



FINAL REPORT
DEVELOPMENT OF CONSTRUCTIVE EXISTENCE
THEOREMS IN OPTIMAL CONTROL THEORY

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FOREWORD

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INTRODUCTION

The establishment of existence of a minimizing solution in the calculus of variations has been recognized as a challenging problem. For many problems which satisfy reasonable hypotheses, a minimizing solution can be shown not to exist. In addition, for some problems which have a minimizing solution, a sequence of approximations that approaches the minimum solution pointwise need not have a limit which is minimizing.

In 1937, these considerations resulted in the idea presented by L. C. Young (Reference 1) for modifying the problem to obtain a solution in a larger class of functions called generalized curves. This idea was extended by McShane (References 2, 3 and 4), who obtained an existence theorem for the problem of Bolza, and arrived at the conditions necessary for a generalized curve to be minimizing. In 1949, Hestenes (Reference 5) translated known results on necessary and sufficient conditions into controls language. This study included the Hamiltonian formulation which later became known as the maximum principle.

Existence for the linear time-optimal control problem with constant coefficients was treated by Bellman, Glicksberg, and Gross (Reference 6); the case of time-varying coefficients was treated by La Salle (References 7 and 8), Gamkrelidze (Reference 9), and Neustadt (Reference 10). Reid (Reference 11) treated the linear case for control functions of minimum norm. Lee and

Marcus (Reference 12) and Roxin (Reference 13) obtained existence theorems for the non-linear time-optimal problem under restrictive hypotheses.

In Russia, Filippov (Reference 14) treated the existence problem somewhat differently and observed that, for a special case in which a minimum does not exist, a minimizing sequence does exist, although the limit is not a solution. Filippov called such limits sliding states. Gamkrelidze (Reference 15), following Filippov, obtained an existence theorem and some necessary conditions for sliding states. Independently, Warga (References 16, 17 and 18), by a construction quite like that of Filippov, obtained similar theorems for what he called a relaxed problem. He showed that under certain hypotheses his relaxed solutions are generalized curves.

Most of the above papers on the subject of existence are non-constructive and use such tools as fixed point theorems and compactness of point sets. Such results are interesting and important, but frequently the hypotheses of a given theorem are difficult to verify.

The purpose of this research was to investigate constructive type existence theorems for optimal control problems with the expectation that the results obtained would shed light on some of the inherent difficulties encountered in computation. A constructive existence theorem was obtained for an elementary control problem; however, conditions on the problem which assure

that the underlying hypothesis is satisfied are not readily obtainable. Two related results were obtained which are of interest in their own right. These are the approximation theorems contained in Section 1 and a computational algorithm contained in Section 3.

In Section 1, several results of a topological nature are presented which relate to approximations of functions. These results are utilized in Section 2 and Section 3 to study sequences of functions for solving approximate problems. An algorithm is developed in Section 3 which is expected to be useful in speeding up the convergence of gradient computational methods. The algorithm depends upon replacing the original problem by a new problem which is easier to solve.

1. SOME APPROXIMATION AND DENSITY THEOREMS*

The following sequence of lemmas and theorems are useful in the development of Section 2 as well as being of interest in their own right.

Lemma 1.1. If H is a locally convex separable space, let D be an open connected subset of H and $\{a_i\}$ a sequence of points in D that is dense in D . Then there exists an open subset $U \subset D$ such that

- i) $\{a_i\} \subset U$
- ii) U is homeomorphic to H
- iii) $D - U$ is nowhere dense in D .

We enclose a_1 in a small ball in D . If $\bigcup_1^k a_i$ has been enclosed in a set C_k homeomorphic to the unit ball in H so that $\bigcup_1^k a_i \subset \text{Int } C_k$, let P_{k+1} be a polygonal path from a_{k+1} to a symmetric ball in C_k . We enclose C_k in the interior of a larger copy C_{k+1} of the unit ball by "swelling" P_{k+1} and C_k so that $\overline{C_{k+1} - C_k}$ is homeomorphic to the closed region between two concentric spheres. There is then a homeomorphism between H and $U = \bigcup_1^\infty C_k$. So the lemma follows.

From now on H denotes a Hilbert space.

