difficulty presented by this course depends upon how the position solution was obtained and its form. If a position solution can be obtained solely in the ground frame coordinates the differentiation is particularly straight forward and for kinematic purposes, it is often fruitful to attempt the attainment of a solution in the ground frame. However, for purposes of dynamic analyses and rigid body mechanics, it is necessary to allow for the expression of vectors in auxiliary or body frames whose coordinate axes are functions of time when motion is considered. For this reason, the method of direct differentiation will be developed in a general manner with the inclusion of a moving coordinate frame. It will be seen that reduction of the formulae thus developed to simpler cases is easily accomplished.

1. General Velocity and Acceleration Equations for Spatial Motion

Figure 23 depicts a moving point in space following a path S. At the time of interest the point is at point P as shown and the position of point P may be defined by the vector Ss_i, written with respect to the inertial frame or ground frame X_i, or by the vector Rr_i^r written with respect to the X_i^r frame. The fact that the X_i^r frame is allowed to move with respect to the X_i frame is of consequence only if one is concerned with the relativity of writing quantities as measured in one frame and relating them to the the same quantities as measured in another frame. That is, to an observer in the X_i^r frame, and unaware of the existence of the X_i frame, the sole definition of the point P at the time of interest is the vector Rr_i^r.

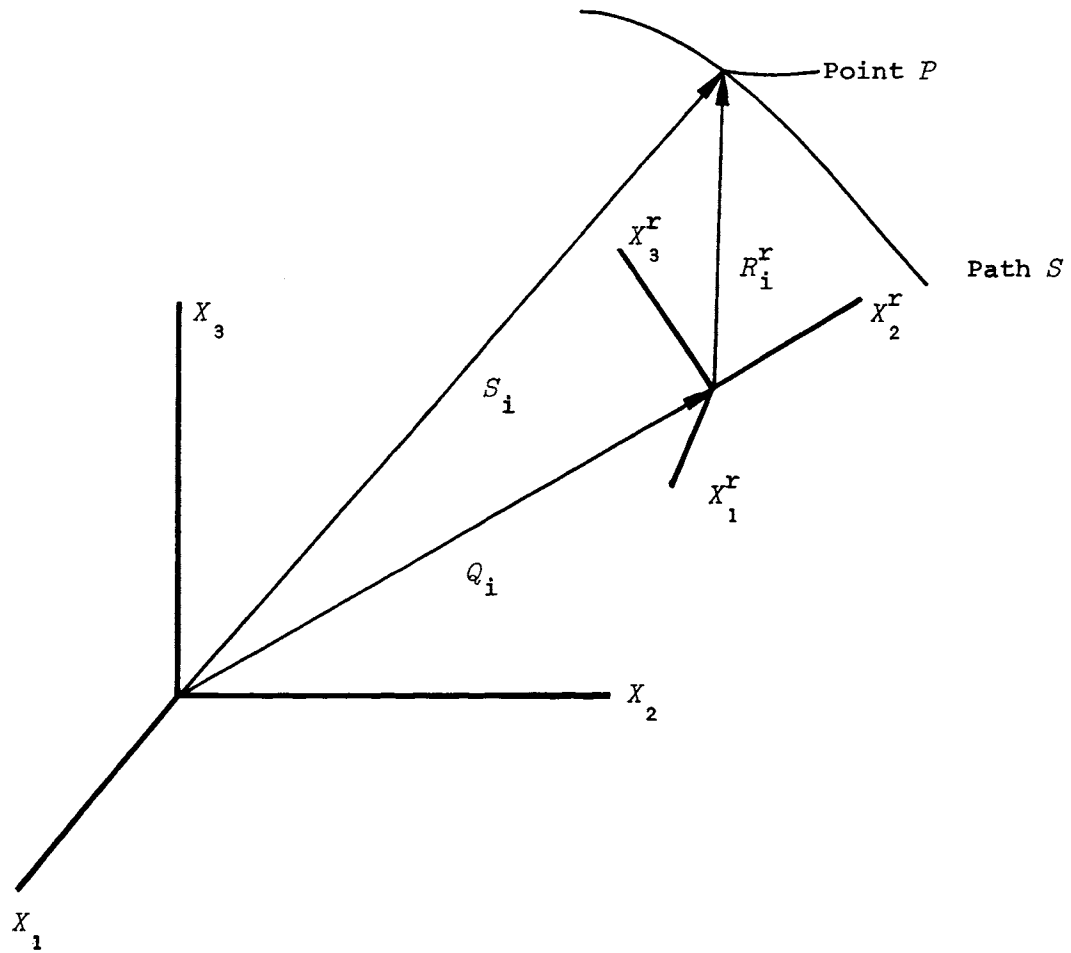'FIGURE 23

THE CASE OF GENERAL SPATIAL MOTION

A MOVING POINT IN SPACE FOLLOWING A PATH S, AT THE TIME OF INTEREST

THE POINT IS AT POINT P AND THE POSITION OF POINT P MAY BE REFERENCED

TO EITHER OF THE TWO COORDINATE FRAMES. THE X_i FRAME IS REGARDED AS

THE GROUND FRAME AND THE X_i^r FRAME IS FREE TO MOVE RELATIVE TO THE X_i

FRAME.

Figure 23

The same may be said for an equivalent observer in the X_i frame and

the vector Ss_i. Now suppose an observer in the X_i frame who is aware

of the X_i^r frame and may write the position relation

$$Ss_i = Qq_i + Rr_i \qquad \text{(IV-1)}.$$

Differentiating equation (IV-1) with respect to time, we may obtain

a relation between the velocity of point P as observed in the ground

frame and that observed in the X_i^r frame if the term Rr_i is replaced

with the transformation relation $A_{ji}^{rx}Rr_j^r$.

It will facilitate the notation to write vectors in unfactored

form as $R_i = Rr_i$. Differentiating equation (IV-1) we have

$$\dot{S}_i = \dot{Q}_i + \dot{R}_i \qquad \text{(IV-2)}.$$

Employing the transformation relation we may write

$$\dot{R}_i = A_{ji}^{rx}\dot{R}_j^r + \dot{A}_{ji}^{rx}R_j^r \qquad \text{(IV-3)}.$$

Examining the terms in equation (IV-3): the term $A_{ji}^{rx}\dot{R}_j^r$ represents

differentiation in the X_i^r frame holding the A_{ji}^{rx} constant and hence

is the velocity of the point P relative to the X_i^r frame. The term

$\dot{A}_{ji}^{rx}R_j^r$ represents differentiation of the A_{ji}^{rx} holding the vector R_j^r

constant and is the contribution to the absolute velocity due to the

rotation of the X_i^r frame relative to the X_i frame.

