

## Development of Quantitative Thinking

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For understanding development of quantitative thinking, the distinction between non-symbolic and symbolic thinking is fundamental. Non-symbolic quantitative thinking is present in early infancy, culturally universal, and similar across species. These similarities include the ability to represent and compare numerosities, the representations being noisy and increasing logarithmically with actual quantity, and the neural correlates of number representation being distributed in homologous regions of fronto-parietal cortex. Symbolic quantitative thinking, in contrast, emerged recently in human history, differs dramatically across cultural groups, and develops over many years. As young children gain experience with symbols in a given numeric range and associate them with non-verbal quantities in that range, they initially map them to a logarithmically-compressed mental number line and later to a linear form. This logarithmic-to-linear shift expands children's quantitative skills profoundly, including ability to estimate positions of numbers on number lines, to estimate measurements of continuous and discrete quantities, to categorize numbers by size, to remember numbers, and to estimate and learn answers to arithmetic problems. Thus, while non-symbolic quantitative thinking is important and foundational for symbolic numerical capabilities, the capacity to represent symbolic quantities offers crucial cognitive advantages.

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Quantitative thinking is central to human life. Whether the situation involves a child recalling which blocks provided her with the most candies on previous Halloweens, a Londoner telling the time by counting the tolls of Big Ben, or a candidate using polls to predict results of an upcoming election, quantitative thinking is important for learning from the past, monitoring the present, and planning for the future.

Quantitative thinking plays an important role in the lives of other animals as well. To project the outcome of a future fight, prides of lionesses compare their pride-number to the number of distinct roars they hear in rival packs (McComb et al., 1994). Similarly,

to learn optimal foraging locations, animals in the wild encode the relative number of food items they have found previously in various locations (Davis, 1993).

The existence of such quantitative abilities makes sense from an evolutionary perspective. Being deprived of a sense of *how many* would deprive an animal of any rationality in its judgments and decision-making. Rational choices among alternative strategies and courses of action would be rendered impossible.

Given the importance of quantitative thinking, it is unsurprising that representations of quantity are a universal property of human cognition. Quantitative representations are present from early infancy (Cordes & Brannon, 2008; Feigenson, Dehaene, & Spelke, 2004), and share striking similarities across human cultures that provide radically different cultural and linguistic experiences (Butterworth et al., 2008; Gordon, 2004; Pica et al., 2004). Moreover, early-developing quantitative representations play a central role in learning a wide range of other quantitative skills that have emerged more recently in human history, such as symbolic arithmetic and algebra (Ifrah, 2000).

The central distinction among numerical capabilities that organizes this review, one that is highly correlated with the distinction between early-developing and later-developing quantitative skills, is that between non-symbolic and symbolic capabilities. Non-symbolic capabilities are ones in which numerical properties are implicit; for example, when sets of 7 and 5 dots are presented, the facts that one set has 7, the other has five, and the first set has more objects than the second, are all implicit. In contrast, symbolic capabilities are ones in which the numerical properties are explicitly expressed as written Arabic numerals, spoken words, or written words (e.g., "7" or "seven"). Non-symbolic and symbolic numerical processing differ in many ways. Non-symbolic processing of numbers is widespread across species; symbolic numerical processing appears to be unique to humans (aside from a small number of primates that have participated in laboratory experiments). Non-symbolic processes emerge in infancy;

symbolic processing does not emerge until later in childhood. Non-symbolic processing is approximate; symbolic processing allows precision. Moreover, symbolic processing of numbers shows wide variations among different cultures and historical periods, whereas variation in non-symbolic processing is less marked.

One reason to be interested in the development of quantitative abilities is that findings from this area often shed light on general theoretical issues in cognitive development. These issues include the potential existence of innate representational abilities, the extent to which early-developed capacities are sufficient to support the development of later abilities, and whether experience creates new representational resources or selects among pre-existing ones for use in novel contexts. Among the reasons why studies of quantitative abilities have been so fruitful is that such studies permit a degree of mathematical precision that is much more typical of psychophysics than of studies of children's concepts in other areas (e.g., theory of mind, biology, and moral development), thereby allowing researchers to test models that generate competing quantitative predictions. An equally important reason for the rapid expansion of this area is that valuable practical applications for improving children's mathematical understanding have arisen from the theoretical work.

Within the distinction between non-symbolic and symbolic quantitative thinking, we focus on three main questions: (1) What quantitative abilities exist in infancy that provide the foundation for later, more advanced abilities; (2) How are later abilities continuous, and how are they discontinuous, with these early emerging abilities; and (3) How do developmental changes in these abilities affect other aspects of quantitative thinking, such as mental arithmetic?

#### **(H1) NON-SYMBOLIC QUANTITATIVE THINKING**

#### **(H2) FOUNDATIONS OF QUANTITATIVE THINKING IN INFANCY**

Three key issues arise in studying the early foundations of human non-symbolic quantitative abilities. First, at what point in development do children reliably *discriminate* among quantities that differ in number and *represent* the number of entities in a set? Second, in early development, what mental mechanisms initially represent numerical values? Third, at what point in development can children *mentally manipulate* numbers (as measured, for example, by their ability to recognize impossible arithmetic transformations)? It turns out that all of these abilities are present early in infancy.

### **(h3) Discrimination of Numerical Quantities**

Without the ability to perceive the difference between two sets that differ only in number, it would be impossible to understand the similarity of sets that have only number in common (cross-modal mapping), to distinguish between possible and impossible arithmetic transformations of sets of objects (non-symbolic arithmetic), to link numeric symbols to their approximate or exact referents (estimation, counting), or to engage in economic transactions. For these reasons, research on the development of numerical cognition begins with efforts to establish the capacities (and limits) of infants' discrimination among quantities.

*(h4) Infants' Discrimination Among Numbers of Objects.* A consistent finding in research on development of quantitative abilities is that human infants notice changes in number in sets of 1-3 objects (Starkey & Cooper, 1980; Strauss & Curtis, 1981; Starkey, Spelke, & Gelman, 1983). In one early study of infants' numerical capacities and a number of later ones, researchers observed a spontaneous preference for the larger set when two sets differed in number by a factor of 2 or more, even with very large sets (e.g., 128 versus 32 elements in Fantz & Fagan, 1975; 32 versus 16 in Xu, Spelke, & Goddard, 2005; 16 versus 8 in Lipton & Spelke, 2003). In other studies, investigators used habituation paradigms in which infants were repeatedly presented a particular number of objects until their looking time decreased, and then were presented with a

different number of objects. Recovery of looking time to novel stimuli (dishabituation) was observed when the ratio between original and novel sets was 2:1 or 3:2 (2 versus 3 in Strauss & Curtis, 1981; 8 versus 4 in Xu, 2003). The earliest signs of this kind of numerical discrimination appeared in 21- to 44-hour-old neonates, who—having been habituated to a display of 2 (or 3) dots—recovered interest when shown 3 (or 2) dots (Antell & Keating, 1983).

