Development Length Requirements for Fully Developed Laminar Pipe Flow of Yield Stress Fluids

R. J. Poole
Department of Engineering,
University of Liverpool,
Brownlow Street,
Liverpool L69 3GH, United Kingdom
e-mail: robpoole@liv.ac.uk

R. P. Chhabra
Department of Chemical Engineering,
Indian Institute of Technology,
Kanpur 208016, India
e-mail: chhabra@iitk.ac.in

In this technical brief, we report the results of a systematic numerical investigation of developing laminar pipe flow of yield stress fluids, obeying models of the Bingham-type. We are able to show that using a suitable choice of the Reynolds number allows, for high Reynolds number values at least, the development length to collapse to the Newtonian correlation. On the other hand, the development length remains a weak, nonmonotonic, function of the Bingham number at small values of the Reynolds number (Re≤40). [DOI: 10.1115/1.4001079]

1 Introduction

Notwithstanding the long (and continuing) debate about the very existence of a "true" yield stress, it is readily acknowledged that the notion of an apparent yield stress is a very useful engineering empiricism for a wide range of materials [1–3]. These materials appear solidlike below some critical (yield) stress but flow above this value. Here we are interested in the development length problem for such fluids, i.e., the length of pipe required for the flow of such fluids to become "fully developed." Although this problem is classical, and has been investigated repeatedly, only relatively recently have accurate results become available even for Newtonian [4] and non-Newtonian inelastic fluids (obeying the power-law model [5]). For pipe flow, Durst et al. [4] proposed the following correlation for Newtonian fluids

$$X_D/D = [(0.619)^{1.6} + (0.0567\ \mathrm{Re})^{1.6}]^{1/1.6} \quad (1)$$

where X_D is the development length (m), D is the pipe diameter (m), and Re is the Reynolds number. For power-law fluids a modification to Eq. (1) was proposed to account for the low Reynolds number dependence on the power-law index (consult Ref. [5] for details).

For visco-plastic fluids, although a number of studies have investigated this issue either analytically or numerically [6–11], with the exception of Ookawara et al. [11], the results of these studies have ignored the diffusion-dominated case (i.e., low Reynolds number) and proposed correlations of the form $X_D/D = C(\mathrm{Re})$, where C is a function of the nondimensional yield stress (usually represented as a Bingham number). Thus they incorrectly predict that for creeping flows (i.e., $\mathrm{Re}\rightarrow 0$) the flow instantaneously develops. In addition, as Ref. [11] highlight, the occurrence of the "plug" region means that although the centerline velocity reaches 99% of its fully developed value in a rather shorter distance than the Newtonian case, the complete radial variation in the velocity at this location is not yet fully developed (see their Fig. 1 for example). Thus one of the conclusions of Vradis et al. [9] that "the velocity profiles develop faster with higher values of the yield (Bingham) number" is essentially incorrect. To overcome this difficulty, Ookawara et al. [11] redefined the entry length as the axial distance where the velocity at a radial position of 95% of the plug radius reaches 99% of the calculated maximum velocity (at the same radial location). Although Ookawara et al. [11] highlighted this important issue and provided a correlation that also predicts the development length for low Reynolds numbers (<10), their yield stress results (obtained using the Bingham model) are restricted to just five simulations and their correlation is independent of the Bingham number, something which, at low Reynolds number at least, seems unrealistic, given the results for the power-law model [5]. In this technical brief we report the results of a detailed numerical study, of quantified accuracy, which attempts to reconcile these issues.

2 Nondimensional Groups

To investigate the yield stress effects in the laminar developing pipe flow, we use here a Bingham-type approach, e.g., a model of the form $\tau=\tau_0+\mu_p\dot{\gamma}$, where τ is the shear stress (Pa), τ_0 is the yield stress (Pa), $\dot{\gamma}$ is the shear rate (s^{-1}), and μ_p is the plastic viscosity (Pa s). To quantify the importance of the yield stress we use the well-known Bingham number

$$\mathrm{Bn} = \frac{\tau_0 D}{\mu_p U_B} \quad (2)$$

where D is the pipe diameter (m), U_B is the bulk velocity (m/s), and ρ is the density (kg/m^3).