To further examine the character of this last term, suppose

that the R_j^r vector is fixed in the X_i^r frame. Then equation (IV-3)

becomes

$$\dot{R}_i = \dot{A}^{rx}_{ji} R^r_j \tag{IV-4}$$

Substituting the transformation relation $R^r_j = A^{rx}_{jm} R_m$ into equation (IV-4) we have

$$\dot{R}_i = \dot{A}^{rx}_{ji} A^{rx}_{jm} R_m \tag{IV-5}$$

The product $\dot{A}^{rx}_{ji} A^{rx}_{jm}$ may be denoted by ω^{rx}_{im}, then

$$\dot{R}_i = \omega^{rx}_{im} R_m \tag{IV-6}$$

Multiply equation (IV-6) by R_i

$$R_i \dot{R}_i = \omega^{rx}_{im} R_m R_i \tag{IV-7}$$

Now $R_i \dot{R}_i$ represents the scalar product of the position vector with the velocity vector of a point which is in circular motion with respect to the origin. Hence these vectors must be perpendicular and their scalar product is zero, therefore

$$0 = \omega^{rx}_{im} R_m R_i \tag{IV-8}$$

Since $R_m R_i$ is a symmetric tensor of second order, it follows from equation (IV-8) that ω^{rx}_{im} is a skew-symmetric tensor of second order. Recalling the duality property discussed in section II, it will be remembered that (in three-dimensional space) there may be associated with any skew-symmetric second order tensor, a vector. The relations were

$$\omega_{im} = \varepsilon_{ijm} \omega_j \tag{IV-9},$$

$$\omega_i = \tfrac{1}{2}\varepsilon_{ijk}\omega_{kj} \tag{IV-10},$$

explicitly $\omega_1 = \omega_{32}$, $\omega_2 = \omega_{13}$, $\omega_3 = \omega_{21}$

and from equation (IV-5)

$$\omega_{im} = \dot{A}^{rx}_{ji} A^{rx}_{jm} \tag{IV-11}.$$

It may be concluded that ω^{rx}_{im} represents the angular velocity tensor containing the rotation relations between the X^r_i and X_i frames and that

$$\omega^{rx}_i = \tfrac{1}{2}\varepsilon_{ijk}\omega^{rx}_{kj} \tag{IV-12}$$

is the angular velocity vector of the X^r_i frame with respect to the X_i frame. Equation (IV-6) may be written

$$\dot{R}_i = \varepsilon_{ijm}\omega^{rx}_j R_m \tag{IV-13},$$

which states that the velocity of the point P in the X_i frame due to the rotation of the X^r_i frame is equal to the vector product of the angular velocity vector of the X^r_i frame and the position vector of the point P, a result familiar from vector kinematics. Equation (IV-3) may be stated as

$$\dot{R}_i = A^{rx}_{ji}\dot{R}^r_j + \omega^{rx}_{im} R_m \tag{IV-14},$$

and equation (IV-2) as

$$\dot{S}_i = \dot{Q}_i + A^{rx}_{ji}\dot{R}^r_j + \omega^{rx}_{im} R_m \tag{IV-15}.$$

Equation (IV-15) may be regarded as the velocity equation for the case of general spatial motion. It is easily reduced for application to simpler cases.

The acceleration equation for the case of general spatial motion may be obtained by again differentiating equation (IV-2).

$$\ddot{S}_i = \ddot{Q}_i + \ddot{R}_i \tag{IV-16}$$

To obtain the term $\ddot{R}_i$ differentiate equation (IV-14) term by term

$$\ddot{R}_i = A_{ji}^{rx}\ddot{R}_j^r + \dot{A}_{ji}^{rx}\dot{R}_j^r + \omega_{im}^{rx}\dot{R}_m + \dot{\omega}_{im}^{rx}R_m \tag{IV-17}$$

Substituting equation (IV-14) for $\dot{R}_m$ we have

$$\ddot{R}_i = A_{ji}^{rx}\ddot{R}_j^r + \dot{A}_{ji}^{rx}\dot{R}_j^r + \omega_{im}^{rx}A_{jm}^{rx}\dot{R}_j^r + \omega_{im}^{rx}\omega_{mn}^{rx}R_n + \dot{\omega}_{im}^{rx}R_m \tag{IV-18}$$

Now from equation (IV-11) we have

$$\omega_{im}^{rx}A_{jm}^{rx} = \dot{A}_{ki}^{rx}A_{km}^{rx}A_{jm}^{rx} = \dot{A}_{ki}^{rx}\delta_{jk} = \dot{A}_{ji}^{rx} \tag{IV-19}$$

Then equation (IV-18) may be written

$$\ddot{R}_i = A_{ji}^{rx}\ddot{R}_j^r + 2\omega_{im}^{rx}A_{jm}^{rx}\dot{R}_j^r + \omega_{im}^{rx}\omega_{mn}^{rx}R_n + \dot{\omega}_{im}^{rx}R_m \tag{IV-20}$$

Equation (IV-16) then becomes

$$\ddot{S}_i = \ddot{Q}_i + A_{ji}^{rx}\ddot{R}_j^r + 2\omega_{im}^{rx}A_{jm}^{rx}\dot{R}_j^r + \omega_{im}^{rx}\omega_{mn}^{rx}R_n + \dot{\omega}_{im}^{rx}R_m \tag{IV-21}$$

Equation (IV-21) is the acceleration equation for the case of general spatial motion. Examination of the terms in equation (IV-21) leads to the identifications

$A_{ji}^{rx}\ddot{R}_{j}^{r}$ acceleration of the point of interest relative to the X_{i}^{r} frame,

$2\omega_{im}^{rx}A_{jm}^{rx}\dot{R}_{j}^{r}$ coriolis acceleration,

$(\omega_{im}^{rx}\omega_{mn}^{rx}R_{n} + \dot{\omega}_{im}^{rx}R_{m})$ centrifugal acceleration.

a. Moving Frames

When a moving frame is defined as in the development of the previous section the general velocity and acceleration equations may be applied as given. Modifications follow for less general cases. Referring again to Figure 23, the following special cases may be considered:

X_{i}^{r} frame not rotating relative to the X_{i} frame $(\omega_{im}^{rx} = 0)$

$$\dot{S}_{i} = \dot{Q}_{i} + A_{ji}^{rx}\dot{R}_{j}^{r} \qquad \text{(IV-22)}.$$

R_{j}^{r} vector not rotating relative to X_{i}^{r} frame $(\dot{r}_{j}^{r} = 0)$, from equation (IV-22)

$$\dot{S}_{i} = \dot{Q}_{i} + A_{ji}^{rx}\dot{R}r_{j}^{r} \qquad \text{(IV-23)}.$$

R_{j}^{r} vector rotating relative to X_{i}^{r} frame but with constant magnitude $(\dot{R} = 0)$, from equation (IV-22)

$$\dot{S}_{i} = \dot{Q}_{i} + A_{ji}^{rx}R\dot{r}_{j}^{r} \qquad \text{(IV-24)}.$$

The above special cases are easily applied to the general acceleration equation (IV-21)

b. <u>Ground Frame Differentiation</u>

When a problem (position solution) is obtained entirely in terms

of ground frame variables equation (IV-2) and (IV-16) may be employed

as they are. Differentiation of the velocity equation for example

takes the form

$$\dot{S}s_i + S\dot{s}_i = \dot{Q}q_i + Q\dot{q}_i + \dot{R}r_i + R\dot{r}_i \qquad \text{(IV-25)}.$$

c. <u>Differentiation of Unit Vectors</u>

Differentiation of a unit vector referenced to the ground **frame**

follows directly and has the form

$$\dot{r}_i = (\dot{\phi}^r \cos\phi^r \cos\theta^r - \dot{\theta}^r \sin\phi^r \sin\theta^r),$$

$$(\dot{\phi}^r \cos\phi^r \sin\theta^r + \dot{\theta}^r \sin\phi^r \cos\theta^r),$$

$$(-\dot{\phi}^r \sin\phi^r) \qquad \text{(IV-26)}.$$

Differentiation of a unit vector referenced to a rotating frame

follows from equation (IV-14) and has the form

$$\dot{r}_i = A_{ji}^{rx} \dot{r}_j^r + \omega_{im}^{rx} r_m \qquad \text{(IV-27)}$$

where

$$\dot{r}_j^r = (\dot{\phi}^{rr} \cos\phi^{rr} \cos\theta^{rr} - \dot{\theta}^{rr} \sin\phi^{rr} \sin\theta^{rr}),$$

$$(\dot{\phi}^{rr} \cos\phi^{rr} \sin\theta^{rr} + \dot{\theta}^{rr} \sin\phi^{rr} \cos\theta^{rr}),$$

$$(-\dot{\phi}^{rr} \sin\phi^{rr}) \qquad \text{(IV-28)}.$$

If the r_j^r unit vector is fixed in the X_i^r frame the term r_j^r is zero and

$$\dot{r}_i = \omega_{im}^{rx} r_m$$

(IV-29)

2. Relative and Angular Velocities and Accelerations

The relative velocity and acceleration of any point in a mechanism with respect to another point may be obtained by writing the position vector from one point to the other and differentiating. When a closed form solution is not available for the vectors involved the relative velocity and acceleration expressions may be obtained and solved at the points of the cycle of operation for which the position solutions were numerically obtained.