* This section was co-authored by P. H. Doyle, Mathematics Department, Michigan State University and will appear under the same title in "Proceedings of the International Symposium on Topology and Its Applications."

Lemma 1.2. Let A and B be countable dense subsets of H. Then there is a homeomorphism h of H onto H corresponding to any preassigned $\epsilon > 0$ such that

1. $h(A) = B$
2. $\rho(x, h(x)) < \epsilon$

The argument is just that of (Reference 19) pp. 44-46 except that in placing A and B similarly the condition 2 above is forced in the induction.

Lemma 1.3. If U is the interior of the unit ball in H, let A and B be countable dense subsets of U and $\epsilon > 0$. Then there exists a homeomorphism h of H onto H such that

1. $h(A) = B$
2. $h|_{E^n - U} = I$
3. $\rho(x, h(x)) < \epsilon$.

Let $h_1 : U \rightarrow H$ be the homeomorphism given by $h_1(x) = \frac{x}{1-|x|}$, while h_2 is the homeomorphism of Lemma 1.2, carrying $h_1(A)$ onto $h_1(B)$, with condition 2) satisfied for $\epsilon/3$. Define h of H onto H by $h|_U = h_1^{-1}h_2h_1$ and h is the identity otherwise.

Note that

$$x - h(x) = x - \frac{x + \delta \frac{x}{|x|} (1 - |x|)}{1 - |x| + |x + \delta \frac{x}{|x|} (1 - |x|)|}$$

where δ_x is a vector depending on x with the property that $|\delta_x| < \epsilon/3$. From this equation we have that

- a) $\lim_{|x| \rightarrow 1} (x - h(x)) = 0$
 b) $|x - h(x)| < \epsilon,$ all $x \in U$.

Here a) establishes the continuity of h on $\bar{U} - U$, and b) shows condition 2 is satisfied.

Corollary. Let M^n be a topological n -manifold (Reference 19), $U \subset M^n$ is an open set while A and B are countable dense subsets of U . Then there exists a homeomorphism h of M^n onto M^n such that $h(A) = B$, $h|_{M^n - U}$ is the identity map, and $\rho(x, h(x)) < \epsilon$ for $\epsilon > 0$.

If U_1 is a component of U , then by construction in Reference 19, $U_1 = E^n \cup R$ where E^n is topologically E_n , R is a closed subset of U_1 of dimension at most $n-1$ and $(A \cup B) \cap U_1 \subset E^n$. In case $n = 1$, $R = \emptyset$ and $E^n = E^1 = U_1$, so that theorem is true for $n = 1$. For $n > 1$, we note that E^n is a strictly increasing union of closed, flat n -cells $\bigcup_1^\infty B_i$ such that $\overline{B_{i+1} - B_i}$ is the product of a sphere and an interval while for each i , the boundary of B_i , $\text{Bd } B_i \cap (A \cup B) = \emptyset$.

Each closed annular region $\overline{B_{i+1} - B_i}$ can be split into two closed n -cells so that neither meets $A \cup B$ on its boundary and each n -cell meets the other only on its boundary. By the preceding lemma, there exists a homeomorphism of

$A_{i+1} = \overline{B_{i+1} - B_i}$ onto itself that is the identity on its boundary, carries $A \cap A_{i+1}$ onto $B \cap A_{i+1}$ and moves no point more than ϵ_i say. Thus there exists a homeomorphism of U_1 onto U_1 that can be extended to one on M^n by the identity map. Hence the h of the above corollary exists.

Theorem 1.1. If U is an open subset of a Hilbert space H while A and B are dense countable sets in U then there exists a homeomorphism h of H onto H such that for $\epsilon > 0$,

- i) $\rho(x, h(x)) < \epsilon$
- ii) $h(A) = B$
- iii) $h|_{H - U} = I.$

If U_1 is a component of U then by Lemma 1.1, $U_1 \cap (A \cup B)$ lies in an open copy of Hilbert space in U_1 and the argument proceeds as in the preceding Corollary.