Guided by the results of Ookawara et al. [11] we use the following definition for the Reynolds number based on the momentum correction coefficient method[1] [11]

$$\mathrm{Re}(\mu_a,\zeta) = \frac{\rho U_B D}{\mu_a}\zeta \quad (3)$$

where $\mu_a=3\mu_p/(a^4-4a+3)$, $\zeta=9(5+6a-11a^2)/5(3+2a+a^2)^2$, and a is the relative plug radius for the Bingham model [12].

3 Numerical Method

We assume that the flow is laminar, incompressible, steady, and axisymmetric (i.e., two-dimensional). We utilize the commercial package FLUENT to solve the governing equations of the conservation of mass and momentum. The differencing schemes used are both formally second-order in accuracy: central differencing is used for the diffusive terms and a second-order up-winding scheme for the convective terms. Coupling of the pressure and velocity was achieved using the well-known semi-implicit method for pressure-linked equations (SIMPLE) implementation of Patankar [13]. The default "Bingham" model in FLUENT utilizes a biviscosity model (see Ref. [14] for example) of the form

$$\tau = \mu_{\text{yield}}\dot{\gamma} \qquad \dot{\gamma} < \frac{\tau_0}{\mu_{\text{yield}}}$$

$$\tau = \tau_0 + \mu_P\left[\dot{\gamma} - \frac{\tau_0}{\mu_{\text{yield}}}\right] \qquad \dot{\gamma} \geq \frac{\tau_0}{\mu_{\text{yield}}} \quad (4)$$

i.e., for low shear rates the material acts as a very viscous liquid (equal to the "yielding" viscosity μ_{yield}) rather than a true solid.

Contributed by the Fluids Engineering Division of ASME for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received July 22, 2009; final manuscript received December 14, 2009; published online March 17, 2010. Assoc. Editor: Hassan Peerhossaini.

[1]This is the Reynolds number obtained when the friction-factor Reynolds number relationship is forced to be equal to the Newtonian one in a laminar flow (e.g., $f(\mathrm{Re})=16$ for a pipe). It is the Bingham model equivalent of the Metzner-Reed Reynolds number for power-law fluids.

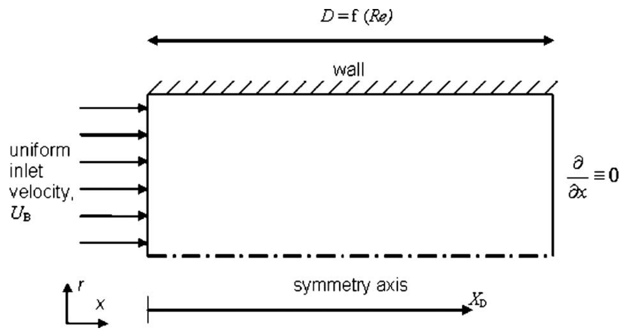

Fig. 1 Schematic of the computational domain and boundary conditions

We also investigate here the role played by the exact value of this yielding viscosity. As the default model is discontinuous, we also developed a user defined function (UDF) for the apparent viscosity to incorporate the exponential model due to Papanastasiou [15]

$$\tau = \tau_0(1 - e^{-m\dot{\gamma}}) + \mu_P \dot{\gamma} \quad (5)$$

where m is the stress growth exponent, which has the dimensions of time. Again this model transforms the "solid" region to a viscous one of high viscosity. It is usually thought that $mU/D > 500$ is sufficient for Eq. (5) to mimic the Bingham model successfully [15]. We also note that although Eqs. (4) and (5) approximate the true Bingham model over a wide range of shear rates [3], both models are fundamentally different to a true "yield stress" model as deformation occurs below the yield stress. As recent analytical work [16] has suggested that for a true Bingham model the development length may be "infinitely delayed," and the continuing debate regarding the existence of a true yield stress [2], we feel that the use of Eqs. (4) and (5) to approximate a yield stress fluid is reasonable.

A schematic representation of the computational domain is given in Fig. 1. At inlet ($x=0$), we apply a uniform velocity U_B and as discussed in the introduction, we define the development length X_D as the axial distance required for the velocity to reach 99% of the calculated maximum value at a radial location corresponding to 95% of the plug radius. We use the well-known no-slip boundary condition at the wall and impose zero axial gradients at the outlet. The length of the domain is dependent on the Reynolds and Bingham numbers of the flow in question ($L = f(\mathrm{Re}, \mathrm{Bn})$). Broadly, longer domain lengths were necessary than in the case of Newtonian or power-law fluids [5].