It often occurs that the angular velocity and acceleration of a particular link or links of a mechanism is of interest. The angular velocities and accelerations of interest may be relative to ground or relative to another point in the mechanism. Both of these quantities for any link may be easily obtained once the position solution is available. Recall the angular velocity relation previously developed

$$\omega_{im}^{rx} = \dot{A}_{ji}^{rx} A_{jm}^{rx}$$

(IV-30).

Differentiating equation (IV-30)

$$\dot{\omega}_{im}^{rx} = \ddot{A}_{ji}^{rx} A_{jm}^{rx} + \dot{A}_{ji}^{rx} \dot{A}_{jm}^{rx}$$

(IV-31).

From equations (IV-30) and (IV-31) the angular velocity and acceleration of any link vector in the mechanism loop relative to ground

may be obtained.

If the appropriate A_{ij} for the vector of interest have not been developed in the course of the position solution they may be constructed using the link vector itself in the vector product manner developed in the discussion of the tetrahedron problem. If the position solution has been accomplished, it is always possible to construct the A_{ij} for that link and hence the angular velocity and acceleration for any link relative to ground may be constructed.

It can be shown that there exists a first order motion analogue of the zeroth order condition, sum of position vectors equal zero. This condition is the sum of relative angular velocities of link vectors around the mechanism loop is zero. That is

$$\omega_i^{rc} + \omega_i^{sr} + \omega_i^{ts} + \ldots + \omega_i^{cn} = 0 \qquad (IV-32).$$

Equation (IV-32) follows from the conditions

$$\omega_i^{rc} = \omega_i^{rx} \qquad (IV-33),$$

$$\omega_i^{rc} + \omega_i^{sr} = \omega^{sx} \qquad (IV-34),$$

$$\omega_i^{rc} + \omega_i^{sr} + \omega_i^{ts} = \omega_i^{tx} \qquad (IV-35).$$

Equation (IV-30) and equations (IV-33, 34, 35) may be used to obtain explicit expressions for any of the relative angular velocities. Similar conditions may be derived for angular accelerations by differentiating equations (IV-32 through 35) and employing equation (IV-31).

B. General Solution Procedure

Assume that the position problem has been solved either in closed form or for discrete steps of the input variables. Write the position vector for each point of interest and differentiate. Define the problem as to input velocities and identify the unknowns. In general, the unknowns will be terms such as $\dot{S}$ and $\dot{s}_i$ which are functions of the unknown vector variables $\dot{\theta}^S$ and $\dot{\phi}^S$. These may be obtained as functions of the input velocity terms by differentiating the position solution expressions for θ^S and ϕ^S. The problem of obtaining explicit expressions for the velocity vectors of interest is a linear algebraic one and may proceed without difficulty. Acceleration solutions may be obtained by once again differentiating the required expressions and assigning input accelerations. Angular quantities of interest may be obtained as outlined in section IV-A-2.

Where a position solution has been obtained at discrete points of the mechanism cycle, the instantaneous velocity and acceleration solutions may be obtained as outlined above at the same point of cycle.

C. Application to Velocity and Acceleration Solutions

1. RSSP Mechanism

Figure 24 depicts a RSSP mechanism which was analyzed as Case 1 of section III-B. The unknowns in the problem were T and s_i. The solutions obtained were

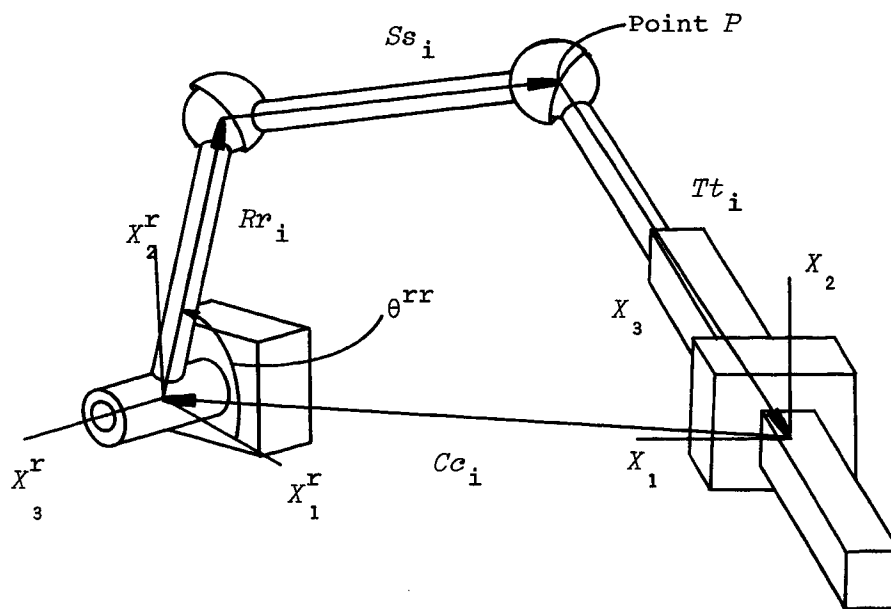$$s_1 = \frac{-K_1}{S} = \sin\phi^S \cos\theta^S \qquad\qquad \text{(IV-36)},$$

FIGURE 24

RSSP MECHANISM IN THE CONFIGURATION USED FOR THE VELOCITY AND

ACCELERATION ANALYSIS OF THE POINT P.

