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DEVELOPMENT OF ANGLE CONCEPTS BY PROGRESSIVE ABSTRACTION AND GENERALISATION

ABSTRACT. This paper presents a new theory of the development of angle concepts. It is proposed that children progressively recognise deeper and deeper similarities between their physical angle experiences and classify them firstly into specific situations, then into more general contexts, and finally into abstract domains. An angle concept is abstracted from each class at each stage of development. We call the most general angle concept the *standard angle concept.* To investigate the role of the standard abstract angle concept in conceptual development, 192 children from Grades 2 to 8 were tested to find how they used it in modelling 9 physical angle situations and in expressing similarities between them. It was found that the standard angle concept first develops in situations where both arms of the angle are visible. Even at Grade 8, there are still significant proportions of students who do not use standard angles to represent turning and sloping situations. Implications for theory and practice are explored.

KEY WORDS: abstraction, angle, concepts, conceptual development, Grades 2–8, physical angle situations

1. INTRODUCTION

There is no doubt that angle is a multifaceted concept. Strehl (1983) noted the wide variety of definitions of angle given in German school textbooks of the time, and Lo, Gaddis and Henderson (1996) reported a similar phenomenon in textbooks intended for US preservice elementary teachers. Close (1982) and Krainer (1989) have surveyed the various angle definitions from a historical perspective, and others (Mitchelmore, 1989; Freudenthal, 1983; Krainer, 1989; Roels, 1985; Schweiger, 1986) have classified these definitions on mathematical grounds. Three particular classes of angle definition occur repeatedly: an amount of turning about a point between two lines; a pair of rays with a common end-point; and the region formed by the intersection of two half-planes. Other authors have preferred to base their classifications on physical properties of angle, noting in particular the difference between dynamic (involving movement) and static (configurational) aspects of the concept (Close, 1982; Kieran, 1986; Scally, 1986).

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It is clear from the research literature that school students have great difficulty coordinating the various facets of the angle concept. For example, students do not readily incorporate turning into their angle concepts: Mitchelmore and White (1998a) found that less than 10% of Grade 4 students mentioned turning when asked to give examples of angles, and Foxman and Ruddock (1983) reported that only 4% of 15 year olds spontaneously mentioned rotation when asked to define an angle. Students also do not find it easy to relate turning around a point to turning along a bent path: Clements, Battista, Sarama and Swaminathan (1996) found that many Grade 3 children could not link their body turns to the LOGO turn parameter, and Mitchelmore (1998) found many indications that students in Grades 2–6 did not relate bending to turning. The consistent finding that activities with LOGO have little effect on other aspects of children's angle knowledge (Clements and Battista, 1989; Cope, Smith and Simmons, 1992; Horner, 1984; Hoyles and Sutherland, 1989; Kelly, Kelly and Miller, 1986; Kieran, 1986; Noss, 1987; Scally, 1987) confirms the separation of turning and bending in students' conceptualisations of angle.

Another aspect which students find difficult to link to their other angle concepts is that of slope. For example, Douek (1998) recently described how students in Grades 3 and 4 learned to analyse the inclination of the sun in terms of angles. She writes, "it is astonishing how often inclination and plane angles intervened in . . . similar situations, and how difficult they were to integrate" (p. 271). Secondary school mathematics teachers would probably echo this sentiment in relation to the difficulties of applying trigonometry to angles of inclination.

Despite much research on conceptual development and the formulation of a number of general theories, "the key issue that has plagued work on angles is the lack of a theoretical framework from which to consider the results of various studies" (Davey and Pegg, 1991, p. 1). The only example we have been able to find of an attempt to develop a specific theory of angle concept development is Scally's (1986, 1987) application of van Hiele theory. However, her approach is limited to features of abstract angle diagrams and does not deal with the genesis of these abstractions.

The purpose of the present paper is to present a new theory which attempts to relate children's angle concepts explicitly to their physical angle experiences and to account specifically for their difficulties in coordinating different aspects of the angle concept. Our theoretical basis will be the process of concept formation by abstraction, and we shall describe angle conceptual development as a three-stage sequence of abstractions. We shall also describe an empirical study intended to use the theory to throw light on the path by which students might come to integrate their various angle concepts.

2. Concept formation by abstraction

The key ideas underlying our approach to concept formation are classification, similarity, abstraction and concept. The relation between these ideas has been concisely summarised by Skemp (1986):

> *Abstracting* is an activity by which we become aware of similarities . . . among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. . . . To distinguish between abstracting as an activity and abstraction as its end-product, we shall . . . call the latter a *concept*. (p. 21, italics in original)

We shall also argue that the formation of everyday concepts, elementary mathematical concepts, and advanced mathematical concepts proceeds in similar but clearly different ways. In the following sections, we summarise some general aspects which will be crucial to our theory of angle concept formation.

2.1. *Everyday concepts*

Children learn to classify objects at an early age. The similarities which link objects in everyday classes frequently relate to the objects' purpose and are not usually definable in terms of single attributes. "Real-world attributes, unlike the sets often presented laboratory subjects, do not occur independently of one another" (Rosch, 1977, p. 213). Everyday objects are classified on the basis of a cluster of attributes. Some objects (prototypes) clearly possess all the most important attributes of the class; others possess enough of the relevant attributes to a sufficient degree to be judged a member of that class. Yet other objects have a weak selection of the relevant attributes, and there may be arguments as to how they should be classified.

At some point, children start to treat classes of objects as single entities. Children who say 'A cup holds water', or play the 'scissors, paper, stone' game with their hands, are operating with classes and not with individual objects. Words such as *cup* and *scissors* now denote concepts – new mental objects, created out of classes of concrete objects, which can be related to each other without reference to specific, concrete objects (Greeno, 1983).

Piaget calls the process of forming everyday concepts from a class of everyday objects *empirical abstraction*. Piaget's view of concept formation as abstracting or detaching the essential qualities of a class of objects from the individual objects (Piaget, 1970) has been common since Aristotle (Damerow, 1996). It is difficult to reconcile this view with the finding that everyday concepts are not definable in terms of clear attributes. We shall simply regard abstraction as the formation of a new mental object that represents a class of objects or experiences, consistent with von Humboldt's view of abstraction as "grasping as a unit what was just presented" (cited by von Glasersfeld, 1991, p. 47).