A preliminary series of calculations at low Re (=0.001) were carried out to assess the effects of mesh refinement, domain length, choice of Bingham-like model (i.e., Eq. (4) and (5)), value of yielding viscosity, and nondimensional value of the m parameter. Our coarse mesh, which is 10 diameters in length, corresponds to mesh M2 of our previous study [5] and comprises 20×200 cells, our base mesh "M3" comprises 40×400 cells, and our refined mesh "M4" comprises 80×800 cells. The cells are quadrilateral and of constant dimension $\Delta x = 2\Delta r$. We note that our base mesh here is of lower refinement than our results for the power-law model: a consequence of the longer domain lengths and the significantly increased time for convergence required for the Bingham-type models used here. In addition to the variation in X_D, to allow us to estimate the accuracy of the various conditions, we define a relative error

$$E = \frac{u_c - U_{C,\mathrm{FD}}}{U_{C,\mathrm{FD}}} \quad (6)$$

where u_c is the calculated centerline velocity (m/s) at the outlet plane and $U_{C,\mathrm{FD}}$ is the corresponding fully developed analytical value (m/s). The analytical solution for fully developed pipe flow of a Bingham fluid is well-known (see Ref. [12] for example) and so is not repeated here. The results of this series of calculations are shown in Tables 1–4 and from these information, we conclude the following: (a) Our coarse mesh (M2) shows discrepancies with the analytical fully developed solution, and therefore, all remaining calculations were conducted using M3 (simulations using M4 were prohibitively expensive and were not pursued above creeping-flow conditions, limited results are provided in Table 4); (b) although for a given mesh E exhibited little sensitivity to the yielding viscosity and stress growth exponent, the development length was more effected; (c) despite the bi-viscosity and exponential models becoming independent of these parameters in a given mesh, there is still a difference of about 2% between the development lengths predicted for nominally identical conditions (Re and Bn); and (d) although at low Reynolds numbers X_D is shorter for yield stress fluids than in the Newtonian case, longer

Table 1 Effect of various parameters using mesh M2 and domain length of 10*D* (NC=4000 cells, Re=0.001)

	u_c/U_B	E (%)	X_D/D
	Bi-viscosity (Bn=1)		
$\mu_{\mathrm{yield}}/\mu_p=10^1$	1.865	0.26	0.6282
$\mu_{\mathrm{yield}}/\mu_p=10^2$	1.852	−0.45	0.5832
$\mu_{\mathrm{yield}}/\mu_p=10^3$	1.850	−0.53	0.5757
$\mu_{\mathrm{yield}}/\mu_p=10^4$	1.850	−0.53	0.5753
	Exp. model (Bn=1)		
$mU/D=37$	1.852	−0.43	0.6269
$mU/D=370$	1.850	−0.53	0.5916
$mU/D=3700$	1.850	−0.54	0.5899
	Bi-viscosity (Bn=10)		
$\mu_{\mathrm{yield}}/\mu_p=10^4$	1.427	−0.59	0.5834
$\mu_{\mathrm{yield}}/\mu_p=10^5$	1.426	−0.60	0.5804
	Exp. model (Bn=10)		
$mU/D=1050$	1.426	−0.61	0.6267
$mU/D=10,500$	1.426	−0.60	0.6044
$mU/D=105,000$	1.426	−0.63	0.5985