Figure 24

RSSP MECHANISM

$$s_2 = \frac{-K_2}{S} = \sin\phi^s \sin\theta^s \qquad \text{(IV-37)},$$

$$T = \left| K_3 + S s_3 \right| \qquad \text{(IV-38)}.$$

The r_i vector is defined to be at 90° to its revolute axis so that the vector r_i^r may be expressed as a function of the single azimuthal angle θ^{rr}

$$r_i^r = \cos\theta^{rr}, \ \sin\theta^{rr}, \ 0 \qquad \text{(IV-39)},$$

where θ^{rr} is then the input variable. Since the ground frame and the X_i^r frame as shown in Figure 24 are fixed relative to each other the A_{ij}^{rx} are constant. They may be obtained through the use of vector products. The $(X_3^r)_i$ vector is known in the ground frame and may be described as

$$(X_3^r)_i = \sin\phi \ \cos\theta, \ \sin\phi \ \sin\theta, \ \cos\phi \qquad \text{(IV-40)},$$

where ϕ and θ are known. Define then

$$(X_1^r)_i = \frac{\varepsilon_{ijk}(X_3^r)_j (X_3)_k}{\left| \varepsilon_{ijk}(X_3^r)_j (X_3)_k \right|} = \sin\theta, \ -\cos\theta, \ 0 \qquad \text{(IV-41)},$$

$$(X_2^r)_i = \frac{\varepsilon_{ijk}(X_3^r)_j (X_1^r)_k}{\varepsilon_{ijk}(X_3^r)_j (X_1^r)_k} = \cos\phi \ \cos\theta, \ \cos\phi \ \sin\theta, \ - \sin\phi \qquad \text{(IV-42)}.$$

The A_{ij}^{rx} are thus defined and the vector r_i may be written

$$r_i = A_{ji}^{rx} r_j^r \qquad \text{(IV-43)},$$

where

$$r_1 = A_{11}\cos\theta^{rr} + A_{21}\sin\theta^{rr} \qquad \text{(IV-44)},$$

$$r_2 = A_{12}\cos\theta^{rr} + A_{22}\sin\theta^{rr} \qquad \text{(IV-45)},$$

$$r_3 = A_{23}\sin\theta^{rr} \qquad \text{(IV-46)}.$$

The input link is now expressed in the ground frame as a function of the single variable θ^{rr} which is the input parameter. The direction of the $(X_1^r)_i$ axis is specified by equation (IV-41) and θ^{rr} is measured from this axis.

It is desired to find the velocity and acceleration of point P as shown in Figure 24. The position vector of this point may be written as

$$P_i = Cc_i + Rr_i + Ss_i \qquad \text{(IV-47)},$$

or as $\qquad P_i = -Tt_i \qquad \text{(IV-48)}.$

Differentiation of either equation will result in the velocity of the point P relative to ground. Differentiating equation (IV-47) we have

$$\dot{P}_i = R\dot{r}_i + S\dot{s}_i \qquad \text{(IV-49)}.$$

The terms $\dot{r}_i$ may be obtained from equations (IV-44, 45, 46) as functions of the input parameters θ^{rr} and $\dot{\theta}^{rr}$. The terms $\dot{s}_i$ are obtained as follows. From equation (IV-36 and 37)

$$\dot{s}_1 = \frac{-\dot{K}_1}{S} \qquad \text{(IV-50)},$$

$$\dot{s}_2 = \frac{-\dot{K}_2}{S} \qquad \text{(IV-51)}.$$

Now $K_i = Cc_i + Rr_i$ then

$$\dot{K}_i = R\dot{r}_i \qquad \text{(IV-52)}.$$

Also $s_3 = \cos\phi^S$, and

$$\dot{s}_3 = -\dot{\phi}^S \sin\phi^S \qquad \text{(IV-53)}.$$

Squaring and adding equations (IV-36 and 37) results in

$$\sin\phi^S = \frac{\sqrt{k_1^2 + k_2^2}}{S} \qquad \text{(IV-54)}.$$

Differentiating equation (IV-54) results in an expression for $\dot{\phi}^S$ in terms of known parameters. Expanding equation (IV-42) and employing equations (IV-50, 51 and 52) we have for the components of the velocity of point P

$$\dot{P}_1 = 0 \qquad \text{(IV-55)}$$

$$\dot{P}_2 = 0 \qquad \text{(IV-56)}$$

$$\dot{P}_3 = R\dot{r}_3 - S\dot{\phi}^S \sin\phi^S \qquad \text{(IV-57)}.$$

The acceleration of point P may be obtained by differentiating equation (IV-57)

$$\ddot{P}_3 = R\ddot{r}_3 - S[(\dot{\phi}^S)^2 \cos\phi^S + \ddot{\phi}^S \sin\phi^S] \qquad \text{(IV-58)}.$$

The terms $\ddot{r}_3$ and $\ddot{\phi}^s$ may be obtained by differentiating equations (IV-46 and 54) respectively. In this simple case, the velocity and acceleration of interest were found in closed form due to the fact that the position solution was effected in closed form.

2. RCCC Mechanism

Next consider the RCCC mechanism of case 8, section III-B-1. As defined the unknowns were S, T, D, t_i. The position solution resulted in a single transcendental equation in the unknown θ^t. In order to effect a numerical solution, the parameters considered to be known must be assigned values. Referring to Figure 25 and case 8 of section III-B-1 the following values were assigned

$C\ = 4$

$R\ = 1$

$C_i\ = 0,\ -\cos 45°,\ \sin 45°$

The axis of the r_i vector, labeled ξ is selected to be in the X_1, X_2 plane and at the 30° orientation shown so that the input variable may be measured directly as ϕ^r. The r_i vector is defined to be at 90° to the ξ axis, therefore, the azimuthal angle θ^r is a known constant. The bend in the driving link is such that r_i, s_i and   are coplanar. The angle between the R and S links is assigned the value 120°. Therefore, s_i may be written as a function of r_i. The constant parameters, P and Q, are defined to be
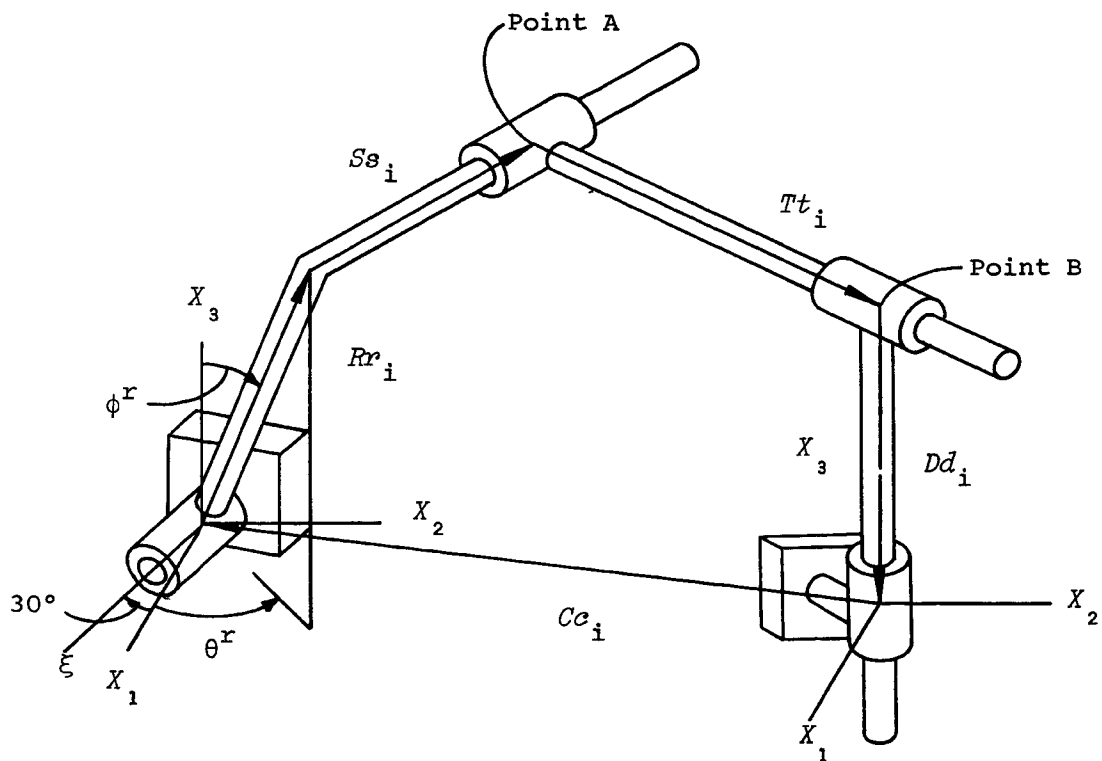
$P\ =\ \cos 90°$

$Q\ =\ \cos 60°$

FIGURE 25

RCCC MECHANISM IN THE CONFIGURATION USED FOR THE VELOCITY AND ACCEL-
ERATION ANALYSIS OF THE POINTS A AND B.