Theorem 1.2. Let f be a continuous function from a Hilbert space H_1 into another H_2 while $A \subset H_1$ is a dense set in H_1 . Then given any open set U in H_1 that contains $f^{-1}(o)$, there is an ϵ -approximation f_1 to f such that

- i) $f_1^{-1}(o) \cap A$ is dense in $f_1^{-1}(o)$
- ii) $f_1|_{H_1 - U} = f|_{H_1 - U}.$

In contrast to Theorem 1.2, we prove an approximation theorem for a rather general domain. $1/2 E^1$ is the closed half-line.

Theorem 1.3. Let X be a perfectly normal space and $D \subset X$ is a dense subset. If $f : X \longrightarrow 1/2 E^1$ is a map there exists a map $g : X \longrightarrow 1/2 E^1$ such that

$$i) \quad g^{-1}(0) \cap D = f^{-1}(0)$$

$$ii) \quad \rho(f, g) < \varepsilon.$$

If $f^{-1}(0) \cap D = f^{-1}(0)$ there is nothing to prove. Otherwise, consider $f^{-1}([0, \frac{\varepsilon}{2}]) \supset f^{-1}(0)$. Let $K = f^{-1}([0, \frac{\varepsilon}{4}])$. The closed set $C = f^{-1}([0, \frac{\varepsilon}{2}])$ is carried by f into $[0, \frac{\varepsilon}{2}]$. But by (Reference 21, p.148), there is a map f_1 of C to $[0, \frac{\varepsilon}{2}]$ such that $f_1^{-1}(0) = K$ and $f_1^{-1}(\frac{\varepsilon}{2}) = f^{-1}(\frac{\varepsilon}{2})$, define $g : X \longrightarrow 1/2 E^1$ by

$$g \mid f^{-1}[0, \frac{\varepsilon}{2}] = f_1$$

$$g \mid f^{-1}[\frac{\varepsilon}{2}, \infty] = f.$$

2. AN ELEMENTARY CONSTRUCTIVE EXISTENCE THEOREM

Let I be a lower semi-continuous (lsc) function from $L_2^m [a,b]$, the set of m -dimensional square integrable functions on the interval $[a,b]$ to the real line, and let K be a compact subset of E^m . Let \mathcal{U} be a closed and bounded subset of $L_2^m [a,b]$ with the property that for u in \mathcal{U} , $u(t)$ is in K for t in $[a,b]$. We will further assume that \mathcal{U} is equal to the closure of its interior symbolically

$$(1) \quad \overline{\mathcal{U}^\circ} = \mathcal{U}.$$

We consider the problem

$$(2) \quad \begin{array}{l} \min I(u) \\ u \in \mathcal{U} \end{array}$$

Let S_n be the set of step functions in \mathcal{U} with discontinuities at $k(b-a)/2^n$, $k=1, \dots, 2^n-1$. Then $S_k \supset S_j$ for $k > j$ and $S = \cup S_n$ is dense in \mathcal{U} as a consequence of (1). We can replace (1) by the condition that S is dense in \mathcal{U} . The approximate problem

$$(3) \quad \begin{array}{l} \min I(u) \\ u \in S_n \end{array}$$

has a solution u_n for each n since on S_n , I is a lsc function of a finite number of variables and K is compact. Clearly

$$I(u_n) \leq I(u_j), \quad n > j$$

Again, because \mathcal{U} is bounded the sequence $I(u_n)$ has a limit. The function*

$$u_0(t) = \inf u_n(t)$$

belongs to \mathcal{U} and has the property that

$$I(u_0) \leq \inf_{u \in S} I(u)$$

Hence u_0 is a solution of (2) and we have shown

Lemma 2.1. There exists a function u_0 in \mathcal{U} satisfying

$$I(u_0) = \min_{u \in \mathcal{U}} I(u)$$

The sequence u_n determined by (3) approximates u_0 in the sense that

$$\lim I(u_n) = I(u_0) .$$

In the proof of Lemma 2.1, we have used the fact that \mathcal{U} has a dense subset S .