Everyday classes may be formed at various levels, and may be separated into subclasses or combined into superordinate classes. The concept associated with a superordinate class (e.g., colour) is said to be a higher order than the concepts associated with the component classes (e.g., red, blue, . . .), which are in turn of a higher order than the subconcepts (e.g., red shoe, red tie, . . .). In this way, hierarchies of everyday concepts may be formed.

Classes may also grow by accretion. For example, having formed the concept of *colour* by abstraction from familiar examples such as red, blue, green and yellow, a child may later add new examples such as turquoise, magenta and mauve. As a result, the initial concept becomes richer without changing in any substantial manner. Such a process is called *generalisation.*

We note that the terms generalisation and abstraction are often used interchangeably in the literature. The essential difference, as we see it, is that abstraction creates a new mental object (a concept) whereas generalisation extends the meaning of an existing concept.

2.2. *Elementary mathematical concepts*

Elementary mathematical concepts are formed through the same process of classification and abstraction as everyday concepts, but with one important difference: the objects which are classified are not only concrete objects nor even mental objects (everyday concepts), but also relations between everyday objects or concepts. For example, the number concept is the result of abstracting, not only properties of sets of objects, but also properties of operations on those sets. Piaget wrote extensively on the resulting difference between logico-mathematical and everyday concepts, referring particularly to the child's (physical or mental) action in manipulating (concrete and mental) objects in order to bring them into relation: "The [mathematical] abstraction is drawn not from the object that is acted upon, but from the action itself" (Piaget, 1970, p. 16). Piaget called this kind of abstraction *reflective abstraction*, and emphasised that it was essentially a constructive process – not only constructing new mental objects but also building up a structure relating them (Piaget, 1975, p. 206). Vygotsky (1987) makes a similar distinction between everyday concepts and scientific concepts.

As with everyday concepts, generalisation plays an important role in the formation of elementary mathematical concepts. For example, children's first experience of fractions is limited to the familiar half and quarter. At some point, a generalisation to unitary fractions with an arbitrary denominator occurs, followed later by a further generalisation to composite fractions. Progressive abstraction also plays an important role as higher-level mathematical concepts are formed by abstraction from existing concepts. For example, young children often form separate concepts of *whole number* and *fraction*; they make a significant advance when they form a superordinate concept of *number* which includes both whole numbers and fractions. In both cases, the difference between mathematical and everyday concept formation is that concepts and relations are abstracted simultaneously.

Another difference is the increasing use of definitions in elementary mathematics. For example, young children may only be taught to recognise a circle visually – but by secondary school they are expected to be able to define it in terms of its centre and its radius. As analysis proceeds, the essential attributes of existing mathematical concepts begin to be embedded in definitions. However, verbal definitions are ineffective in teaching new concepts – not only in mathematics (Miller and Gildea, 1987; Vinner, 1991). As Skemp (1986, p. 25) puts it, "concepts of a higher order than those which people already have cannot be communicated to them by a definition".

2.3. *Formal mathematical concepts*

Advanced mathematical concepts can be regarded as abstractions from elementary mathematical concepts. For example, *group*, *ring* and *field* may be regarded as abstractions from structured sets of numbers, functions, permutations and other transformations. However, these objects are now defined in terms of properties of undefined operations on undefined elements. We shall call such a construction a formal mathematical concept.

It is important that the definition of a formal mathematical concept captures the essence of the elementary mathematical concepts from which it is abstracted. "An abstract definition of a concept . . . provides no more than a starting point – a starting point, furthermore, that must already have proved its usefulness in concrete situations in order to serve as a truly useful point of departure" (Damerow, 1996, pp. 77–78).

2.4. *Summary*

We have outlined a constructivist theory of mathematics learning by progressive abstraction. In our view, concepts which occur late in this progression supplement and do not replace concepts acquired earlier: the links between a concept, relations involving that concept, and the class of objects from which it is abstracted remain crucial. Our theory therefore has much in common with the idea of *situated abstraction* described by Noss and Hoyles (1996).

A fuller description of concept learning would require, in addition to our theory, a complementary sociocultural perspective which describes "the conditions for the possibility of learning" (Cobb, 1994, p. 13). Nevertheless, we believe that progressive abstraction provides an adequate framework for describing cognitive growth and hence for interpreting learning in any particular social, cultural or pedagogical environment. In particular, we shall now show how it provides a framework for describing the development of students' angle concepts.

3. ANGLE LEARNING BY PROGRESSIVE ABSTRACTION

Our theory (Mitchelmore and White, 1998a) describes angle concept development in terms of three, overlapping stages of abstraction which represent a progressively more refined classification of students' experience. We shall briefly describe the three stages below and then show how the theory allows an integration of the research literature.

3.1. *Stage 1: Situated angle concepts*

Many of the everyday concepts preschool children learn (e.g., scissors, tile, cross, merry-go-round, slide, hill, roof, pencil point, bent stick and crane) can be seen by an adult to involve angles. We call these *situated angle concepts*, since they derive from children's mental classification of situations they have experienced. We call the corresponding experiences *physical angle situations*. The term 'situated' is intended to have the same meaning as its use in the term 'situated knowing' (Greeno, 1991). A situated angle concept is strictly limited to situations which look alike, involve similar actions, and are experienced in similar social circumstances.

Situated angle concepts may generalise a little over time as the initial similarity changes to focus more on the physical situation and the actions performed and less on the social circumstances. For example, the concept of a hill may initially be restricted to paths which children have walked up but may be extended later to include roads they have driven up. But there are limits to this development. Children never call a roof a hill – even if they see a tradesman walk up one.

Our investigations (Mitchelmore, 1997; Mitchelmore and White, 1998a) have confirmed that children have formed many situated angle concepts by the start of schooling. For example, 6 year olds readily recognise a model of a hill and show an excellent understanding of the effect of variations in its steepness.

3.2. *Stage 2: Contextual angle concepts*

Our investigations also show that during early elementary school most children learn to use words such as 'slope' and to classify physical angle situations into what we call *physical angle contexts* using these terms. For example, when they are asked to given examples of 'things which slope like this hill', they mention a wide range of sloping situations – including roofs. We interpret this to mean that children have at some stage recognised a similarity between hills, roofs and so on, and have come to call this similarity 'slope'. (No doubt they first recognised the similarity between a small number of situations and then generalised it to others.) We have also found some evidence that children can reason about slopes in general, thus indicating that 'slope' has become a mental object in its own right – a concept. We call such concepts *contextual angle concepts*.