Figure 25

RCCC   MECHANISM

The known parameters are now suitably defined. The unknowns may be obtained from the position solution as follows:

$$t_3 = -Q = \cos\phi^t \qquad\qquad\text{(IV-59)}$$

$$s_1 \sin\phi^t \cos\theta^t + s_2 \sin\phi^t \sin\theta^t + s_3 t_3 = P \qquad\text{(IV-60)}$$

Equation (IV-60) may be solved for the unknown θ^t by the Newton iterative technique as outlined in appendix I. The remaining unknowns are from the position solution

$$S = \frac{K_1 t_2 - K_2 t_1}{s_1 t_2 - s_2 t_1} \qquad\qquad\text{(IV-61)}$$

$$T = \frac{k_1 s_2 - k_2 s_1}{t_1 s_2 - t_2 s_1} \qquad\qquad\text{(IV-62)}$$

$$D = K_3 + S s_3 + T t_3 \qquad\qquad\text{(IV-63)}$$

A solution by the above procedure was obtained for each discrete increment of the input variable ϕ^r in steps of 5° and through a complete cycle of operation $(0° \simeq \phi^r \simeq 360°)$. Once this set of solutions is complete the instantaneous velocities and accelerations of the points of interest may be derived and numerical solutions obtained using the position solution at each of the points of cycle. Points of interest are selected as point A and B as shown in Figure 25. Writing the position vectors we have

$$A_i = C c_i + R r_i + S s_i \qquad\qquad\text{(IV-64)},$$

$$B_i = -D d_i \qquad\qquad\text{(IV-65)}.$$

The velocities are

$$\dot{A}_i = \dot{R}r_i + \dot{S}s_i + S\dot{s}_i \qquad \text{(IV-66)},$$

$$\dot{B}_i = -\dot{D}d_i \qquad \text{(IV-67)},$$

and the accelerations

$$\ddot{A}_i = \ddot{R}r_i + \ddot{S}s_i + S\ddot{s}_i + 2\dot{S}\dot{s}_i \qquad \text{(IV-68)},$$

$$\ddot{B}_i = -\ddot{D}d_i \qquad \text{(IV-69)}.$$

The various derivatives in the above expressions are readily obtained by direct differentiation from expressions already established. These velocities and accelerations were programmed and solutions obtained at the 5° steps using the input data $\dot{\phi}^r = 1$ radian / sec and $\ddot{\phi}^r = 0$. The components of the velocity and acceleration as well as the magnitudes are shown in Figures 26, 27 and 28.

3. Canard Mechanism

Returning to the canard deployment mechanisms of section III-B-2 it is now possible to examine some velocity and acceleration characteristics of the mechanism configuration as defined in the position solution. Of primary interest is the angular velocity and acceleration of the spar vector w_i.

Referring to Figure 21, it can be seen that the angular velocity of the spar vector may be measured in the X_i^t frame directly as $\dot{\theta}^{wt}$. Recall that the t_i^t and w_i^t vectors are both in the X_1^t, X_2^t plane and

FIGURE 26

VELOCITY OF POINT A

THE MAGNITUDE $(\dot{A})$ AND THE THREE COMPONENTS OF THE VELOCITY OF POINT A
VERSUS THE INPUT PARAMETER ϕ^r FOR A COMPLETE CYCLE OF OPERATION OF
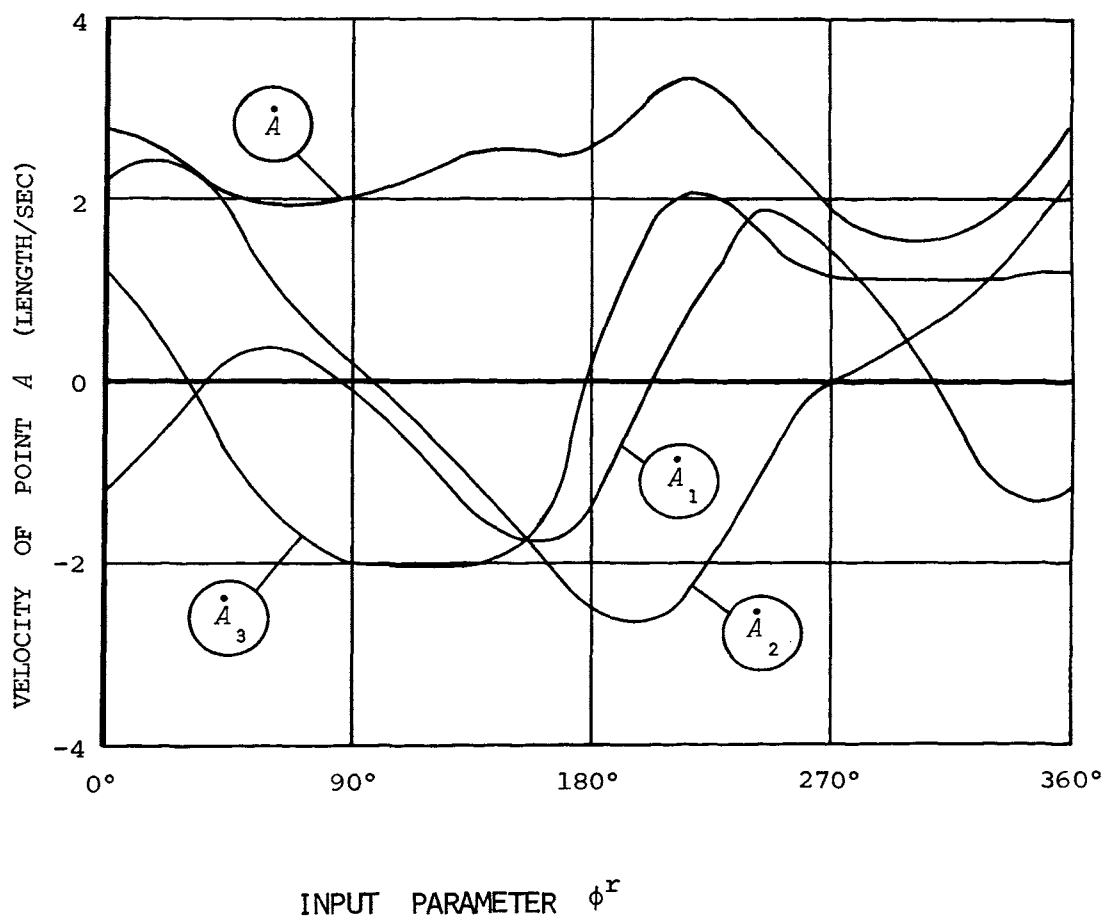THE RCCC MECHANISM.