To extend this result, let \mathcal{V} be a closed and bounded subset of \mathcal{U} which need not intersect S . We consider the problem

$$(4) \quad \min_{u \in \mathcal{V}} I(u) .$$

We will further assume that I has the property that if u_n and v_n are sequences in \mathcal{U} for which

$$(5 a) \quad \|u_n - v_n\| \rightarrow 0$$

* inf applying to each component.

then

$$(5b) \quad |I(u_n) - I(v_n)| \rightarrow 0$$

where $||\cdot||$ denotes the norm in $L_2^m [a,b]$.

Let \mathcal{U}_k be a nested sequence of closed and bounded sets, which satisfy (1) and such that

$$\mathcal{V} = \bigcap \mathcal{U}_k$$

Each of the problems

$$(6) \quad \begin{array}{l} \min I(u) \\ u \in \mathcal{U}_k \end{array}$$

has a solution w_k by Lemma 2.1 and

$$I(w_k) \geq I(w_j) \quad k \geq j$$

since $\mathcal{U}_k \subset \mathcal{U}_j$.

Hence the sequence $I(w_k)$ has a limit. For each k let v_k be a function in \mathcal{V} determined by

$$||v_k - w_k|| = \min_{v \in \mathcal{V}} ||v - w_k||$$

Then since $U_k \rightarrow \mathcal{V}$,

$$\|w_k - v_k\| \rightarrow 0,$$

hence by (5)

$$|I(w_k) - I(v_k)| \rightarrow 0 .$$

Let

$$v_0(t) = \inf v_k(t)$$

Then clearly $v_0 \in \mathcal{V}$ and

$$\lim I(w_k) = I(v_0) .$$

Further, v_0 is a solution of (4) since if there was a $\bar{v} \in \mathcal{V}$ such that

$$I(\bar{v}) < I(v_0)$$

then for k sufficiently large

$$I(w_k) > I(\bar{v}) .$$

Since $U_n \supset \mathcal{V}$, this would contradict the fact that w_k is a solution of (6).

Hence we have proved

Lemma 2.2. There exists a function v_0 in \mathcal{V} satisfying

$$I(v_0) = \min_{v \in \mathcal{V}} I(v) .$$

The sequence w_k determined by (6) approximates v_0 in the sense that

$$\lim I(w_k) = I(v_0) .$$

We can now apply these results to the linear isoperimetric optimal control problem. Let

$$(7) \quad J_\alpha = \int_a^b f_\alpha(t, u(t)) dt \quad \alpha=0, 1, \dots, p.$$

We consider the problem of

$$(8a) \quad J_0(u) = \min$$

subject to

$$(8b) \quad \begin{aligned} J_\alpha(u) &= 0 & \alpha &= 1, \dots, p' \\ J_\alpha(u) &\leq 0 & \alpha &= p' + 1, \dots, p, \end{aligned}$$

where the functions $u(t)$ are in a subset \mathcal{V} of $L_2^m [a, b]$ with values in a compact set K . Let

$$(9) \quad \begin{aligned} I(u) &= J_0(u) \\ \mathcal{W} &= \{u(t) \in \mathcal{V} : u \text{ satisfies (8b)}\} \end{aligned}$$

Clearly \mathcal{W} is a closed subset of the closed and bounded set \mathcal{V} . Suppose that \mathcal{W} satisfies (1). Then we can take $\mathcal{U} = \mathcal{W}$ and the approximation problem (4) becomes

(10a)

$$\min_{u \in S_n} \sum_{i=0}^{2^n-1} \int_{a_i}^{b_i} f_0(t,u) dt$$

subject to

$$\sum_{i=0}^{2^n-1} \int_{a_i}^{b_i} f_\alpha(t,u) dt = 0 \quad \alpha = 1, \dots, p'$$

(10b)

$$\sum_{i=0}^{2^n-1} \int_{a_i}^{b_i} f_\alpha(t,u) dt \leq 0 \quad \alpha = p' + 1, \dots, p$$

where

$$a_i = a + \frac{i(b-a)}{2^n}$$

$$b_i = a_{i+1}, \quad i = 0, \dots, 2^n-1.$$

Theorem 2.1. Let W satisfy (1). There exists a function u_0 in W satisfying

$$J_0(u_0) = \min J_0(u)$$

and

$$\begin{aligned} J_{\alpha}(u_0) &= 0 & \alpha &= 1, \dots, p' \\ J_{\alpha}(u_0) &\leq 0 & \alpha &= p'+1, \dots, p. \end{aligned}$$

The sequence u_n determined by (10) approximately converges to u_0 in the sense that

$$\lim J_0(u_n) = J_0(u_0)$$

and

$$\begin{aligned} J_{\alpha}(u_n) &= 0 & \alpha &= 1, \dots, p' \\ J_{\alpha}(u_n) &\leq 0 & \alpha &= 1, \dots, p \end{aligned}$$

This is an immediate consequence of Lemma 2.1.