We initially hypothesised the existence of 14 physical angle contexts involving slopes, turns, intersections, corners, bends, directions and openings (Mitchelmore and White, 1998a). Subsequent empirical investigations (Mitchelmore, 1997, 1998; Mitchelmore and White 1998a) have confirmed the existence of many of these contexts. They have also indicated that some physical angle contexts overlap to an extent which diminishes with age, whereas others never overlap. For example, children often fail to distinguish plane corners (such as the corner of a table top) from solid corners (such as the edge of a table top); X-shaped from V-shaped intersections; and limited from unlimited turns. But no child has ever suggested that a bent object (such as a bend in the road) turns in the same way as a fan and a door. Our findings suggest that most children have formed clear and distinct contextual angle concepts of slope, turn, intersection and corner by the age of 9 years but that their concept of bend is still vague.

Physical angle contexts are formed on the basis of a common geometrical configuration and similar physical actions, and not on similarities between physical or mental operations on these configurations. For example, the concept of turn is abstracted from the movement of turning – but this is an action of a physical object and not an action imposed by the learner on a physical object. Contextual angle concepts are therefore formed by empirical abstraction and are best regarded as second-order everyday concepts. There is little to indicate a mathematical concept of angle at this stage. For example, children rarely need to define terms such as *corner, slope* and *turn*.

3.3. *Stage 3: Abstract angle concepts*

Although children regard angle contexts as distinct, they also recognise similarities between them. Studies with small samples of Grade 4 children (Mitchelmore, 1997; Mitchelmore and White, 1998a) have indicated that most recognise a similarity between intersections and bends, and about half recognise similarities between slopes and corners or between turns, intersections and bends.

We claim that the recognition of similarities between angle contexts is the beginning of an elementary mathematical concept of angle. For, unlike similarities between situations in a physical angle context, the similarities between contexts in an abstract angle domain may not be at all obvious. Recognition of such a similarity often requires a physical or mental action on the part of the learner. For example, students often demonstrate the similarity between a tile and a ramp by physically moving a corner of the tile to fit into the space between the ramp and the horizontal base. Again, a turn is seen to be similar to a corner by holding the initial position of the turning object in memory. The recognition of similarities between different angle contexts is therefore a constructive process requiring reflective abstraction.

We call a class of physical angle contexts which a child recognises as similar an *abstract angle domain*. If the similarity becomes abstracted to form a concept, we call it an *abstract angle concept*. Table I represents a hypothetical cognitive structure resulting from such abstractions. This child recognises (1) a similarity between intersections, corners, bends and slopes, and (2) a similarity between limited and unlimited turns. This hypothetical child thus recognises two abstract angle domains – but, at this point in time, sees no relation between them.

By asking children to show or tell us what they find similar between different physical angle contexts, we have found evidence of several abstract angle concepts. One is that of a point, sometimes used by young children to relate corners and intersections. Another is that of a single sloping line, sometimes used to relate sloping and turning objects. But by far the most common is that of two inclined lines meeting at a point, used to relate a wide variety of contexts. This angle concept is implied, for example, in the first abstract angle domain in Table I.

We shall call the last-mentioned concept the *standard angle concept* and the class of contexts it relates the *standard angle domain*. The signi-

TABLE I

Hypothetical example of a student's abstract angle domains

Domain	Physical angle contexts	Sample physical angle situations
1	Intersection	Road junction, scissors, hands of a clock
	Corner	Table top, pattern block, pencil point, table edge
	Bend	Road bend, boomerang, bent limb
	Slope	Slanting pole, railway signal, roof, hillside
2	Limited rotation	Door, windscreen wiper, water tap
	Unlimited rotation	Ceiling fan, revolving door, lighthouse beam

ficance of this concept is that it has the potential to effectively relate all physical angle contexts. In different angle contexts, the two lines forming the angle may be indicated by linear objects, by straight edges, or by imaginary lines – but there are always two lines. Also, the inclination between the two lines represents different things in different contexts – for example, sharpness, steepness or amount of turning – but there is always some significance attached to the relative inclination. The standard angle concept is also the most general in the sense that, apart from these two features (two lines meeting at a point and their relative inclination), there is nothing else which is common to all physical angle contexts. The other abstract angle concepts we have found (angle-as-point and angle-as-sloping-line) are ineffective because they cannot always relate the sizes of angles in different contexts.

It would appear that the standard angle concept develops slowly. For example, we have found (Mitchelmore, 1998) that about a third of Grade 6 students are unable to recognise any angular similarity between turns and bends; so their standard angle domain must be rather limited in scope. We conjecture that the students' standard angle concept generalises during secondary school, and by adulthood may or may not include all common physical angle contexts.

We claim that the angle concept dealt with in elementary mathematics is the standard angle concept we have defined above. The construction of the corresponding formal mathematical concept could be considered as the fourth stage of angle concept development, but that would take us beyond the confines of this paper.

3.4. *Summary*

Our theory comprises three successive stages of abstraction. Similarities between experiences are abstracted to form situated angle concepts, similarities between situated angle concepts are abstracted to form contextual angle concepts, and similarities between contextual angle concepts are abstracted to form abstract angle concepts. These are not exclusive stages: children may still be in the process of forming contextual angle concepts at the time they form their first abstract angle concept. Further, each concept is in a constant state of generalisation as new experiences, situations, or contexts are recognised as similar.

4. Interpreting the Literature

Our theory allows us to view the multifaceted nature of the angle concept in a new light. The standard angle concept is that which is common to a large number of angle contexts which are superficially not at all similar. Each facet of the concept is a set of related contexts.

Each of the various textbook definitions of angle seems to correspond to a different physical angle context. For example, the turning definition of angle clearly models the turns context, the ray pair definition models the intersections context, and the region definition models the corners context. The reason why each definition can also be applied to other angle contexts is that all the definitions embody the standard angle concept, although it is expressed in different ways. The fact that no one definition appears to match all physical angle contexts emphasises the difficulty of forming a general standard angle concept. The reason why several definitions exist is that each one must fit into a different formal mathematical structure.

Children's definitions and examples of angles also appear to describe various angle contexts or domains. For example, Davey and Pegg (1991) obtained a sequence of four definitions of angle: (a) a corner which is pointy or sharp; (b) a place where two lines meet; (c) the distance or area between two lines; and (d) the difference between the slope of the two lines. Clements and Battista (1989) found that Grade 3 children most commonly described an angle as a sloping line, a place where two lines meet, or the two lines themselves; children who had studied LOGO also said it was a turn. Matos (1994) reports that Grade 4–5 children conceived of angles as points, as turning bodies, as a source of two rays, as a bent path, and as two lines connected at their end-points. The point and line conceptions may be non-standard abstract domains or single angle contexts – it is not possible to tell without eliciting examples. The turns and paths conceptions are very likely contexts. The two lines conceptions seem to refer to the standard angle domain, but may be limited to contexts where both lines are physically present (e.g., including corners but excluding slopes).