Findings from studies of infants' discrimination of sets of objects suggest three general conclusions. First, at all ages, the number required to discriminate between two relatively large sets (i.e., 4 or more members) is not an absolute number but rather a ratio (e.g., discrimination of 4 versus 8 objects is typical at ages where discrimination of 8 versus 12 objects is not). Rather, as in the Weber-Fechner psychophysical function, *the probability of discrimination is proportional to the difference in the logarithms of the numbers*, where, for example,  $\ln(32) - \ln(16) = \ln(16) - \ln(8) = \ln(8) - \ln(4) > \ln(12) - \ln(8) = \ln(6) - \ln(4)$ . Second, the difference in logarithms required to discriminate between numbers decreases with age. Thus, older infants discriminate ratios that younger infants do not (Cordes & Brannon, 2008). Finally, for very small numbers (i.e., 3 or less), the probability of discrimination is uniformly high, higher than would be predicted from considering the differences in logarithms alone. Thus, discriminating 2 versus 3 is easier for infants than discriminating 4 versus 6.

Infant's discrimination of auditory sequences (Lipton & Spelke, 2003; vanMarle & Wynn, 2009), temporal intervals (Brannon et al., 2008; vanMarle & Wynn, 2006), events (Wynn, 1996; Wood & Spelke, 2005,) and collective entities (Wynn et al., 2002) conforms to these generalizations about number discrimination. To cite one example, the Weber-Fechner law applies to discriminations among number of sounds. When Lipton and Spelke (2003) familiarized 6-month-old infants to a sequence of 8 or 16 sounds, infants more often turned their heads to hear a novel number of sounds (16 or 8) than to

hear the original number of sounds. As with objects, the ratio of the sets is what matters; 6-month-olds discriminate between 32 and 16, 16 and 8, and 8 and 4 sounds, but not between 4 and 6 or 8 and 12 sounds (Lipton & Spelke, 2004; Xu, 2003; Xu et al., 2005). Also as with objects, the difference in logarithms required to discriminate between two numbers of sounds decreased with age: older infants discriminated ratios (e.g., 4:6) that younger infants did not (Lipton & Spelke, 2003). And as with objects, discrimination of very small numbers of sounds is consistently high, higher than would be expected from considering the ratio in isolation. When four-day-old infants were presented multisyllabic utterances, discriminations between 2 and 3 syllables were more likely than between 4 and 6 (Bijeljic-Babic et al., 1993).

### **(h3) Do Infants Represent Number *Per Se*?**

Whenever infants react to changes in a stimulus, a number of possible mechanisms might give rise to the reaction. In the case of numerical discrimination, different mechanisms might process numbers of discrete entities than other quantitative dimensions that often are correlated with number of discrete entities, such as summed area of the objects or their contour length). Because numerical and non-numerical parameters are inextricably linked in sets of objects, it is sometimes impossible to uniquely identify the factors that allowed infants to dishabituate to a display within a specific experiment (Mix, Huttenlocher, & Levine, 2002). Thus, infants' ability to discriminate sets of items that differ in number may not necessarily mean that infants possess mental mechanisms that code number *per se*. In the next two sections, we examine this issue, first by reviewing whether infants' sensitivity to non-numerical features of sets is sufficient to explain their ability to discriminate between sets that differ numerically, and then by examining evidence from cross-modal mapping studies that we view as decisive on this issue.

*(h4) Non-numerical Cues to Numerical Quantity.* One class of non-numerical cues that might cause an infant to dishabituate to a novel number of objects is continuous quantitative cues, such as surface area, volume, and contour length. A number of investigators have posited that discrimination among values of these continuous dimensions, rather than of number as such, explains findings that others have interpreted as indicative of numerical discrimination abilities (Clearfield & Mix, 1999, 2001; Gao, Levine, & Huttenlocher, 2000; Mix, Huttenlocher, & Levine, 2002; Feigenson, Spelke, & Carey, 2002). In an important challenge to early research on infants' numerical abilities, Clearfield and Mix (1999) habituated 6-month-olds to a series of stimuli that shared a set size (e.g., 3 objects) and a constant cumulative contour length. Infants were then presented a dishabituation trial, either with number held constant and cumulative contour length changed or vice versa. Infants responded as if they detected the change in contour length but not the change in number. Results of subsequent studies indicated that infants' discrimination of surface area resembles that of the Weber-Fechner psychophysical function, with novelty preference increasing linearly with the ratio of the surface areas, a result that paralleled previous findings with discrete objects (Clearfield & Mix, 2001). One interpretation of these results (e.g., Mix et al., 2002; Newcombe, 2002) is that young infants do not represent number and instead respond solely to non-numerical properties of the set.

On the other hand, several considerations suggest that non-numerical cues are insufficient to account for infants' ability to discriminate sets of objects. First, when non-numerical properties of a large set of objects (e.g., 8 or more) are varied in the habituation display while number is held constant, infants in the test phase of the procedure reliably dishabituate when shown a set with a novel number of objects that shares non-numerical properties of the habituation sets (Brannon, Abbott, & Lutz, 2004; Xu & Spelke, 2000). If infants were sensitive only to the non-numerical properties of a

set, this pattern of results would not occur. Second, discriminating non-numerical values of a set is often more difficult for infants than discriminating the numbers of objects in the set (Brannon et al., 2004; Cordes & Brannon, 2008, 2009). In Brannon et al.'s (2004) study, for example, 6-month-olds detected a 2-fold change in number after being habituated to a 5-fold change in surface area of the objects, but failed to detect a 2-fold change in surface area of the objects after being habituated to a 5-fold change in their number. Third, the similarity of findings on visual and auditory stimuli cannot be explained by spatial dimensions, such as contour length and surface area, which do not apply to sounds. Thus, rather than number being detected as a last resort, it seems more likely that infants simultaneously track the number of discrete objects in a set and the non-numerical characteristics of those objects. Consistent with this interpretation, combining auditory and visual cues to number improves the Weber-Fechner ratio that infants can discriminate over that which they can discriminate on the basis of visual cues alone (Jordan, Suanda, & Brannon, 2008).