In the event that \mathcal{W} does not satisfy (1) there are two possibilities. We can obtain the sequence of sets \mathcal{U}_n specified in Lemma 2.2 either by modifying the functions f_{α} ($\alpha = 1, \dots, p$) or by modifying the constraints (8b). The first alternative is assured of success in that by Theorem 1.3 of Section 1, there exists a sequence of functions $f_n(t, u)$ such that setting

$$(11) \quad J_{n\alpha}(u) = \int_a^b f_{n\alpha}(t, u(t)) dt$$

Replacing (7) by (11) and setting

$$(12) \quad \mathcal{U}_n = \{u(t) \in \mathcal{Y} : u \text{ satisfies (8b)}\}$$

the sets \mathcal{U}_n satisfy the hypotheses of Lemma 2.2. However, Theorem 1.1 only assures the existence of the sequence. The construction is not clear and is currently under investigation. Likewise, Theorem 1.3 assures that there exists a sequence $J_{n\alpha}(u)$ obtained by modifying the functions $J_\alpha(u)$ directly with the resulting sets \mathcal{U}_n defined as in (12) also satisfying the conditions of Lemma 2.2. A method which suggests itself is to choose a sequence of vectors $E_n = (E_n^1, \dots, E_n^p)$, $E_n^j \geq 0$, $j = 1, \dots, p$, converging to the zero vector and consider the sequence of problems with (8b) replaced by

$$(13) \quad \begin{aligned} J_\alpha(u) &\leq E_n^\alpha & \alpha = 1, \dots, p' \\ - J_\alpha(u) &\leq E_n^\alpha & \alpha = 1, \dots, p' \\ J_\alpha(u) &\leq E_n^\alpha & \alpha = p' + 1, \dots, p. \end{aligned}$$

The question of when the sets \mathcal{U}_n defined by

$$\mathcal{U}_n = \{u(t) \in \mathcal{V} : u(t) \text{ satisfies (13)}\}$$

satisfies (1) is also under investigation.

A problem which can be cast in the form (8) is the linear optimal control problem. Let

$$(14) \quad \begin{aligned} \dot{x} &= A(t)x + g(t,u) & a \leq t \leq b \\ x(a) &= \alpha, x(b) = \beta \end{aligned}$$

be a system of differential equations in vector form where x is a q -dimensional vector. For convenience, we assume that $A(t)$ and $g(t,u)$ are continuous. We wish to minimize (8a) subject to (14) and a condition on the controls $u(t)$. Let $\Phi(t)$ be the fundamental solution matrix of

$$\dot{x} = A(t)x .$$

Then (14) can be rewritten as

$$x(t) = \Phi(t)\alpha + \int_a^t \Phi(t)\Phi^{-1}(s) g(s,u)ds$$

and in particular at $t = b$

$$(15) \quad \int_a^b \Phi(b)\Phi^{-1}(s)g(s,u)ds + \Phi(b)\alpha - \beta = 0$$

Then setting $f_\alpha = \alpha^{\text{th}}$ component of the vector

$$\Phi(b)\Phi^{-1}(s)g(s,u) + \frac{\Phi(b)\alpha - \beta}{b - a}$$

we see that (14) can be written as

$$\int_a^b f_\alpha(t,u)dt = 0 \quad \alpha = 1, \dots, q .$$

A case of particular interest is that where

$$(16) \quad g(t,u) = B(t)u$$

where $B(t)$ is an $q \times m$ matrix and the controls $u(t)$ are required to satisfy

$$(17) \quad |u(t)| \leq 1 .$$

Let \mathcal{U} be the set of controls satisfying (15) and (17) with g as in (16).