Our theory also provides a definite answer to Douek's (1998, p. 271) question, "Does embodiment constitute a negative element in the perspective of building decontextualised concepts, or rather an inevitable, and positive, phase on the road towards complex and long concept building?" The difficulties which students experience in coordinating various angle concepts are part of a natural developmental process. The initial links between different angle contexts may be very weak or non-existent, and developing a mature abstract angle concept depends *essentially* on learning to link them together.

In particular, the failure of LOGO training to 'transfer' to other angle contexts suggests a weak linkage of the bent path context to other physical angle contexts. If our theory is correct, conceptual development consists of strengthening the links between different contexts. Providing vivid experiences in one angle context cannot therefore be expected to transfer to another context unless the link is already present – or is constructed as a central part of the instruction. Most of the LOGO training studies neglect this link and may be regarded as attempting the impossible.

A number of studies have attempted to measure children's abstract angle understanding by posing contextual tasks. For example, Piaget and Inhelder (1946) used lazy tongs, Beilin (1984) used biscuit cut-outs, Kieran (1986) and Noss (1987) used bends in a road, and Mitchelmore (1992) used colliding children. The present theory suggests that the findings of these studies gave interesting information on children's understanding of various angle situations (or possibly contexts), but – in the absence of any information on how the children linked various angle situations and contexts together – say absolutely nothing about their abstract understanding.

The converse criticism can be made of studies which have investigated children's understanding of angle diagrams (e.g., Piaget and Inhelder, 1946; Piaget, Inhelder and Szeminska, 1960; and studies following the van Hiele model). These studies all assume that they are investigating children's abstract angle concepts. But without any evidence as to the current structure of students' angle concepts, the results may be completely uninterpretable. At worst, when students have a poorly developed abstract angle concept, the findings may be limited to the specific context of angle diagrams and say absolutely nothing about their understanding of physical angle contexts.

4.1. *The present study*

Angle research is interested in the general question, how does the structure of children's angle concepts develop? The theory outlined above enables us to refine this question into specific questions such as the following: How do situations tend to cluster together into contexts, which contexts cluster into domains, and at what ages? Do contexts cluster into multiple domains (e.g., contexts containing fixed situations might merge together in one cluster and moveable situations in another) before these merge into one, or do they develop in some other pattern? What characteristics of angle contexts determine how well children tend to recognise their similarity? How can we design ways of helping children to recognise similarities between angle contexts? We can find no research apart from our own which has addressed such questions.

For the present study, the following research questions were posed:

- How well do children of various ages recognise the standard angle concept in various contexts?
- How well do children of various ages recognise standard angle-related similarities between different contexts?
- What factors determine whether children find angle contexts easy or difficult to relate in terms of the standard angle concept?

Our previous studies (Mitchelmore, 1997, 1998; Mitchelmore and White, 1998a) have included only small numbers of contexts and have had either a very small sample size for each pair of contexts or few pairs of contexts. The present study was conducted with a sufficiently large sample to obtain an adequate sample size for all the pairwise comparisons between a larger number of physical angle contexts.

5. METHOD

5.1. *Sample*

The sample was gender-balanced and consisted of 144 children in each of Grades 2, 4 and 6 chosen from six schools in Sydney. A further 48 Grade 8 students from two high schools were also interviewed.

5.2. *Materials*

Nine situations were used, chosen from a variety of physical angle contexts: wheel, door, scissors, hand fan, signpost, hill, road junction, tile and wall. The first four situations are movable, while the last five are fixed. Each situation was represented by one or more models, illustrated in Figure 1. Each movable situation (wheel, door, scissors and fan) was represented by a single adjustable model. Each of the fixed situations (signpost, hill, road junction, tile and wall) was represented by a set of three models representing a 'neutral' configuration (angle 0° or 90°), an angle

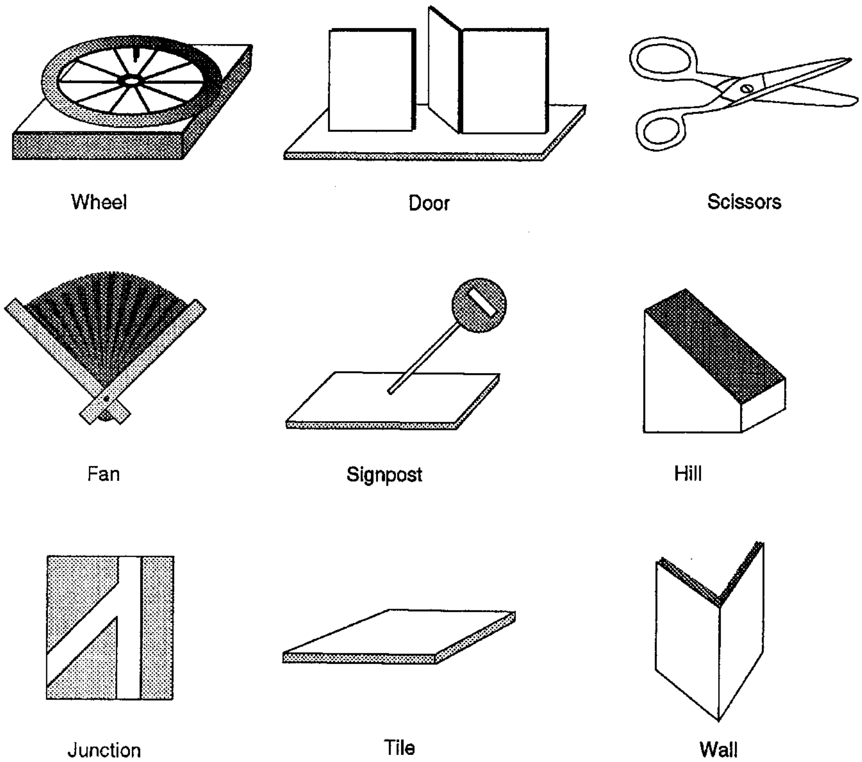

Figure 1. Models used to represent the nine physical angle situations.

of 45°, and a 'middle' angle (22.5° or 67.5°). Only one of the three models is shown in Figure 1. Adjustable models of the fixed situations were deliberately not used, in order to avoid artificially suggesting a dynamic interpretation.