Table 2 Effect of various parameters using mesh M3 and domain length of 10*D* (NC=16,000 cells, Re=0.001)

	u_c/U_B	E (%)	X_D/D
	Bi-viscosity (Bn=1)		
$\mu_{\mathrm{yield}}/\mu_p=10^1$	1.872	0.67	0.6305
$\mu_{\mathrm{yield}}/\mu_p=10^2$	1.859	−0.06	0.6000
$\mu_{\mathrm{yield}}/\mu_p=10^3$	1.857	−0.14	0.5923
$\mu_{\mathrm{yield}}/\mu_p=10^4$	1.857	−0.16	0.5922
	Exp. model (Bn=1)		
$mU/D=37$	1.859	0.04	0.6297
$mU/D=370$	1.858	−0.13	0.6077
$mU/D=3700$	1.858	−0.13	0.6066
	Bi-viscosity (Bn=10)		
$\mu_{\mathrm{yield}}/\mu_p=10^4$	1.434	−0.10	0.5852
$\mu_{\mathrm{yield}}/\mu_p=10^5$	1.431	−0.30	0.5695
$\mu_{\mathrm{yield}}/\mu_p=10^6$	1.432	−0.23	0.5731
	Exp. model (Bn=10)		
$mU/D=10,500$	1.432	−0.23	0.5916
$mU/D=105,000$	1.432	−0.23	0.5910

Table 3 Effect of domain length and the Bingham number for M3 using the bi-viscosity model and $\mu_{yield}/\mu_p=10^4$ (Re=0.001)

Bn=1	X_D/D	Bi=3.16	X_D/D	Bi=10	X_D/D
2*D*	0.5703	2*D*	0.4839	2*D*	0.5371
5*D*	0.5904	5*D*	0.5033	5*D*	0.5620
10*D*	0.5922	10*D*	0.5087	10*D*	0.5852
Bn=2		Bi=5			
2*D*	0.5008	2*D*	0.4997		
5*D*	0.5352	5*D*	0.5182		
10*D*	0.5440	10*D*	0.5258		

Table 4 Effect of mesh refinement using the bi-viscosity model and Re=0.001

Bn=1 ($\mu_{yield}/\mu_p=10^4$)	X_D/D	Bn=3.16 ($\mu_{yield}/\mu_p=10^4$)	X_D/D	Bn=10 ($\mu_{yield}/\mu=10^5$)	X_D/D
M2 10*D*	0.5753	M2 10*D*	0.5238	M2 10*D*	0.5804
M3 10*D*	0.5922	M3 10*D*	0.5087	M3 10*D*	0.5695
M4 10*D*	0.6011	M4 10*D*	0.5113	M4 10*D*	0.5766
Extrapolated	0.6047	Extrapolated	0.5123	Extrapolated	0.5975
Bn=2 ($\mu_{yield}/\mu_p=10^4$)		Bn=5 ($\mu_{yield}/\mu_p=10^4$)			
M2 10*D*	0.5417	M2 10*D*	0.5361		
M3 10*D*	0.5440	M3 10*D*	0.5258		
M4 10*D*	0.5498	M4 10*D*	0.5320		
Extrapolated	0.5521	Extrapolated	0.5345		

domain lengths are required for X_D to become independent of this domain length (see data in Table 3).

We use the bi-viscosity model for the rest of our simulations with a yield viscosity ratio (μ_{yield}/μ_p) of at least 10^4 and 10^5 for Bn=10. The data in Tables 1 and 2 show that the use of the Papanastasiou model would produce very similar results. Given all of the effects investigated we believe that the uncertainty in our estimation of X_D is no better than 2%.

To highlight the quality of the simulations, in Fig. 2, we show the radial variation in the outlet axial velocity profile from our simulations for Re=0.001 (bi-viscosity model, $\mu_{yield}/\mu_p=10^4$, mesh M3 and domain length of 10*D*) compared with the analytical solutions for the fully developed flow. Excellent agreement can be seen with $|E|$ at most 0.3%.

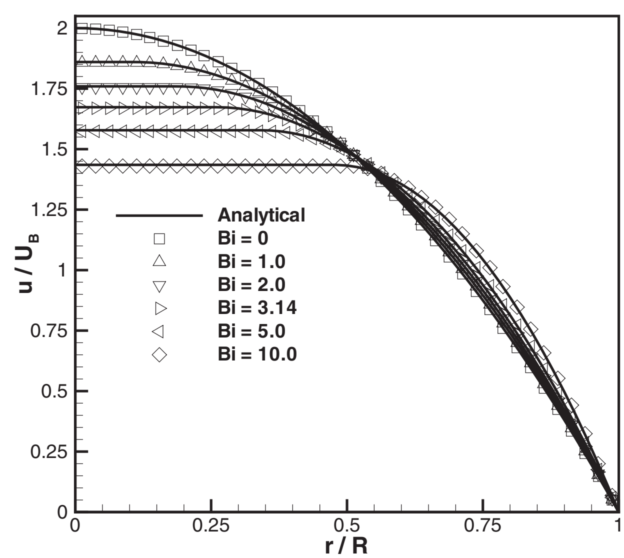

Fig. 2 Comparison of the numerical simulation velocity profiles at the pipe exit with the fully developed analytical solution at Re=0.001 for a range of Bingham numbers (bi-viscosity model, $\mu_{yield}/\mu_p=10^4$, mesh M3 10*D* length)