*(h4) Numerical Representation in Infants: Evidence from Cross-Modal Mapping.* When infants are habituated to a display of 8 objects or 8 tones and then recover interest in a display of 4 objects or 4 tones, is it because they recognize that the “eightness” of the habituation displays had been violated by the “fourness” of the test displays? This question is crucial because it gets at the heart of whether infants have a concept like “four” or “eight”, as when adults generalize the word “four” to four votes, four bell-tolls, and four candies. To determine whether infants also have an abstract concept to which adults and older children would sensibly apply number words, researchers have tested infants’ ability to generalize across numerical groups that—like candies and bell-tolls—are perceived by different sensory modalities (Izard et al., 2009; Jordan & Brannon, 2006; Kobayashi et al., 2005; Lourenco & Longo, 2010). An important feature of these cross-modal mapping tests is that they obviate objections that infants’ reactions are



driven only by non-numerical features of the input – contour length usually increases with the number of candies, but it doesn't increase with the number of bell-tolls. Similarly, temporal length of a sequence of bell tolls generally increases with number of tolls but not with objects' contour length.

In cross-modal mapping studies, infants were initially reported to look longer at a set of 2-3 objects that matched the number of sounds that were played simultaneously (Starkey, Spelke, & Gelman, 1983). However, subsequent studies reported no such preference (Mix et al., 1997) or a reversed preference (Mix et al., 1997; Moore et al., 1987). In all cases, the effect sizes were quite small, leading to the hypothesis that the contradictory findings were due to participants on the tails of a distribution whose means were in the middle (Mix et al., 2003).

Recent studies, however, have reported robust evidence of cross-modal number matching for sets of 2-3 s in 6-month-olds (objects and tones in Kobayishi et al., 2005; faces and voices in Jordan & Brannon, 2006) and of larger sets in 2-day-olds (Izard et al., 2009). An interesting feature of Izard et al. (2009) is that newborns' looking time was greater when viewing congruent displays (4 syllables/4 objects or 12 syllables/12 objects) than incongruent displays (4 syllables/12 objects or 12 syllables/4 objects). The difference in that study between congruent and incongruent displays was nearly identical for another 3-fold change in number (6 vs 18), and both differences were much greater than those evoked by a 2-fold change in number (4 vs 8). These findings are thought provoking, because they suggest that when newborns represent number in different modalities, their number representations are subject to the same ratio dependence observed in number discrimination within a single modality.

### **(h3) Origins of the Mental Number Line**

If infants are capable of representing the number of items in a set, how might they do it? An important proposal for how this occurs—the mental number line

hypothesis—came from findings on number matching in non-human animals (Mechner, 1958; Platt & Johnson, 1971; for review, see Boysen & Capaldi, 1993).

*(h3) Parallels between Infants' and Non-human Animals' Numerical Magnitude Processing.* To examine rats' matching of number of physical actions to a criterion number (i.e., the number of actions rewarded on previous trials), Mechner (1958) and Platt and Johnson (1971) developed paradigms in which food was dispensed to a rat after it had pressed a lever a criterion number of times and then stopped pressing the lever, either in order to press another lever (Mechner, 1958) or to enter a feeding area (Platt & Johnson, 1974). Results of the two studies were similar, with the modal number (and standard deviation) of lever presses increasing with the criterion number (4, 8, 12, and 16). To ensure that the rats were estimating the number of lever presses, rather than the time since the trial had started, subsequent studies varied the degree of food deprivation imposed on the rats, on the logic that the hungriest rats would press the lever faster than the less-hungry rats (which they did). Speed of bar pressing did not affect matching of bar presses to the criterion number; the hungry rats still pressed the lever the same *number* of times as the less-hungry rats for each criterion number (Mechner & Guevrekian, 1962).

The similarity of rats' numerical processing to that of infants can be seen in the rats' pattern of errors. Specifically, the likelihood of pressing the lever a given number of times in response to a particular criterion number decreased *as a function of the difference in the two numbers' logarithms*. For example, as in the previously described studies of human infants, Mechner found that rats were more likely to confuse 6 with 4 lever-presses than to confuse 8 with 4, and rats were more likely to confuse 16 with 12 than 8 with 4. As Gallistel (1993) wrote, "It is as if the rat represented numerosity as a position on a [logarithmic!] mental number line (that is, a continuous mental/neural quantity or magnitude), using a noisy mapping process from numerosity to values on this

continuum, so that the one and same numerosity would be represented by somewhat different mental magnitudes (positions on the number line) on separate occasions" (p. 215).

Why might a "mental number line" (illustrated in Figure 1) be useful for thinking about infants' and non-human animals' encoding of number? One reason is that the mental number line provides a non-verbal mechanism for a true number concept, thereby accounting for prelinguistic infants' (and other animals') ability to generalize across perceptual modalities (Dehaene, 1992). If infants automatically convert the sight of six apples or the sound of six tones (Fig. 1, bottom panel, "Stimuli") to the same position on a mental number line (Fig. 1, middle panel, "Mental Number Line"), then—long before they learn conventional verbal counting—they could recognize an equivalent "sixness" of the two sets, to which a written or oral symbol ("six" or "6") could later be associated.

The mental number line metaphor is also useful for illustrating how number discrimination follows the Weber-Fechner law. That is, if mental magnitudes increase logarithmically with actual numeric value (as numbers on a slide rule), then numeric intervals on the lower end of the number line (like 1 and 2) are further apart and—given noisiness in the mapping—easier to discriminate than the same intervals at higher ends of the range, due to reduced overlap of signals.

The idea that the mapping of quantities to the number line is noisy provides an interesting way to think about what changes in infants' development. If the noisiness of the mappings to the logarithmic number line decreases with age, then the difference in logarithms required to discriminate any two quantities would also decrease with age. Thus, a logarithmically-scaled mental number line provides a simple way to conceptualize the numerical abilities of infants, how those abilities are limited by the Weber-Fechner Law, and how those abilities change with age and experience.