A drinking straw which could be bent at various angles was used as an abstract angle model. A second straw fixed at 45° was also used.

5.3. *Procedure*

Each student was shown one of the following combinations of the situations shown in Figure 1:

Wheel, door, scissors	Door, signpost, tile
Fan, signpost, hill	Door, hill, junction
Junction, tile, wall	Door, fan, wall
Wheel, fan, junction	Scissors, hill, wall
Wheel, signpost, wall	Scissors, fan, tile
Wheel, hill, tile	Scissors, signpost, junction

These combinations were chosen is such a way that each of the 36 possible pairs of situations occurs exactly once. (Each individual situation occurs 4 times.) Each of the 12 combinations was presented to 12 students in each of Grades 2, 4 and 6, with order counterbalanced. Thus 144 students were interviewed at each grade level, giving 48 responses for each situation and 12 responses for each situation pair.

When a preliminary data analysis showed that the wheel, door and hill were still causing difficulty to Grade 6 students, these three situations were also administered to 48 students in Grade 8.

Individual interviews were conducted in a quiet room by a trained research assistant. She followed a fixed protocol, using neutral prompts such as ‘Good; anything else?’ or ‘OK; show me how’ whenever necessary to obtain a better understanding of children’s responses. The word ‘angle’ was used only if students introduced it. All interviews were audiotaped and transcribed the same day on to a response sheet; the authors later categorised and coded non-quantitative responses.

Each interview proceeded as follows. Firstly, the interviewer presented the three physical models separately. For each model, she briefly described the situation the model was intended to represent and then administered three angle recognition tasks (Tasks 1–3 in Figure 2). She then presented the models again in pairs. For each of the three pairs of models, she administered three angular similarity tasks (Tasks 4–6).

5.4. *Scoring*

Task responses were scored according to whether they showed an interpretation of the given situation(s) in terms of the standard angle concept – that is, with the vertex and the two arms of the abstract angle model appropriately matching a point and two lines on the physical model. For the movable situations, the vertex is the point of rotation and the opening of the arms represents the amount of rotation. For the fixed situations, the two arms represent two specific lines and the vertex is the point where these two lines meet.

Task 1 was scored correct if students used both arms of the abstract angle model and explained what they represented in general terms which clearly indicated use of the standard angle concept. In Tasks 2 and 5, an error of $\pm 15°$ in setting the movable models was allowed. In Tasks 3 and 6, specific error limits were defined for each model; for example, a placement of the straw on the scissors was scored as correct if the arms were placed anywhere along the blades with the vertex in the region where the blades crossed.

Task 1: Global angle representation. The interviewer laid the flexible drinking straw on the table in front of the student, showed how it bent, and asked the student to demonstrate how it could be used to represent the general situation in the given model. For example, the student was asked how the straw could be used to show "how the wheel turns" or "how the hill slopes".

Task 2: Size matching (recognition). The interviewer bent the two parts of the flexible drinking straw together, laid the straw on the table in front of the student, opened it through an angle of about 45°, and then replaced it by the fixed straw. The student was then asked to demonstrate the same angle on the physical model. For example, the student was asked to "open the door by that amount" or to "point to the tile whose corner is like that".

Task 3: Angle matching (recognition). The student was asked to place the 45° bent straw on the physical model to explain how he or she had solved the previous task.

Task 4: Global angular similarity. The interviewer asked if the student could see "anything the same" about the two situations. Neutral prompts such as "Anything else?" were given until the student reported no further similarities.

Task 5: Size matching (similarity). The interviewer set one situation to show an angle of 45° by moving a movable model or selecting a fixed model. She then asked the student to set the other situation to "show the same as this".

Task 6: Angle matching (similarity). The student was asked to use the flexible straw to explain how he or she knew the two settings were the same.

Figure 2. Interview tasks.

Task 4 evoked two standard angle similarity interpretations: dynamic and static. A response was judged to indicate a dynamic similarity if it referred to the same angular movement in both situations, such as turning or opening. A response was judged to indicate a static similarity if it referred to a common geometrical configuration. In both cases, a response was only scored correct if it implicitly or explicitly related two appropriate lines. For example, in relation to the tile and the scissors, many students said "they both have two lines" or "you can put the tile in between the scissors", both of which were clearly static similarities. Responses such as "the scissors can be opened to show the corners of the tile" were also classified as static because (a) they referred to the resulting position of the movable model and not to the movement which produced it, and (b) there was no implication that the fixed situation had any movement associated with it.

6. Results

6.1. *Standard angle recognition*

Table II shows the rates of standard angle usage in Tasks 1 to 3. It appears that students interpreted the nine situations quite differently.

Five situations were most frequently interpreted using standard angles: scissors, fan, junction, tile and wall. With a few exceptions (notably size matching in the junction, tile and wall situations in Grade 2), standard angles were used by at least 80% of the students on all these tasks at all grade levels – in many cases over 95%. A sixth situation, the signpost, was not often interpreted using standard angles in Grade 2 but by Grade 6 rose to similar levels as the other five situations.

Two situations, wheel and door, showed a different pattern: mostly frequent use of standard angles (over 85%) on Tasks 1 and 2, but much less frequent use on Task 3 (from an average of about 30% in Grade 2 to about 55% in Grade 8). On Task 3, many students had difficulty identifying the center of rotation on the wheel, despite the fact that it was marked with a prominent bolt; and several used only one arm of the bent straw to represent the rotation. For the door, the main difficulty lay in identifying its initial position as one arm of the standard angle.

In the ninth situation, the hill, standard angles were used with similar frequencies on all three tasks, varying from about 45% in Grade 2 to about 80% in Grade 8. The most common non-standard methods of representing the hill in Task 1 were to use only one, sloping arm of the bent straw (only in Grade 2) or both arms in an inverted V-shape. Furthermore, only 19% of students in Grade 4, 27% in Grade 6, and 17% in Grade 8 represented the angle between the horizontal and the slope of the hill (as most mathematics teachers probably would). The rest of the students who correctly represented the slope using a standard angle indicated the angle between a sloping edge and one of the vertical edges on the hill model. In Task 2, almost all of the errors consisted of choosing the hill with a slope of 22.5°; students seemed to be sure that the bent straw represented a sloping hill, but were uncertain as to the amount of slope. In Task 3, most standard interpretations again used a sloping edge and a vertical edge of the hill model.