4 Results and Conclusions

The nondimensional development length is plotted as a function of the modified Reynolds number, Eq. (3), in Fig. 3 for a range of Bingham numbers $1<\text{Bn}<10$. At higher values of Bn convergence became increasingly difficult and, as discussed in Ref. [3], as most interesting viscoplastic phenomena occuring in the range $1<\text{Bn}<10$ simulations at higher Bn were not pursued. For Bn =0.1 we found that the results were practically indistinguishable from our Newtonian data (Bn=0) and are, therefore, not included

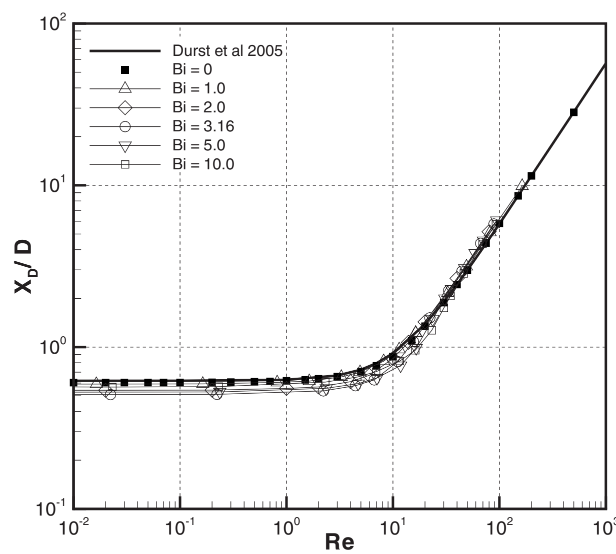

Fig. 3 Development length variation for Bingham fluids

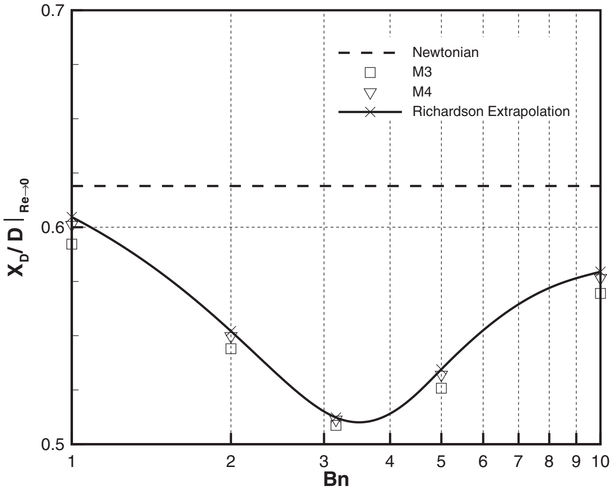

Fig. 4 Variation in the creeping-flow development length with the Bingham number

here. Much as was observed for non-Newtonian power-law fluids [5], above a critical value of Re, the data collapse to the Newtonian correlation [4]. Below this critical value of Re, which appears to be about 40 at the highest Bn, the development length departs from the Newtonian correlation in a nonmonotonic fashion dependent on the Bingham number. To highlight this variation in Fig. 4 we plot the creeping flow development length (i.e., $Re \to 0$) versus the Bingham number (the Richardson extrapolation values shown in Fig. 4 and quantified in Table 4 were determined using the method outlined in Ref. [5]). The variation in the development length with Bn is complex; as the yield stress effects increase (i.e., as Bn increases) the central solid "plug" region grows—effectively changing the pipe diameter that the fluid "sees"—and the characteristic diffusion velocity also changes. Simple scaling arguments such as that used for power-law fluids [5] did not help reconcile this behavior. As the differences are relatively small—the maximum departure from the Newtonian correlation being about 20%—we propose that the Newtonian correlation (Eq. (1)) can be used for engineering purposes (estimating entrance effects, for example, when designing pipe flow systems [17]), provided that the momentum corrected Reynolds number is used in lieu of the Newtonian Reynolds number.