*(h4) The Neural Basis of the Mental Number Line.* What physical mechanisms generate the findings that are explained metaphorically by the mental number line construct? An idea that proved prescient was formulated by Dehaene and Changeux (1993). Working from the pattern of errors and solution times in numerosity discrimination studies that we reviewed earlier, as well as Dehaene's own research on symbolic number comparison in human adults, Dehaene and Changeux proposed the existence of innate numerosity detectors, that is, neural units that directly code numerosity (cf. "numeros" in Gelman & Gallistel, 1978).

In the Dehaene-Changeux model, objects of various sizes and locations are initially presented to a retina-like module (or to echoic auditory memory in the case of sounds), and then normalized for size and location on an intermediate topographical map of object locations. Number is registered from a map of numerosity detectors that sum all outputs from the intermediate topographical map of object locations. An important feature of such a mechanism is that the internal representation of numerosity would be highly correlated with the number of objects in a set, regardless of their physical characteristics. Additionally, simulations that embodied the model revealed that the activations evoked by different input numerosities overlapped with one another, with the degree of overlap being proportional to the difference between the logarithms of the numerosities. In this way, the performance of the Dehaene-Changeux neural model suggested explanations of the experimental findings from newborns and rats, and it provided a powerful theoretical model for how a mental number line might be realized.

Independent empirical support for a neural mechanism like that hypothesized by Dehaene and Changeux (1993) came from a series of later findings by Nieder, Miller, and colleagues (for a review, see Nieder, 2005). These investigators obtained single-cell recordings of neural activity as an awake animal (a macaque monkey) tracked the

number of objects in a set. In a typical task, a set of  $N$  objects (dots in various positions, configurations, and sizes) was presented to the monkey in a sample display, a brief delay was imposed, and then a test display of objects was presented that had either the same number of objects as, or a different number than, the original display. The monkey's task was to respond if the number in the test display matched the number in the sample display

Like the hypothetical 'numerons' in the Dehaene-Changeux model, many neurons were found to be tuned to a particular numerosity (Nieder & Merten, 2007; Nieder & Miller, 2003, 2004). These neurons maintained their numerical selectivity in the face of variation in the position, size, and configuration of the objects in the display (Nieder et al., 2002). For example, some neurons were found to show peak firing for a set of 1, other neurons peak responding at a set size of 2, others at 3, and so on, all the way to numbers in the 30s (which were the highest numbers tested). These neurons were involved in numerical memory as well as perception: When the monkey saw a set of 4 objects that then was hidden and the monkey had to maintain a representation of the number in memory, the 4-neurons maintained their activity more than competing neurons did.

Further, the tuning curves of these neurons showed Gaussian variability on a logarithmic scale (as in Fig. 1). This means that neurons that peaked when four objects were presented to the monkey would also respond (somewhat less) if three or five objects were presented, and would respond much less when 1 or 10 objects were presented. Consequently, when the monkey needed to judge whether he was being shown a new number or an old number of objects, accuracy was linked to the difference in logarithms between the two numbers. Thus, collectively, the number-tuned neurons identified by Nieder and colleagues formed a physical basis for the Weber-Fechner law.

These number-tuned neurons were most abundant in the lateral prefrontal cortex (PFC) and in the fundus of the intraparietal sulcus (IPS) (Nieder & Miller, 2004). Neurons in the IPS are thought to code number first, because these neurons required shorter latencies on each trial to become numerosity selective than did PFC neurons (Nieder & Miller, 2004). Based on previous work showing that posterior parietal cortex (PPC) and PFC are functionally interconnected (Cavada & Goldman-Rakic, 1989, Chafee & Goldman-Rakic, 2000, Quintana et al., 1989), it also seems likely that numerical information first encoded in PPC might be conveyed directly or indirectly to the PFC, where the signals are amplified and maintained to gain control over behavior. Finally, in a remarkable similarity to the mental number line metaphor, the number-tuned neurons in PPC were found to be so intermingled with neurons that code for line length that number and length sensitive neurons were sometimes under the same electrode tip (Tudusciuc & Nieder 2007)! Thus, the distribution and timing of activation of number-selective neurons suggests that the most likely site for a neural “mental number line” is a fronto-parietal circuit.

Could the mental mechanism serving infants’ representation of number be the same ones serving monkeys’ numerical representations? Because obtaining single-cell recordings requires neuro-surgical implantation of electrodes, obtaining directly equivalent measures in human infants is unethical. Therefore, to obtain unobtrusive measures of human infants’ neural coding of number, Izard and colleagues (2008) obtained event-related potentials from 3-month-old infants while they were presented with a succession of sets that either changed in number but not object type (e.g., from 4 ducks to 12 ducks) or in object type but not number (e.g., from 4 ducks to 4 balls). The study was designed so that most sets had the same number and type of objects (e.g., repeatedly presenting non-identical images of 4 ducks). Occasionally, however, a test image appeared that broke this regularity in either number, object identity, or both. The

brain response to this disruption was recorded in order to measure the event-related potential (ERP). Across different groups of infants, the numeric changes were 2 versus 3, 4 versus 8, or 4 versus 12.

In all cases, the ERPs revealed that infants' brains detected both types of changes (number and object). To examine the underlying circuitry, the investigators used a source reconstruction method that was based on a model of the infants' cortical folds. This analysis suggested that whereas the infants' left occipitotemporal cortex responded to object novelty, the infants' right parietal cortex responded to numerical novelty, like the corresponding area of the monkeys studied by Nieder.

This method for localizing infants' neural activity is admittedly coarse, due to the relatively poor spatial resolution of EEG. However, the results were quite similar to those observed in adults and 4-year-olds in studies using more spatially precise fMRI methods (Cantlon et al., 2006; Piazza et al., 2004). In these studies too, regions of the occipitotemporal area reacted to changes in object identity but not to changes in number, whereas posterior parietal regions reacted to changes in number but not to changes in object identity. Both EEG and fMRI data are consistent with the conclusion that number is among the dimensions that are quickly extracted when processing visual stimuli.

### **(h3) Arithmetic in Infancy: Travels on the Mental Number Line**

An important property of a number line is that it makes basic addition and subtraction trivial. That is, traveling four spaces forward from four registers the sum of four and four; traveling four spaces back from eight registers the difference between eight and four, and so on.