Let $u(t)$ and $v(t)$ be in \mathcal{U} , $c \geq 0$, $d \geq 0$ and $c + d = 1$. Clearly $cu(t) + dv(t)$ satisfies (17). Also

$$\begin{aligned} & \int_a^b \phi(b)\phi^{-1}(s)B(s)(cu(s) + dv(s))ds + \phi(b)\alpha - \beta \\ &= c \left[\int_a^b \phi(b)\phi^{-1}(s)B(s)u(s)ds + \phi(b)\alpha - \beta \right] \\ &+ d \left[\int_a^b \phi(b)\phi^{-1}(s)B(s)v(s)ds + \phi(b)\alpha - \beta \right] = 0 \end{aligned}$$

so \mathcal{U} is convex. Hence if \mathcal{U} has a non-empty interior, it satisfies (1). If the dimension m of u is one and more than one control satisfying (17) also satisfies (15), then the conditions of Lemma 2.1 are satisfied.

3. A COMPUTATIONAL ALGORITHM

In attempting to extend the ideas in Section 2 to non-linear control problems, a variety of transformations were investigated. Although none have yet proved completely successful for this purpose, one appears to have the possibility of speeding the convergence of gradient-type computational methods. This transformation, due to M. R. Hestenes (Reference 22), was used by him to facilitate proofs of necessary conditions.

The motivation for such transformations is simple: a solution to the given problem is also a solution of many other problems and among these, one is chosen which is computationally simpler.

Suppose we are given functions $f_i(t,x,u)$, $i = 0, \dots, q$ defined on a region R in $(1+q+m)$ -dimensional euclidean space. We desire to minimize

$$(1) \quad I(x) = \int_a^b f_0(t,x,u) dt$$

subject to

$$(2a) \quad \dot{x}^i = f_i(t,x,u), \quad a \leq t \leq b; \quad i = 1, \dots, q,$$

$$(2b) \quad x(a) = \alpha, \quad x(b) = \beta .$$

To incorporate additional constraints, it may also be required that $(t, x(t), u(t))$ lie in a subset R_0 of R for $a \leq t \leq b$ and $(x(t), u(t))$ a solution of (2).

In many computational schemes, the first step is to linearize the problem by replacing (1) and (2a) by*

$$(1') \quad I'(x) = \int_a^b f_{ox^j} \delta x^j + f_{ou^k} \delta u^k$$

and

$$(2'a) \quad \delta \dot{x}^i = f_{ix^j} \delta x^j + f_{iu^k} \delta u^k$$

where f_{ix^j} , f_{iu^k} are partial derivatives of f_i with respect to x^j , u^k respectively.

Here it would be convenient for computation if the coefficients of δx^j vanished. One cannot hope in general to modify the problem so that this occurs in the large, but it can be done along any given solution to (2).

Suppose that $(x_n(t), u_n(t))$ is a solution to (2). We designate this solution by the symbol x_n and will perform a transformation so that along x_n , the δx^i coefficients vanish. To this end, let

* Repeated indices are summed in all equations.

$$(3a) \quad f_{n\alpha}(t) = f_{\alpha}(t, x_n(t), u_n(t)) \quad \alpha = 0, \dots, q$$

$$(3b) \quad A_{nj}^i(t) = f_{ix^j}(t, x_n(t), u_n(t)), \quad i, j = 1, \dots, q$$

and

$$(3c) \quad B_{nj}(t) = f_{ox^j}(t, x_n(t), u_n(t)).$$

Let

$$(4) \quad J_0(x) = \int_a^b \left[f_0(t, x, u) - \frac{d}{dt}(r^i x^i) \right] dt$$

where r is a q -dimensional vector function to be chosen. For any r ,

$$(5) \quad J_0(x) = I(x) - r^i(b)\beta^i + r^i(a)\alpha^i$$

and since only a constant has been added to I , a minimum of I also is a minimum of J_0 . Now we set

$$(6) \quad F_0(t, x, u) = f_0(t, x, u) - r^i f_{x^i}(t, x, u) - \dot{r}^i x^i.$$

Then along x_n we have that

$$(7) \quad F_{ox^j}(t, x_n(t), u_n(t)) = B_{nj}(t) - r^i A_{nj}^i(t) - \dot{r}^j.$$

Then (7) vanishes if $r(t)$ satisfies the differential equation

$$(8) \quad \dot{r}^j + r^i A_{nj}^i = B_{nj}(t), \quad r(a) = 0.$$

In this event

$$(9) \quad J(x_n) = I(x_n) - r^i(b)\beta^i$$

and in particular if x_n minimizes I , it minimizes J_0 .