References

[1] Bird, R. B., Dai, G. C., and Yarusso, B. J., 1983, "The Rheology and Flow of Viscoplastic Materials," Rev. Chem. Eng., **1**, pp. 2–70.
[2] Barnes, H. A., 1999, "The Yield Stress—A Review or '$\pi\alpha\nu\tau\alpha$ $\rho\epsilon\iota$'—Everything Flows?," J. Non-Newtonian Fluid Mech., **81**, pp. 133–178.
[3] Mitsoulis, E., 2007, "Flows of Viscoplastic Materials: Models and Computations," *Rheology Reviews*, D. M. Binding, N. E. Hudson, and R. Keunings, eds., The British Society of Rheology, Glasgow, pp. 135–178.
[4] Durst, F., Ray, S., Unsal, B., and Bayoumi, O. A., 2005, "The Development Lengths of Laminar Pipe and Channel Flows," ASME J. Fluids Eng., **127**, pp. 1154–1160.
[5] Poole, R. J., and Ridley, B. S., 2007, "Development Length Requirements for Fully-Developed Laminar Pipe Flow of Inelastic Non-Newtonian Liquids," ASME J. Fluids Eng., **129**, pp. 1281–1287.
[6] Chen, S. S., Fan, L. T., and Hwang, C. L., 1970, "Entrance Region Flow of the Bingham Fluid in a Circular Pipe," AIChE J., **16**, pp. 293–299.
[7] Shah, V. L., and Soto, R. J., 1975, "Entrance Flow of a Bingham Fluid in a Tube," Appl. Sci. Res., **30**, pp. 271–278.
[8] Nowak, Z., and Gajdeczko, B., 1983, "Laminar Entrance Flow of a Bingham Fluid," Acta Mech., **49**, pp. 191–200.
[9] Vradis, G. C., Dougher, J., and Kumar, S., 1993, "Entrance Pipe Flow and Heat Transfer for a Bingham Plastic," Int. J. Heat Mass Transfer, **36**, pp. 543–552.
[10] Min, T., Choi, H. G., Yoo, J. Y., and Choi, H., 1997, "Laminar Convective Heat Transfer of a Bingham Plastic in a Circular Pipe—II. Numerical Approach—Hydrodynamically Developing Flow and Simultaneously Developing Flow," Int. J. Heat Mass Transfer, **40**, pp. 3689–3701.
[11] Ookawara, S., Ogawa, K., Dombrowski, N., Amooie-Foumeny, E., and Riza, A., 2000, "Unified Entry Length Correlation for Newtonian, Power Law and Bingham Fluids in Laminar Pipe Flow at Low Reynolds Number," J. Chem. Eng. Jpn., **33**, pp. 675–678.
[12] Chhabra, R. P., and Richardson, J. F., 2008, *Non-Newtonian Flow and Applied Rheology*, 2nd ed., Butterworth-Heinemann, Oxford.
[13] Patankar, S. V., 1980, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, Washington, DC.
[14] O'Donovan, E. J., and Tanner, R. I., 1984, "Numerical Study of the Bingham Squeeze Film Problem," J. Non-Newtonian Fluid Mech., **15**, pp. 75–83.
[15] Papanastasiou, T. C., 1987, "Flow of Materials With Yield," J. Rheol., **31**, pp. 385–404.
[16] Al Khatib, M. A. M., and Wilson, S. D. R., 2001, "The Development of Poiseuille Flow of a Yield Stress Fluid," J. Non-Newtonian Fluid Mech., **100**, pp. 1–8.
[17] Escudier, M. P., Poole, R. J., Presti, F., Dales, C., Nouar, C., Desaubry, C., Graham, L., and Pullum, L., 2005, "Observations of Asymmetrical Flow Behaviour in Transitional Pipe Flow of Yield-Stress and Other Shear-Thinning Liquids," J. Non-Newtonian Fluid Mech., **127**, pp. 143–155.