Therefore, if infants possess a mental number line and encode numerical values of stimuli, they should register sums and differences of numeric quantities (at least approximately). This hypothesis led Wynn (1992) to conduct a series of experiments on infants' arithmetic capacities. To test ability to compute  $1 + 1$ , for example, Wynn

recorded infants' looking times as they watched one object appear to be placed behind an opaque screen and then another object added to it behind the screen. When the screen dropped, seeming to reveal the arithmetically impossible event  $1 + 1 = 1$ , infants looked longer than when the screen dropped and revealed an outcome consistent with the arithmetically realistic event  $1 + 1 = 2$ . To be sure that this difference in looking wasn't caused by infants simply preferring to look at one object rather than two, Wynn also examined two objects initially seeming to be placed behind a screen and one object then seeming to be removed. In this situation, infants looked longer when they were shown the arithmetically impossible event of two objects being present after one seemed to be removed, rather than the arithmetically realistic event of one being present. This pattern of findings has since been replicated by other investigators (Koechlin et al., 1997; Simon et al., 1995).

Testing the idea that infants correctly registered the outcome of the arithmetic operation and were surprised by the arithmetically impossible event, Berger, Tzur, & Posner (2006) collected EEG recordings of infants' brain activity during presentation of correct and incorrect arithmetical operations. In addition to replicating Wynn's finding of infants looking longer at the seemingly impossible number of objects, infants' brain activity indicated an error detection process in the arithmetically impossible condition. Specifically, the topography and frequency ( $\theta$ -band effects) of the infants' brain activity were quite similar to what had been found previously in adults when they observe or make responses that they know are wrong, though the error detection signal emerged more slowly in infants than in adults.

An alternative to Wynn's interpretation was that the source of infants' surprise was not violation of their *numeric* expectations but rather a violation of their expectations about objects (Simon, 1997; Uller, Carey, Huntley-Fenner, & Klatt, 1999). In this account, babies used the appearance of the objects to track their individual identities



and were surprised when a new individual—not a new number—appeared or went missing. This is a reasonable distinction: When determining whether everyone is present in a small research group, for example, the professor might notice that Tom is missing from the normal attendees of Tom, Dick, Harry, without bothering to encode how many people were present. As we have seen, infants' abilities to compare sets of four or fewer objects is much greater than would be expected by a mental number line; thus, a subitizing mechanism for tracking small numbers of objects (Kahneman, Treisman, & Gibbs, 1992; Scholl & Leslie, 1999; Trick & Pylyshyn, 1994) might explain infants' error detection on this task.

One way to test whether babies' *numeric* expectations are violated by arithmetically impossible events is to test whether this surprise is also evident when babies witness arithmetically impossible transformations of large sets, where all the individuals couldn't be represented through subitizing (McCrink & Wynn, 2004). Consistent with Wynn's original interpretation that infants were surprised by the numerical outcome, when 9-month-old babies were confronted with arithmetic transformations of large sets (e.g.,  $5 + 5 = 10$  versus  $5 + 5 = 5$ , or  $10 - 5 = 5$  versus  $10 - 5 = 10$ ), they also looked longer at the arithmetically-impossible events than the arithmetically-possible ones.

One reason why infants might look longer at the "impossible" outcome is that a logarithmically compressed mental number line representation supports babies' ability to register approximate sums and differences. If adding  $n_2$  to  $n_1$  involves traveling  $n_2$  spaces forward from  $n_1$  on the mental number line, then babies would arrive at the position  $\log(n_1) + \log(n_2)$ , which would be a considerable overestimation of the actual result. Conversely, subtraction through traversing a mental number line would yield a considerable underestimation of the actual results. From this perspective, babies would

have no doubt found  $5 + 5 = 5$  surprising because they would have experienced it as  $\log(5) + \log(5)$  and thus expected to see  $\log(25)$ !

Consistent with this compressed mental number line interpretation, McCrink & Wynn (2009) reported that 9-month-old babies' expectations of arithmetic transformations overestimate the results of additive operations and underestimate the results of subtractive operations. Thus, when babies were initially shown a sequence of events equivalent to  $6 + 4$ , they looked significantly longer when the raised screen revealed 5 objects than when it revealed 20 objects. Similarly, when babies were shown a sequence equivalent to  $14 - 4$ , they looked significantly longer when the raised screen revealed 20 objects than when it revealed 5. In the same study, the infants looked equally long at 20 objects as 10 at the end of the first example, and looked as long as 5 objects as at 10 at the end of the second example. McCrink and Wynn (2009) explained these findings in terms of "operational momentum." In particular, they suggested that infants moved too far in a positive direction along the mental number line when they added sets of dots and too far in a negative direction when they subtracted sets of dots because they didn't know where to stop their travel along the mental number line. Another possibility is that the results reflected the infants using logarithmically compressed representations of numerical quantity. In particular, the arithmetically impossible events (e.g.,  $6 + 4 = 20$ ;  $14 - 4 = 5$ ) that babies looked at as much as the correct answers were those answers predicted by the view that the infants were mentally representing the problem on a logarithmically compressed mental number line.

## **(H2) CHANGES BEYOND INFANCY IN NON-SYMBOLIC NUMERICAL PROCESSING**

*(h3) Numerical Discrimination.* Infants' discrimination between sets of objects that differ in number predict several features of older children's and adults' performance on similar, non-verbal numerosity discrimination tasks. Recall the conclusion that the probability of infants' discrimination was related to the difference between the

logarithms of the number of objects in the two sets. When prevented from counting sets of objects (e.g., through tight time limits or interpolated tasks), older children's and adults' reaction times show the same pattern. For example, the time required for children and adults to select the larger set is proportional to the logarithm of the distance between the numbers of objects in the two sets, both for sets of less than 10 objects (Buckley & Gillman, 1974; Huntley-Fenner & Cannon, 2000) and for sets of 10 or more objects (Birnbau, 1980; Piazza et al., 2004; Ratcliff et al., in press).

Another conclusion from infants' performance was that the difference in logarithms needed to discriminate the number of objects in two sets decreases with age. The relation between set size and solution times shows the same pattern in childhood. The time required to discriminate between two numbers that are close together decreases from early childhood to later childhood to adulthood (Ratcliff et al., in press). The "internal Weber fraction," which indicates the difference in ratios needed to reliably discriminate set sizes, has proved useful for quantifying this type of developmental change (Izard et al., 2008; Piazza et al., 2004). Piazza et al. (2010) reported an exponential decline in the internal Weber fraction from an average of 0.34 for kindergarteners, down to 0.25 for 10-year-olds, to 0.15 for adults. Similarly, Halberda and Feigenson (2008) reported internal Weber fractions of 0.38 for kindergarteners and 0.11 for adults. Interestingly, individual differences in Weber fractions within an age group are correlated with the students' math achievement test scores (Halberda, Mazocco, & Feigenson, 2008). This pattern supports the hypothesis that the precision of the mental number line is foundational to other quantitative skills.