TABLE II

Percentage of recognition task responses showing standard angle usage, by grade and situation

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
Task 1: Global angle representation									
2	17	90	100	98	24	48	77	96	96
4	90	96	90	88	67	50	85	100	98
6	94	92	94	92	83	67	96	100	100
8	94	94	–	–	–	59	–	–	–
Task 2: Size matching (recognition)									
2	49	77	87	87	47	43	60	64	64
4	85	100	100	100	88	58	92	96	98
6	98	98	98	98	96	65	96	98	100
8	94	100	–	–	–	75	–	–	–
Task 3: Angle matching (recognition)									
2	21	44	87	79	43	45	94	100	92
4	31	50	89	81	75	60	98	98	92
6	56	65	92	85	88	69	100	96	100
8	42	65	–	–	–	81	–	–	–

Note: In each cell, $n = 48$.

6.2. *Standard angle similarity*

Table III summarises the responses to Tasks 4 to 6. Responses to Task 4 have been separated into dynamic responses (Task 4a) and static responses (Task 4b). To reduce the number of situation pairs to a manageable number, averages have been taken over 6 movable-movable pairs, 20 fixed-movable pairs and 10 fixed-fixed pairs. Variation between individual pairs of situations will be discussed separately below.

On Task 4, students almost never reported a dynamic similarity between a fixed situation and another situation. Also, the dynamic similarity recognition rate between movable situations was fairly constant at about two-thirds. By contrast, static similarities were readily recognised not only between fixed situations but (especially after Grade 2) also between fixed and movable situations and even between movable situations. Within each category of situation pair, the static similarity recognition rate increased monotonically from Grade 2 to Grade 6.

In both Tasks 5 and 6, the percentage of standard angle responses increased steadily from Grade 2 to Grade 6 in all three categories. Also, at each grade level, the order of the standard recognition rates was al-

TABLE III

Percentage of similarity task responses showing standard angle usage, by grade and situation pair

Grade	Movable-movable (n = 72)	Fixed-movable (n = 240)	Fixed-fixed (n = 120)
Task 4a: Global angular similarity (dynamic)			
2	64	2	0
4	60	0	0
6	68	0	0
Task 4b: Global angular similarity (static)			
2	4	32	50
4	74	63	81
6	86	79	89
Task 5: Size matching (similarity)			
2	56	43	73
4	89	75	91
6	90	89	97
Task 6: Angle matching (similarity)			
2	36	61	81
4	61	68	82
6	81	80	89

most always the same: Fixed-fixed similarities were most frequently recognised, then movable-movable, then fixed-movable. It is notable that, with one exception, the same patterns also occur in the similarities reported in response to Task 4b.

The similarities between the patterns of responses to Tasks 4b, 5 and 6 suggested that all three tasks might be measuring a common construct. To investigate this possibility further, three similarity scores were calculated for each pair of situations: the percentage of students making standard angle responses on each task. The correlations between these three scores in Grade 2 (0.61, 0.68 and 0.72), and the correlations between the first and the third scores in Grades 4 and 6 (0.82 and 0.83) supported the hypothesis of a common construct. (The remaining inter-correlations were somewhat smaller, partly due to a ceiling effect in Task 5.) Tasks 4b and 6 clearly involve the recognition of a common configuration, indicating that the common construct is static similarity recognition. We infer, therefore, that Task 5 also measures static similarity recognition – even in the case of

TABLE IV

Average similarity indices (percentages) for each pair of physical angle situations

	Wall	Junction	Tile	Scissors	Fan	Signpost	Door	Hill	Wheel
Wall	–	0	0	0	3	0	0	0	0
Junction	94	–	0	0	3	0	0	0	6
Tile	95	94	–	0	0	0	0	0	0
Scissors	94	74	86	–	83	0	72	3	56
Fan	82	80	67	74	–	0	78	0	50
Signpost	77	76	76	63	59	–	0	0	0
Door	69	58	59	67	68	53	–	0	47
Hill	70	70	69	65	43	60	47	–	0
Wheel	57	53	41	62	62	51	53	38	–

Note. Dynamic similarities are given above the diagonal and static similarities below.

moveable situations. For example, students may match the angle sizes on a wheel and a fan not by comparing the two movements but by comparing the resulting configurations.

The identification of a common construct across the three tasks justified averaging the three similarity scores to obtain a *static similarity index* for each pair of situations at each grade level. For most pairs, the static similarity index increased monotonically from Grade 2 to Grade 6; the average indices were 41%, 74% and 86% respectively. Table IV shows (below the diagonal) the average static similarity indices for each pair, arranged in decreasing order of their average static similarity to the other situations. (Space considerations prohibit presentation of separate tables for each grade level. They are available from the authors.) Table IV also shows average *dynamic similarity indices* calculated in a similar manner from the Task 4a results.

Hierarchical cluster analysis of the static similarity indices in Table IV indicated one main cluster of similar situations at each grade level:

- In Grade 2, there were three situations in the main cluster (junction, tile and wall) with scissors nearby.
- In Grade 4, the main cluster consisted of junction, tile, wall and scissors. There were also two secondary clusters: door and fan, and hill and signpost.
- In Grade 6, there was only one cluster consisting of junction, tile, wall, scissors, fan and signpost.

The static similarity indices were also subjected to multidimensional scaling analysis using a euclidean distance metric, with the following results:

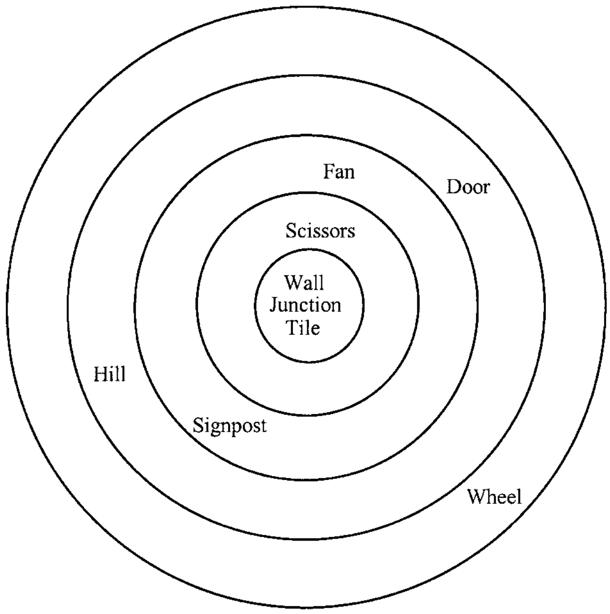

Figure 3. Modulating facet model of static similarity between the nine situations.