Figure 26

FIGURE 27

THE MAGNITUDE $(\ddot{A})$ AND THREE COMPONENTS OF THE ACCELERATION OF POINT A VERSUS THE INPUT PARAMETER ϕ^r FOR A COMPLETE CYCLE OF OPERATION OF THE RCCC MECHANISM.
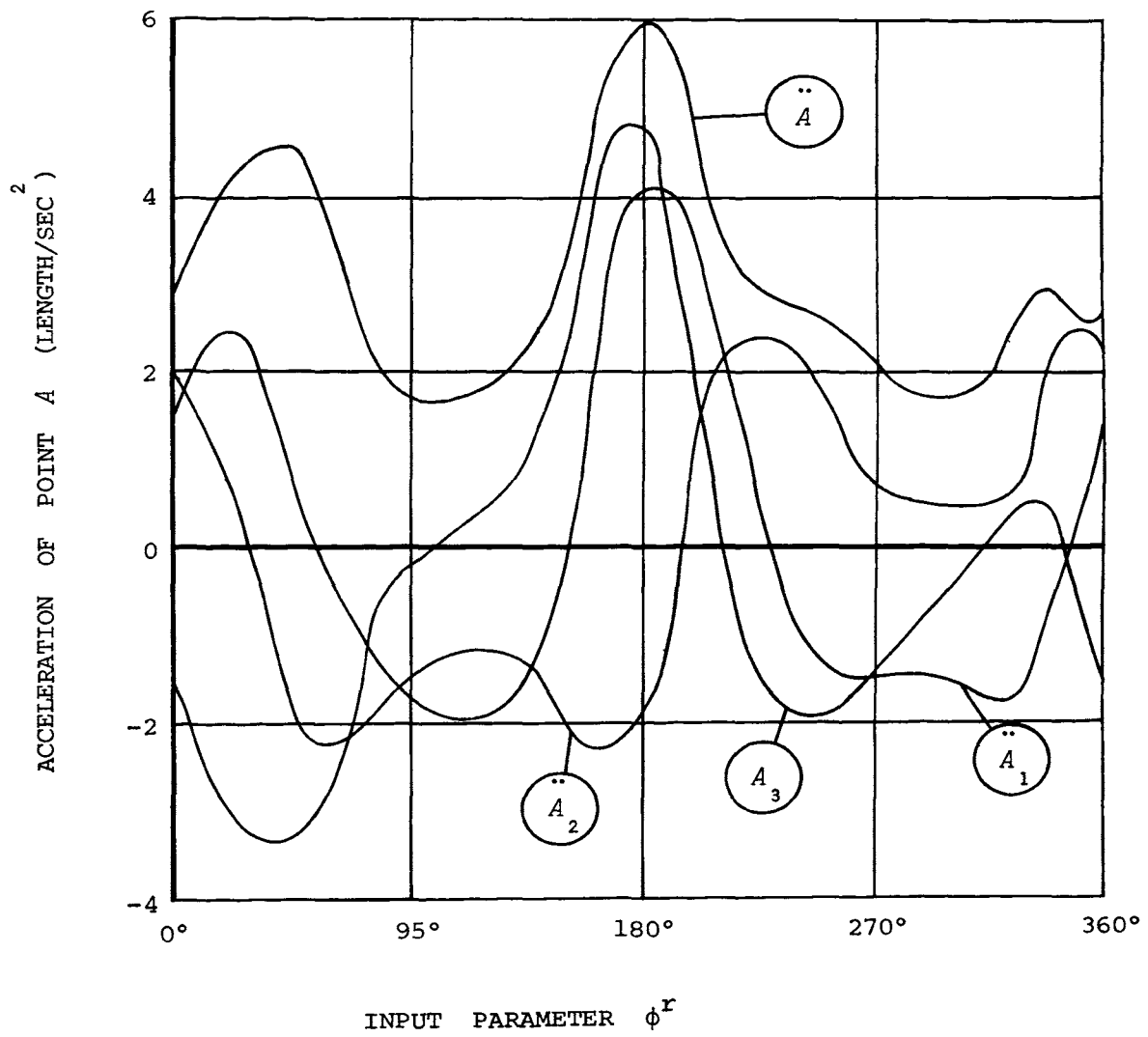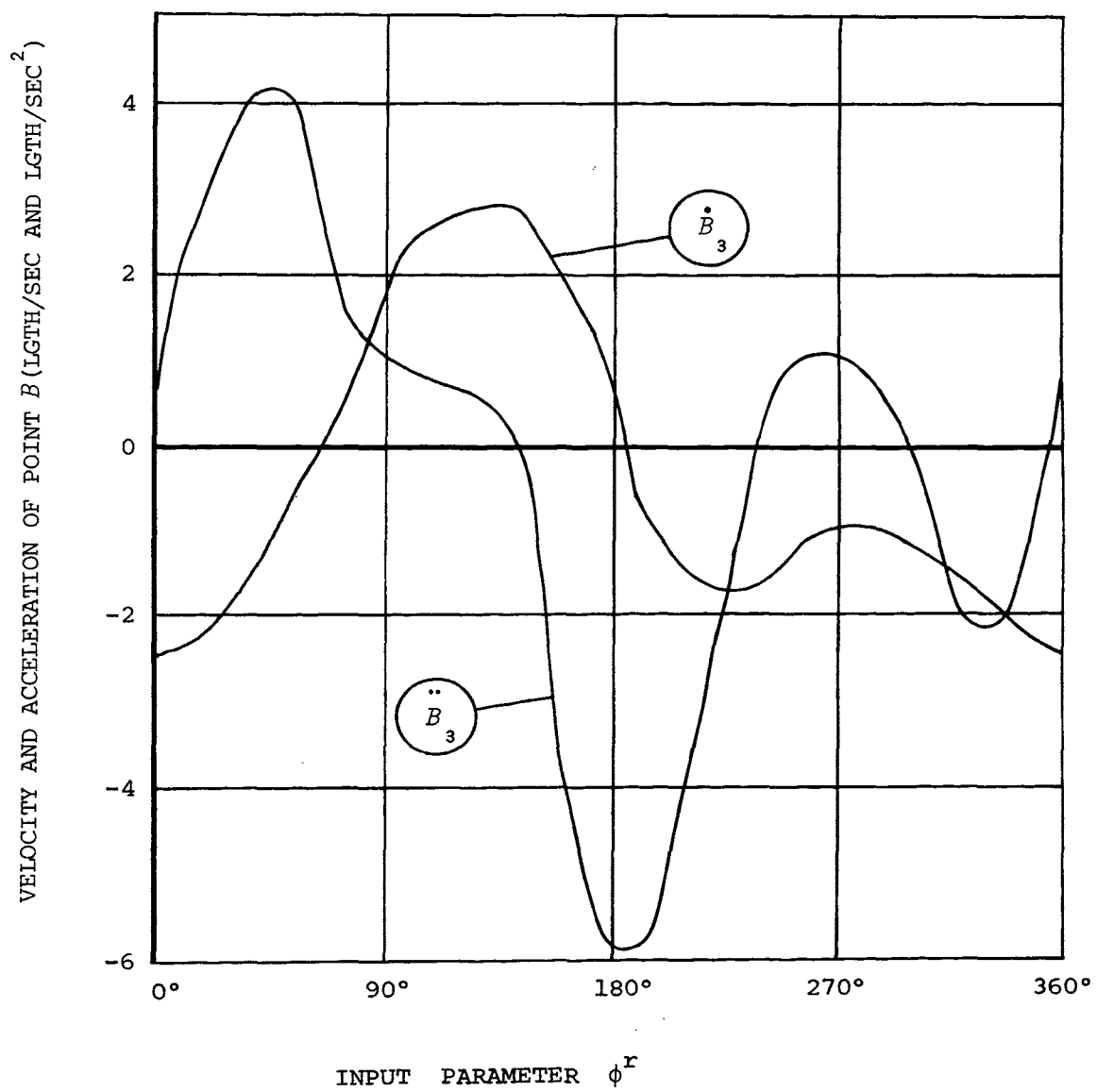
Figure 27

FIGURE 28

THE VELOCITY AND ACCELERATION OF POINT B VERSUS THE INPUT PARAMETER

ϕ^r FOR A COMPLETE CYCLE OF OPERATION OF THE RCCC MECHANISM. ONLY

THE THIRD COMPONENT IS NON-ZERO DUE TO THE MECHANISM CONFIGURATION.