A third conclusion from studies of infants' numerical abilities is that infants' discrimination of very small numbers of objects are consistently higher than would be expected from considering the Weber fractions in isolation. The same is true for older children's and adults' solution times on similar problems. That is, the time required to

judge the greater of two sets of 3 or fewer objects is uniformly low, much lower than would be expected given their ratios alone (Oyama, Kikuchi, & Ichihara, 1981; Chi & Klahr, 1975; Mandler & Shebo, 1982; Trick & Pylyshyn, 1994).

The similarities in performance among infants, older children, and adults suggest that the process of numerosity discrimination is similar, with all groups accessing the mental number line representation. This hypothesis is consistent with findings of substantial overlap between the regions in the intraparietal sulcus that are activated by infants, younger children, older children, and adults when comparing numerosities (Castelli, Glaser, & Butterworth, 2006; Piazza Izard, Pinel, et al. 2004; Piazza, Mechelli, Price, et al. 2006; Piazza, Pinel, Le Bihan, et al. 2007).

*(h3) Development of Non-symbolic Arithmetic Beyond Infancy.* To examine the development of non-symbolic arithmetic beyond infancy, McCrink, Dehaene, & Dehaene-Lambertz (2007) showed adults several hundred videos of two successive sets of dots and asked them to approximate their sum or difference by choosing one of two sets of dots. As with infants, adults almost always overshot the correct outcomes on addition problems, as if they had moved “too far” along the mental number line, whereas they almost always undershot the correct outcomes on the subtraction problems, again moving too far on the mental number line in the other direction. In addition, the researchers found that from adults’ modal response, the distribution of other responses tapered off as a function of the ratio of the true and alternative quantities, just as would be predicted by Weber-Fechner’s law. To illustrate the magnitude of this error, the presented subtraction problem of  $32 - 16 = 8$  was judged to be correct approximately 60% of the time, which is quite a radical departure from moving along a linearly scaled mental number line.

## **(H1) DEVELOPMENT OF SYMBOLIC REPRESENTATION OF NUMBERS IN CHILDHOOD AND BEYOND**

In the previous section, we reviewed evidence that a logarithmically-compressed mental number line is a powerful construct for understanding non-symbolic representations of numerical magnitude, that the construct provides a good analogy to the way that the brain codes numerosity, and that the hypothesis accounts for prominent features of infants', older children's, and adults' non-symbolic arithmetic capacities.

In this section, we examine the extent to which early-developing, non-symbolic representations of quantity serve as a foundation for later-developing, symbolic ones. As we will show, on some tasks, the logarithmic function translating symbolic numeric quantities into a subjective representation is preserved from early childhood through adulthood. We will also see that new ways of thinking about number emerge with age and experience with symbolic expressions of numbers, thereby preparing children to acquire more complex quantitative skills. The new and old numerical representations coexist after the new forms are acquired, with their use varying with the situation and with the child's numerical experience and mathematical aptitude. Evidence for these ideas comes largely from tasks such as number line estimation that require participants to translate between alternative quantitative representations, at least one of which is inexact and at least one of which is a symbolically expressed number. Finally, we examine the influence of internal representations of number on uses of those representations in symbolic arithmetic, their relation to overall mathematical achievement, and ways in which research on the mental number line has led to effective instructional interventions.

### **(h2) The Growth of Quantitative Thinking During Childhood: Understanding and Operating on Numerical Symbols**

In this section, we examine how children learn to use the symbols of the decimal system to think about quantities, and how mastery of this symbol system profoundly expands their mathematical capabilities. Early in the learning process, these symbols — like the Arabic numerals in Diester and Nieder’s study—are meaningless stimuli for young preschoolers. For example, 2- and 3-year-olds who count flawlessly from 1-10 have no idea that  $6 > 4$  and  $8 > 6$ , regardless of whether the number symbols are spoken or written, nor do children of these ages know how many pennies to give an adult who asks for 4 or more (and very early on, not even how many to give an adult who asks for 1-3) (Le Corre, Van de Walle, Brannon, & Carey, 2006; Le Corre & Carey, 2007; Sarnecka & Carey, 2008).

As young children gain experience with the symbols in a given numerical range and associate them with non-verbal quantities in that range, they initially map them to a logarithmically-compressed mental number line (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Booth & Siegler, 2006; Geary, Hoard, Nugent, & Byrd Craven, 2008; Opfer, Thompson, & Furlong, 2010; Siegler & Booth, 2004; Siegler & Opfer, 2003; Thompson & Opfer, 2010). Over a period that typically lasts 1-3 years for a given numerical range (0-10, 0-100, or 0-1000), their mapping changes from a logarithmically compressed form to a linear form, in which subjective and objective numerical values increase in a 1:1 fashion. Use of linear magnitude representations occurs earliest for the numerals that are most frequent in the environment, that is the smallest whole numbers (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2009), and it gradually is applied to increasingly large numbers (Siegler, Thompson, & Opfer, 2009). In this section, we first examine this process of learning representations of symbolic numbers as it applies to numerical magnitude comparison and then as it applies to number line estimation; after this, we examine the consequences of individual differences in these numerical

representation processes and interventions that promote the growth of linear representations.

(h3) *Symbolic number comparison.* When 3-year-olds from middle-income backgrounds or 4- and 5-year-olds from low-income backgrounds are presented numerical magnitude comparison problems involving the numbers 1-9 in symbolic form, their performance is near chance (Ramani & Siegler, 2008; Siegler & Robinson, 1982). In contrast, by kindergarten or first grade, most children are highly accurate on these numerical comparison problems with Arabic numerals. Their solution times, and those of older children and adults, are subject to the same Weber-Fechner law that characterizes numerical comparisons of non-symbolic stimuli, such as dot arrays (Buckley & Gillman, 1974; Moyer & Landauer, 1967; Sekuler & Mierkiewicz, 1977), with the time required to select the larger of two numbers being inversely proportional to the logarithm of the distance between the numbers being compared. Thus, the time required to select the larger of 3\_5 is less than the time required to compare 5\_7, and the time required to select the larger of 30\_50 is less than the time required to compare 50\_70. This finding suggests that Arabic numerals automatically activate the same mental number line representation that encodes non-symbolic numerosity (as in Fig. 1).