- The Grade 2 pattern showed no clear clustering.
- In Grade 4, junction, tile, wall and scissors clustered near the centre of the diagram with fan and signpost further away, door and hill further still (with door near to fan and hill near to signpost), and wheel far away.
- In Grade 6, junction, tile, wall, scissors, fan and signpost formed a central cluster with door, hill and wheel far outside.

These results suggest a core of similar situations gradually expanding to include more situations at successive grade levels. The results are well described by the five-partition model shown in Figure 3, suggested by facet theory (Levy, 1985).[1] We note that the order in which the various situations are absorbed as one moves outward in Figure 3 is precisely the order of situations in Table IV.

The dynamic similarity indices, which in effect only relate the movable situations, showed little variation from grade to grade. The door, scissors and fan situations formed one cluster, with the wheel separate. Only about a half of the students thought that the wheel turned like the other movable situations. In fact, students often referred to the door, scissors and fan as ‘opening’, not ‘turning’.

7. DISCUSSION

We now review our findings in terms of the research questions posed earlier.

7.1. *Standard angle recognition*

Students of all ages found it easy to represent the scissors, fan, junction, tile and wall situations using the standard angle model and to identify the standard angles on the physical models. By Grade 6, the signpost situation had joined this group. From Grade 2, all students matched the sloping signpost to one arm of the abstract angle model; between Grade 2 and Grade 6, they evidently learned to identify a line in the base of the signpost model to match the second arm.

Students also found it easy to represent the wheel and door situations using the abstract angle model, even to the extent of matching sizes correctly. However, a large proportion of students (among the older students, still about a half for the wheel and a third for the door) could not identify the two lines which make up the standard angle. These results suggest that, in Tasks 1 and 2, students were using the arms of the abstract angle model to represent the rotation rather than a standard angle – even though they appeared to use both arms of the model. Rather than interpreting turning in terms of angles, it would seem that students intuitively treat rotation globally, simply as a particular kind of movement. It is impressive that students were able to represent the size of a rotation so accurately without interpreting it as an angle.

The hill showed a third pattern of results. Many students (still about a third in Grade 8) found it difficult both to represent the hill using the abstract angle model and to identify the abstract angle on the physical model. The responses to Tasks 1 and 2 suggest that most students had some global concept of slope but that many did not quantify it by relating the sloping line to a fixed reference line. Unlike the wheel and door, where the second line may be suggested by the initial position and there is a global movement which can be copied, there is in fact very little to help a naïve student interpret a slope in terms of a standard angle. When students placed one arm of the abstract angle model on a sloping edge and one on a vertical edge in Task 3, they could simply have been choosing an available visible line to match the second arm of the straw. Such responses do not necessarily imply any awareness of the significance of the vertical edge as an appropriate reference line. We conjecture that many students have a global conception of slope as a single line and do not conceive it in terms of angles. Had the physical model of the hill consisted simply of a sloping plane without any supporting edges, it is likely that far fewer students would have indicated a standard angle interpretation.

7.2. *Standard similarity recognition*

Children appear to recognise two types of similarity between angle contexts: dynamic and static. However, they only recognise dynamic similarity between moveable situations and therefore cannot use it to classify angle situations in general. We shall, therefore, concentrate on static similarity. By definition, static similarity involves recognition of both lines of the standard angle in both situations and is therefore applicable to all physical angle contexts.

On the basis of the static similarity results summarised in Figure 3, the general development of children's standard angle domain is best described as consisting of one cluster which gradually expands by accretion. We may expect most Grade 2 children to recognise the standard angle similarity between junction, tile and wall, with smaller percentages recognising the similarity between these and other situations. By Grade 6, most children will recognise the standard angle similarity between junction, tile, wall, scissors, fan and signpost, with fewer recognising the similarity between these situations and hill, door and wheel.

Of course, there is individual variation. There are no doubt some Grade 2 children who would recognise the standard angle similarity between all nine situations and some Grade 6 children who would only recognise the standard angle similarity between two or three situations. However, it seems reasonable to assume that individual children would tend to follow the same expanding pattern of development – first junction, tile and wall; then scissors, fan and signpost; then hill, door and wheel – but at varying rates.

We note that the main cluster consists initially of only fixed situations (junction, tile and wall); but by Grade 6, it includes two moveable situations (scissors and fan) as well as a further fixed situation (signpost). Of the situations not in the cluster by Grade 6, two are moveable (wheel and door) and one is fixed (hill). There is therefore absolutely no evidence for clustering of moveable and fixed contexts into separate angle domains.

Two minor clusters were observed in Grade 4: one linking linear and plane slopes (hill and signpost) and one linking two moveable objects (door and fan). However, the fact that neither cluster persisted into Grade 6 suggests that each of them was strongly influenced by global similarities (slope or rotation).

7.3. *Factors influencing standard angle interpretation*

The results of both the recognition and similarity tasks strongly suggest that the major factor influencing students' use of standard angles is the physical presence or absence of the lines which make up its arms.

In all the situations in the central cluster (scissors, fan, signpost, junction, tile and wall), the two arms are physically present. They are not quite so easily identified in the case of the scissors and fan situations (both move and have thick arms), nor in the signpost.

In the three situations outside the central cluster (door, hill and wheel), one or both arms of the physical model are not physically present but have to be constructed by the student. The fact that the standard angle was used more frequently for the door and hill (where one arm must be constructed) than the wheel (where both arms must be constructed) supports the view that the crucial factor accounting for the rate of use of standard angles is the physical presence of the angle arms.

Quantitative information on the effect of this factor may be obtained from the data in Tables III and IV. On Task 3 (preferred to Tasks 1 and 2 because of the possibility that the results on Tasks 1 and 2 might be affected by global recognition of turning or sloping without the use of the standard angle), 88% of the students used standard angle modelling when both lines were visible, 55% when only one was visible, and 36% when no line was visible. Also, the average similarity index between pairs of situations where both lines are visible was 79%. Where one situation had two lines visible and the other only one, the average similarity index was 63%. Other combinations of numbers of visible lines gave similar average similarity indices averaging 46%.

8. IMPLICATIONS

The results of the present study need cautious interpretation for at least three reasons. Firstly, the findings are undoubtedly influenced by children's previous experience, including their school geometry. In particular, the similarity of junction, tile and wall which Grade 2 students report may be attributable to their study of polygons – including activities with pattern blocks (P. Boero, personal communication, 15 July 1998). The developmental pattern we have found cannot therefore be regarded as 'natural'. However, since the curriculum followed by the children in the present study (New South Wales Department of Education, 1989) is in no way exceptional, we could expect the results to be replicable in other systems.