Figure 28

that $\theta^{wt} = \theta^{tt} + \xi$ where ξ is a constant parameter. Therefore, $\dot{\theta}^{wt} = \dot{\theta}^{tt}$ and we may obtain $\dot{\theta}^{tt}$ from the position solution by differentiation. Since the X_i^t frame is fixed with respect to the X_i frame we have from equations (III-212 and 213)

$$\dot{K}_1^t + A(\dot{\phi}^{st} \cos\phi^{st} \cos\theta^{st} - \dot{\theta}^{st} \sin\phi^{st} \sin\theta^{st}) - B\dot{\theta}^{tt} \sin\theta^{tt} = 0 \quad \text{(IV-70)}$$

$$\dot{K}_2^t + A(\dot{\phi}^{st} \cos\phi^{st} \sin\theta^{st} + \dot{\theta}^{st} \sin\phi^{st} \cos\theta^{st}) + B\dot{\theta}^{tt} \cos\theta^{tt} = 0 \quad \text{(IV-71)}$$

The unknowns in equations (IV-70 and 71) are $\dot{\theta}^{st}$ and $\dot{\theta}^{tt}$. These equations are linear in these unknowns and they may be algebraically solved. Eliminating $\dot{\theta}^{st}$ the expression is obtained

$$\dot{\theta}^{tt} = -(\dot{K}^t \sin\theta^{st} + \dot{K}_1^t \cos\theta^{st} + A\dot{\phi}^{st} \cos\phi^{st})/$$
$$B(\sin\theta^{tt} \cos\theta^{st} - \cos\theta^{tt} \sin\theta^{st}) \quad \text{(IV-72)}$$

The various derivatives in equation (IV-72) may be obtained from the position solution equations as functions of the linear velocity of the jack screw length, $\dot{E}$. A second differentiation of equation (IV-72) yields an expression for $\ddot{\theta}^{tt}$ as a function of $\dot{E}$ and $\ddot{E}$.

The above angular velocity and acceleration terms were included in the general computer program for the canard solution. Input parameters were selected as $\dot{E} = .1$ unit/sec and $\ddot{E} = 0$ which represents a reasonable constant linear velocity of the jack screw drive.

Figures 29 and 30 depict the computer plots of $\dot{\theta}^{wt}$ and $\ddot{\theta}^{wt}$ versus the position of the driving link E for a complete cycle of the mechanism (canard fully deployed to fully stowed).

FIGURE 29

PLOT OF ANGULAR VELOCITY $\overset{\bullet}{\theta}{}^{wt}$ VERSUS THE LENGTH OF THE JACKSCREW

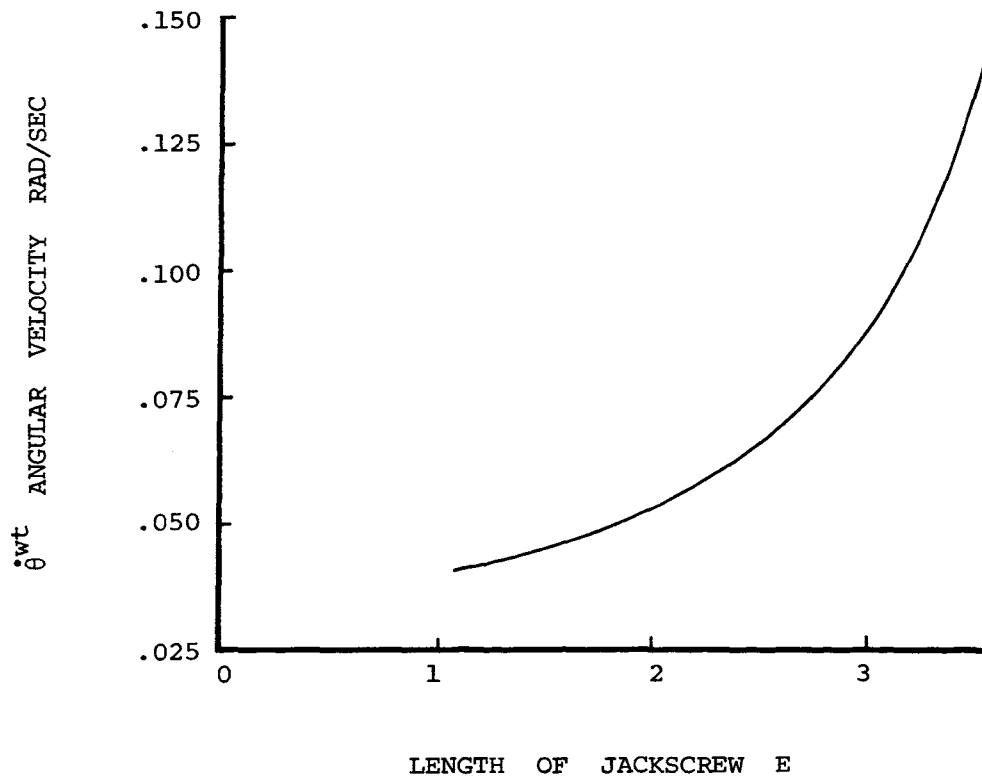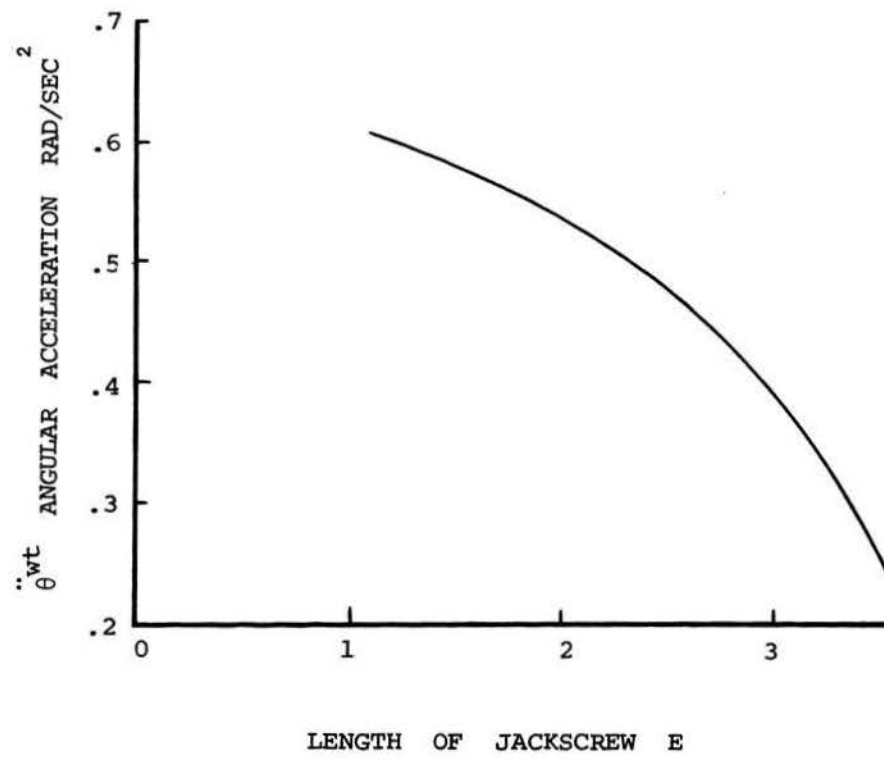LINK E. REPRESENTS THE ANGULAR VELOCITY OF THE SPAR VECTOR w_i^t.