Consistent with this idea, fMRI studies have revealed that even very brief presentations of number symbols evoke number-related activation in the intraparietal cortex of educated adults (Naccache & Dehaene, 2001). The systems for representing non-symbolic and symbolic numbers appear to be closely linked. In addition to the physical overlap, if not identity, between the systems, habituation of one leads to habituation of the other. Thus, after adaptation to 17, 18, or 19 dots, dishabituation (as measured by fMRI activations) is observed when the Arabic numeral 50 is presented, but not when the numeral 20 is presented (Piazza et al. 2007). These results suggest that—at least in educated adults—populations of neurons in parietal and prefrontal

cortex are activated by both non-symbolic numerosities and by number symbols. Put another way, the mental number line appears to link symbolic and non-symbolic numerical representations.

Given that the invention of Arabic numerals only dates back a few thousand years (Ifrah, 2000), it is implausible that the brain evolved specifically to handle them. More likely, the link between the two systems arises through learning during childhood. To investigate how this learning process might occur, Diester and Nieder (2007) trained two monkeys to associate Arabic numerals with the numerosity of multi-dot displays. After the training, a large proportion of PFC neurons encoded numerical values, irrespective of whether the numbers had been presented to the monkeys in the form of dots or Arabic numerals. Moreover, these neurons exhibited similar tuning functions, with activity falling with increasing numerical distance from the neuron's 'preferred' numerical value. Over the course of training, numeral-numerosity associations progressively shifted from the PFC to the IPS.

A similar pattern may exist in human development (Ansari et al., 2005; Kaufmann et al., 2006; Rivera et al., 2005; Houdé et al., 2010). The distance effect observed in monkey's PFC neurons has also been observed in the PFC of preschool children. Moreover, with age and experience, neural activation in response to numerical stimuli shifts to posterior parietal areas, particularly in the left hemisphere, much as the numeral-numerosity associations shifted to IPS during the later stages of training among the monkeys studied by Diester and Nieder (2007). Together, these results suggest that the PFC might be the first cortical area to link numerals to the mental number line, and that with age and experience, the IPS is increasingly involved in linking numerals with the numerosities for which they stand.

*(h3) Number-line estimation.* An alternative method that has been developed to examine the development of representations of numerical magnitudes is the number-



line estimation task (Siegler & Opfer, 2003). On this task, participants are shown a blank line flanked by a number at each end (e.g., 0 and 1,000) and asked where a third number (e.g., 150) would fall on the line. This task is particularly revealing about representations of numerical magnitude because it transparently reflects the ratio characteristics of the number system. Just as 150 is twice as large as 75, the distance of the estimated position of 150 from 0 should be twice as great as the distance of the estimated position of 75 from 0. More generally, estimated magnitude ( $y$ ) should increase linearly with actual magnitude ( $x$ ), with a slope of 1.00, as in the equation  $y = x$ .

Across a number of cross-sectional studies using this number-line estimation task (Booth & Siegler, 2006; Laski & Siegler, 2007; Opfer & Siegler, 2007; Siegler & Booth, 2004; Siegler & Opfer, 2003; Thompson & Opfer, 2008, 2010), a systematic difference between younger and older children's estimates has been evident: Younger children's estimates of numerical magnitude typically follow Fechner's law ( $y = k \times \ln x$ ) and increase logarithmically with actual value (Figure 2). In contrast, older children's estimates for the same range of numbers increase linearly with actual value.

This developmental sequence emerges at different ages with different numerical ranges (Figure 2). It occurs between preschool and kindergarten for the 0-10 range, between kindergarten and second grade for the 0-100 range, between second and fourth grade for the 0-1,000 range, and between third and sixth grade for the 0-100,000 range (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Opfer & Siegler, 2007; Siegler & Booth, 2004; Thompson & Opfer, 2010). Thus, as shown in Figure 2A, on the 0-100 number line estimation task, the logarithmic function fit kindergartners' estimates better than did the linear function, but the linear function fit second-graders' estimates better than did the logarithmic function. The same transition occurs roughly a year later for children with mathematical learning difficulties (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). The timing of the changes corresponds to the periods when children are

gaining extensive exposure to the numerical ranges: through counting during preschool for numbers up to 10, through addition and subtraction between kindergarten and second grade for numbers through 100, and through all four arithmetic operations in the remainder of elementary school.

How might these changes occur? One clue came from the estimates of sixth graders and adults in Siegler and Opfer (2003). The median estimates of both groups increased linearly with numeric value, but the variability of estimates for both was smallest near the *quartiles* on the number line, as if these older children and adults mapped numbers like 0, 250, 500, 750, and 1000 to 0, 1, 2, 3, and 4—that is, numbers that other studies indicated they already represent linearly (Chi & Klahr, 1975; Siegler & McGilly, 1989; Siegler & Robinson, 1982). This observation suggested that children might map early-developing linear representations of numerical magnitudes to guide their learning about the magnitudes of less familiar, larger numerals, a process that might be aided through children learning about fractions and percentages in third through fifth grade.

To test this idea about how the logarithmic-to-linear shift occurs, we conducted three microgenetic studies that allowed us to examine changes in numerical estimation on a trial-to-trial basis (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008). In these studies, we used an experimental manipulation to help students who initially used logarithmic representations to adopt a linear representation. Specifically, we provided children with feedback on their estimates of numbers around 150, the point where the logarithmic and linear functions that pass through 0 and 1,000 are most discrepant. The idea was that this feedback would be highly salient, due to it indicating that the children's estimates were far from correct, and that the experience would lead children to draw an analogy between what the feedback indicated about the correct placement of 150 on a 0-1000 number line and their existing knowledge of the

correct placement of 15 on a 0-100 number line. After receiving feedback on their estimates of numbers around 150, second-graders in all three studies provided estimates that increased linearly with actual value (Figure 3A). This representational change was evident in differences in the median estimates of the treatment groups on pretest and posttest, with pretest estimates being best fit by logarithmic functions and posttest estimates being best fit by linear functions. Providing feedback regarding correct placements at other numerical values, such as 725, also increased the linearity of estimates, but change occurred more slowly and to a somewhat lesser extent (Opfer & Siegler, 2007). Consistent with the interpretation that this change came about through a process of analogy, when children were given an opportunity to compare (1) the placement of 15 cherries on a line flanked by 0 and 100 cherries, and (2) the placement of 1500 on a line flanked by 0 and 10,000, children quickly apprehend the similarity of the two situations, and use it to improve their estimates of the larger numbers (Thompson & Opfer, 2010).