A second limitation lies in the specific physical angle situations used in the present study. In particular, there was only one situation with no lines of the standard angle visible and only two with one line. A different choice of situations may have led to different results. On the other hand, the results are consistent with our previous findings, and replication with other situations is relatively easy.

A third limitation lies in the fact that this was a cross-sectional study. Only a longitudinal study can confirm whether the development of an individual student's standard angle domain tends to follow the path inferred from the averages we have obtained.

Despite these limitations, the results would appear to have some clear implications for both theory and practice.

8.1. *Theoretical implications*

Our results show that, during development, several clusters of physical angle contexts are formed. One cluster (including junction, tile, wall, scissors, fan and signpost) consists of contexts where both angle arms are visible. Another cluster (including hill and signpost) consists of sloping contexts. A third cluster (including wheel, door, scissors and fan) consists of turning and opening contexts. Note that these clusters overlap, and that none of them includes all angle contexts. Children's various angle conceptions, as reported by Davey and Pegg (1991), Clements and Battista (1989) and Matos (1994) and discussed earlier, seem to consist of such clusters and a few others.

The three clusters we have found correspond to the everyday concepts of *corner*, *slope* and *turn*, respectively. They are based on superficial similarities (two visible lines, a single sloping line and rotation) and distinguished from one another by the number of arms of the abstract angle which are visible. All three clusters are therefore most likely formed by empirical rather than reflective abstraction. So, although these clusters relate different angle contexts, it does not seem appropriate to call them abstract angle domains. Instead, we shall call them *physical angle domains*.

Our findings strongly suggest that the standard abstract angle domain develops from the corner physical angle domain. For most children, the corner domain is formed during elementary school as they come to recognise the two arms in confusing situations such as fan and signpost. But even at the beginning of elementary school, there are some children who recognise the standard angle similarity to other contexts – some moveable and some fixed. The proportion grows steadily until, towards the end of elementary school, at least half the children can recognise the more obcure similiarities. Such children can be said to have formed a general (standard) abstract angle concept.

Our results show clearly the significance of constructive activity in the formation of an abstract angle concept: the fewer arms that are present in a particular angle context, the more that has to be constructed to bring it into relation to other angle contexts and, therefore, the more difficult it is to recognise the standard angle. It is only in exceptional cases that the relevant line has to be discovered. In most cases, it has to be invented through conscious mental activity.

The question this poses is, what is the motivation for relating two contexts from different physical angle domains? Physical angle situations, contexts and domains are presumably formed for the sake of economy of communication in everyday circumstances where the individual experiences, situations or contexts need not be distinguished. It is difficult to think of everyday circumstances where corners, slopes and turns would be indistinguishable. They can only be brought into relation by mathematical activities such as measuring and drawing, activities which simultaneously lead to other abstract concepts such as point and line, congruence and similarity. This is the other constructive aspect of reflective abstraction which Piaget refers to: the construction of a system of relations between concepts. A general standard abstract angle concept can only arise as one outcome of a systematic attempt to investigate our spatial environment mathematically.

8.2. *Pedagogical implications*

Our findings show that many children form a standard angle concept as early as Grade 2, but that this concept is likely to be limited to situations where both arms of the angle are visible (the corner domain). If the concept is to develop into a general abstract angle concept, children will need more help than is presently given to identify angles in slope, turn and other contexts where one or both arms of the angle are not visible. The slope and turn domains are particularly important for the secondary mathematics curriculum, the former because of the frequency of angles of inclination in trigonometrical applications and the latter because it provides a valuable aid in teaching angle measurement.

Because the logic for the construction of the missing line(s) varies from context to context, it is probably most effective to treat each slope or turn context separately. We support the idea of organising teaching around *fields of experience* (Boero et al., 1995). For example, the teaching experiment reported by Douek (1998) shows how difficult it can be for students to recognise any similarity between inclination and more familiar physical angle situations. Only repeated practical investigation and intense reflection can relate the height of the sun to an angle and identify the horizontal as the missing line. Similar treatment of other ‘difficult’ situations is probably necessary to incorporate them into the evolving standard angle domain.

Guided investigations of the similarity between different angle contexts or domains could be most beneficial. For example, a student who says that a corner and a hill are similar because “you can fit a corner into the hill” is already on the way to identifying the second line needed for a standard angle conceptualisation of slopes. A teacher could use a tilting window to illustrate corners, slopes and turns, thus illustrating the similarity between the three domains. Anecdotal evidence (Wilson and Adams, 1992) suggests that teaching designed to help young children link different angle contexts can be most successful. Indeed, our theory suggests that it is *only* by recognising the similarity between angle situations with and without both arms visible that the standard angle concept can be generalised to cover the latter type of situation.

A further place where an emphasis on similarity could prove helpful is in the teaching of angle measurement in secondary school. This has always been a difficult topic to teach (Mitchelmore, 1983; Close, 1982). Using a protractor requires the student to identify two lines on the protractor and match them to lines on a given angle diagram. This is already problematic; not only are there several choices for the base line on the protractor, but the second line has to be imagined. Many teachers and textbooks attempt to help students find this second line by interpreting the angle diagram as representing a turn. Given the consistent finding in the present and previous studies that many students are unable to conceive of turning as a relation between two lines, it is not surprising that this is ineffective – although it is more effective if a 360° protractor, which more clearly represents a full turn, is used (Close, 1982). It would seem that the ability to interpret a turn as a relation between two lines, and hence to recognise the angular similarity between a turn and a corner, is an essential prerequisite to angle measurement using a protractor.

A third implication of our study is that verbal definitions of angle are unlikely to be helpful to young children. It is only when students have learned to recognise the similarity between many angle contexts that they are likely to accept a definition which is expressed in terms of a single context as applicable to all angle contexts. The tendency to define an angle as “an amount of turning about a point between two lines” would seem to be particularly inappropriate. If a definition must be used, we would propose “two lines meeting at a point with an angular relation between them”. This is, of course, a circular definition. But, to echo Skemp (1986,

p. 25), the meaning of ‘angular’ – since it relates to a higher-order concept – cannot be communicated through a definition but only through specific characteristics such as sharpness, inclination, rotation, slope and direction which are embedded in the various contextual angle concepts.

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NOTES

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