Figure 29

FIGURE 30

PLOT OF ANGULAR ACCELERATION $\ddot{\theta}^{wt}$ VERSUS THE LENGTH OF THE JACK-
SCREW LINK E. REPRESENTS THE ANGULAR ACCELERATION OF THE SPAR
VECTOR w_i^t.

Figure 30

V

CONCLUSIONS

The cartesian tensor analysis has been shown to be a compact and tractable tool for the study of kinematic concepts and in particular the spatial kinematics. The difficulties of the position solution by other methods are largely alleviated by the brevity of the tensor notation and operations.

The method of approach to a particular problem may be quickly arrived at with a minimum of experience in handling tensor equations. In addition, the tensor analysis reduced to cartesian form has the advantage of being not a new concept but merely a more succinct method of handling familiar concepts.

Many of the familiar vector operations are introduced in the form of definitions with unclear origin. With the vector analysis in tensor form, this is not necessary as the operations arise as natural consequences, for example, scalar and vector products and the formulae for differentiation with respect to moving coordinate frames. Also, the algebraic character of vector equations is explicit in tensor form requiring a minimum of manipulation to arrive at solutions.

The combination of the tensor method and the simple numerical procedures required provide a powerful tool for the solution of a large class of spatial mechanism problems. The tensor notation due to its algorithmic character is extremely conducive to ease of computer programming allowing its use by engineers with lesser sophisticated programming experience.

In todays times significant engineering effort is spent in the kinematic analysis of mechanisms such as gears, cams, and linkages. Although linkages present a more complex problem of analysis than other basic mechanisms, they are widely employed due to their reliability, speed and force transmission properties. Industry continually seeks to devise linkages for new mechanical systems and to improve existing linkages. If engineers have knowledge of spatial concepts and the analytical tools at their disposal, they may be encouraged to try the use of spatial mechanisms which have few joints and few links in an intricate system in order to obtain an optimum design. In practice many spatial motions are arrived at through the over use of spherical joints which are much more difficult to manufacture than the simple revolute and cylindrical joints which might replace them in a properly designed spatial mechanism.

Besides the ordinary machine design, spatial mechanisms can be utilized in many other areas. A recent series of moon landing spacecrafts, for example, were equipped with spatial mechanisms in their solar panels and landing gear actuators. Spatial mechanisms are extensively used in the automotive industry, particularly in suspension systems. In the future, the excursion vehicles of planetary or ocenographic exploration will inevitably use spatial mechanisms.

The practitioners of medical science constantly search for a better understanding of human body motions. The continuing development of spatial kinematics certainly would give an improved knowledge of kinesiology.

VI

APPENDICES

A. Newton - Raphson Method

Consider the set of n equations in n unknowns

$$f^1 (X_1, X_2, \ldots, X_n) = 0$$

$$f^2 (X_1, X_2, \ldots, X_n) = 0$$

(A-1)

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f^n (X_1, X_2, \ldots, X_n) = 0$$

Assume that the set of values $X_1^*, X_2^*, \ldots, X_n^*$ be a solution to this system and let

$$X_1^* = X_1^o + \Delta X_1$$

$$X_2^* = X_2^o + \Delta X_2$$

(A-2)

$$\ldots\ldots\ldots\ldots$$

$$X_n^* = X_n^o + \Delta X_n$$

where $X_1^o, X_2^o, \ldots, X_n^o$ are known approximate solutions.

Expand the original functions about these approximations via Taylor series to yield

$$f^1 (X_i^*) = 0 = f^1 (X_i^o) = f_{x_1}^1 \Delta X_1 + f_{x_2}^1 \Delta X_2 + \ldots + f_{x_n}^1 \Delta X_n$$

$$f^2(X_i^*) = 0 = f^2(X_i^o) + f^2_{x_1}\Delta X_1 + f^2_{x_2}\Delta X_2 + \ldots + f^2_{x_n}\Delta X_n$$

(A-3)

$$\cdots \cdots \cdots \cdots \cdots \cdots$$

$$f^n(X_i^*) = 0 = f^n(X_i^o) + f^n_{x_1}\Delta X_1 + f^n_{x_2}\Delta X_2 + \ldots + f^n_{x_n}\Delta X_n$$

where $f^n_{x_n} = \partial f^n(X_i)/\partial X_n$ and the partial derivatives in equations (A-3) are understood to be evaluated at the values X_i^o. Higher order terms in ΔX_i are neglected.

Rearranging equations (A-3) yields

$$-f^1(X_i^o) = f^1_{x_1}\Delta X_1 + \ldots + f^1_{x_n}\Delta X_n$$

$$-f^2(X_i^o) = f^2_{x_1}\Delta X_1 + \ldots + f^2_{x_n}\Delta X_n$$

(A-4)

$$\cdots \cdots \cdots \cdots \cdots$$

$$-f^n(X_i^o) = f^n_{x_1}\Delta X_1 + \ldots + f^n_{x_n}\Delta X_n$$

Provided the determinant of the above system, equation (A-4), is non-zero, this system of linear equations may be solved for the ΔX_i.

Using Cramer's rule an expression for each of the ΔX_i is obtained, for example, ΔX_1

$$\Delta X_1 = \frac{\begin{vmatrix} -f^1(X_i^\circ) & f^1_{x_2} & \cdots & f^1_{x_n} \\ -f^2(X_i^\circ) & f^2_{x_2} & \cdots & f^1_{x_n} \\ \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ -f^n(X_i^\circ) & f^n_{x_2} & \cdots & f^n_{x_n} \end{vmatrix}}{J(f^1, f^2, \ldots, f^n)}$$

where J is the Jacobian of the system.

Having found the ΔX_i equations (A-2) are used to form the improved value of the root. These improved values are not exact because higher order terms were neglected in the expansion. However, the new values are improved and the process may be repeated to any desired degree of accuracy. The recursion formula takes the form

$$X_n^{i+1} = X_n^i + \left| \frac{\det}{J} \right|_i$$

where X^{i+1} represents the (i + 1) approximation to the nth unknown variable.

While the foregoing development may seem detailed and cumbersome it is developed for a n by n system of equations. In practice most spatial mechanism problems result in a system of few equations for which case the Newton method is quite straightforward and tractable, both in terms of mechanical manipulation and computer programming.

B. Convergence of the Newton - Raphson Method

A set of conditions sufficient to ensure convergence is the following [20].

1. f^1, f^2, ..., f^n and all their derivatives through order n are continuous and bounded in a region R containing the solutions.

2. The Jacobian of the system does not vanish in R.

3. The initial starting values are chosen sufficiently close to the roots.

In the experience of the author starting values are uncritical and may be selected by visualization of the initial configuration of a given mechanism. Convergence of the Newton method is quadratic and solutions are obtained in very little computer time.

VII

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VIII

VITA

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