Next, let $Z(t)$ be the $q \times q$ - matrix solution of

$$(10) \quad \dot{Z}_{ij} + Z_{ih} A_{nj}^h = 0, \quad Z_{ij}(a) = \delta_{ij}$$

where δ_{ij} is the delta function. Let

$$(11) \quad F_i(t, x, u) = Z_{ij} f_j + \dot{Z}_{ij} x^j, \quad i, j = 1, \dots, q.$$

and

$$G_i = \alpha_i - Z_{ij}(b)\beta^j, \quad i = 1, \dots, q.$$

Then setting

$$(12) \quad J_i(x) = G_i + \int_a^b F_i(t, x, u) dt \quad i = 1, \dots, q$$

for every solution $(x(t), u(t))$ of (2a), we have

$$(13) \quad J_i(x) = Z_{ij}(b)(x^j(b) - \beta^j) - (x^i(a) - \alpha^i).$$

Hence since $Z(b)$ is non-singular, we have that any solution of (2a) with $x(a) = \alpha$ satisfies $x(b) = \beta$ if and only if

$$(14) \quad J_i(x) = 0 \quad i = 1, \dots, q$$

Further, F_i has the property that

$$(15) \quad F_{ix^j}(t, x_n(t), u_n(t)) = Z_{ik} A_{nj}^k + \dot{Z}_{ij}$$

$$= Z_{ik} A_{nj}^k - Z_{ik} A_{nj}^k = 0, \quad i, j=1, \dots, q$$

by (10).

Hence we can replace the original problem by the transformed problem

$$J_0(x) = \min$$

$$J_i(x) = 0 \quad i = 1, \dots, q$$

The suggested computational algorithm is then as follows:

- i) Choose any solution $(x_i(t), u_i(t))$ of (2).
- ii) Compute the matrices A_i and B_i using (3) with $n = 1$.
- iii) Compute $r(t)$ using (8).

- iv) Compute the solution Z of (10).
- v) Determine $J_0(x)$, $J_1(x)$..., $J_q(x)$ using (4) and (12).
- vi) Minimize $J_0(x)$ subject to $J_i(x) = 0$, $i = 1, \dots, q$ to obtain a new solution $(x_2(t), u_2(t))$ using any computational technique which profits from the fact that

$$F_{\rho x^i}(t, x_1(t), u_1(t)) = 0 \quad \rho = 0, \dots, q; i = 1, \dots, q.$$

- vii) Repeat ii) through vi) with $n = 2$, etc.

Although additional computation is required, both (8) and (10) are linear equations with fixed initial conditions. It is expected that the additional time for this computation will be more than made up in the simplification of the computation in step vi) of the algorithm. The extension to variable end point problems is quite straight forward and will not be carried out here. Further details, including modification of problems with additional constraints, may be found in Chapter 6 of (Reference 22).

4. SUGGESTED FURTHER RESEARCH

The approximation theorems contained in Section 1 are existence theorems. In a sense, the problem of construction of optimal controls has been transferred to the somewhat more direct problem of construction of the approximations presented in Theorems 1.2 and 1.3. Construction of these approximations is interesting and should have important applications.

The question of what conditions in the problem assure that the set U in Section 2 has the property

$$\bar{u}^0 = u \quad (1)$$

remains open. This fact causes difficulty in studying the linear case and certainly even more so in the non-linear case. Without an answer to the above question, constructive existence theorems with easily verifiable hypotheses appear very difficult to obtain. Thus a large part of the future research effort should be devoted to a deeper study of this problem.

Lastly, the extension of the results of Section 2 to the non-linear problem remains open. Transformations of the type considered in Section 3 seem to offer possibilities of application.

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