As is evident in Figure 3, second-graders' newly adopted linear representation spanned the entire 0–1,000 range and was not simply a “local fix” to the numbers near 150. This effect across the entire numerical range was consistent with the view that the numerical magnitude representation has psychological reality as a coherent unit, rather than simply being a convenient way of summarizing data. Moreover, like an analogical insight (e.g., Gick & Holyoak, 1983; see Holyoak, Chap. 13), the shift occurred abruptly, much more abruptly than what had been observed in studies of transitions in related areas such as addition and numerical insight problems (e.g., Siegler & Jenkins, 1989; Siegler & Stern, 1998). Often, feedback on a single trial was sufficient to yield a shift from a logarithmic to a linear pattern of estimates (Opfer & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Opfer, 2008).

The finding of new symbolic representations arising through analogies to representations of similar relations in simpler contexts is widely evident in cognitive development (for a review, see Gentner, 2010). Such analogy-based representational change is important, because it suggests a potential solution to a problem that is endemic to the mental number line representation discovered by Nieder and colleagues. The problem is that it seems as though one would simply run out of “numerons” to represent large numbers. Do people and other animals possess neurons that fire more in response to seeing 1582 dots than in response to 1583? However, if learners can map small numbers to large ones, they would gain the ability to use the relations among small number representations to represent relations among larger magnitudes (as when adults mapped 250-500-750-1000 to 1-2-3-4 or  $\frac{1}{4}$ ,  $\frac{1}{2}$ ,  $\frac{3}{4}$ , 1 or to 25%, 50%, 75%, and 100%). If this occurs at the neural level (and in some sense it must), then having a limited number of numerons imposes no inherent limit on the number of numeric magnitudes that could be represented.

*(h3) Measurement and numerosity estimation.* To examine the generality of the developmental transition revealed by the number line task, several studies have tested whether similar changes are evident in estimates of line lengths and numbers of discrete objects (measurement estimation and numerosity estimation) (Booth & Siegler, 2006; Laski & Siegler, 2007; Thompson & Siegler, 2010). On the measurement estimation task, children saw an extremely short line, labeled “1 zip”; a long line, labeled “1,000 zips”; and a number indicating the length of a line (in zips) that should be drawn. Children drew a line to approximate the desired length, then were asked to draw a line of a different number of zips, and so on. On the numerosity estimation task, children saw a computer screen that depicted one box with 0 dots, one with 1,000 dots, and a third, initially empty, box that could be filled with the desired number of dots by placing the cursor in the “increase” box or the “decrease” box and holding down the mouse until

the desired number of dots was reached (the time limit was too short for the children to count the dots).

On each task, the same logarithmic to linear shift that had been observed with number line estimation was observed (Booth & Siegler, 2006; Thompson & Siegler, 2010). On the measurement estimation task in Booth & Siegler (2006), for example, the variance accounted for by the best-fitting linear function increased with age (from 85% to 98%), whereas the variance accounted for by the best-fitting logarithmic function decreased with age (from 91% to 74%). Further, individual second graders' measurement estimates were more likely to be better fit by the logarithmic function than by the linear function (70% vs. 30%), whereas fourth graders' estimates were more likely to be better fit by the linear function than by the logarithmic function (78% vs. 22%). Similarly, the percentage of children for whom the linear function provided the better fit on the numerosity estimation task increased from 43% to 82%, whereas the percentage of children for whom the logarithmic function provided the better fit decreased from 57% to 18%.

*(h3) Number categorization.* To examine whether the logarithmic to linear transition extended beyond estimation tasks, Laski and Siegler (2007) presented 5- to 8-year-olds with a numerical categorization task. Children were told that 1 was a "really small number" and that 100 was a "really big number" and then were asked to categorize numbers between 1 and 100 as "really small," "small," "medium," "big," or "really big." Each child's categorization of each number was assigned a numerical value ranging from 1 for the "really small" category to 5 for the "really big" category. Then, the mean value for the categorizations of each number was computed, and the fit of linear and logarithmic functions to the mean categorization scores for the full set of numbers was calculated.

Kindergartners' mean categorizations of the numbers were better predicted by the best-fitting logarithmic function than by the best-fitting linear function. In contrast,

second-graders' mean categorizations were (non-significantly) better predicted by the best-fitting linear function than by the best-fitting logarithmic function. The same change was evident for the number line task. Moreover, the linearity of individual children's number line estimation and categorization patterns were highly correlated, providing additional evidence for the generality of the logarithmic to linear transition in representations of numerical magnitudes.

If number line estimation and categorization reflect the same underlying numerical representation, then experience that leads to improved number line estimation might transfer to numerical categorization. To test this hypothesis, Opfer and Thompson (2008) presented kindergartners who initially produced number line estimates that were more logarithmic than linear with feedback designed to improve the linearity of number-line estimates and tested both number line estimation and categorization. As expected, after the feedback experience, the linear function provided a better fit to the mean category judgments of children who received feedback on their number-line judgments than for those who did not receive such feedback. Thus, the change from a logarithmic to a linear representation on the number line task extended to the categorization task, even without any feedback on that task.

*(h3) Symbolic Arithmetic.* Linear representations of numerical magnitudes also are important for symbolic arithmetic abilities, both for approximating answers and for learning exact sums. One source of evidence regarding approximations to correct sums comes from Gilmore, McCarthy, and Spelke's (2007) study of preschoolers' estimates of answers to arithmetic problems that they had not yet encountered in school. The investigators presented 5- and 6- year-olds with problems such as "Sarah has 21 candies, she gets 30 more, John has 34 candies— who has more?" To insure that the preschoolers understood the symbols that were being used, the problems were simultaneously presented both orally—as spoken numerals—and in writing, as Arabic numerals.

Despite the fact that the preschoolers had received no training with numbers of that size, they spontaneously performed better than chance (60%–75%). This was true regardless of their socio-economic origins. Performance was still approximate, however, and depended on the ratio of the two sums that the children were choosing between, a signature of the Weber-Fechner law.

Other evidence for the relation between the linearity of numerical magnitude representations and arithmetic knowledge comes from positive relations between the linearity of first through fourth graders' estimates on number line, measurement, and numerosity tasks on the one hand and the accuracy of their estimates of answers to two digit plus two digit addition problems on the other (Booth & Siegler, 2006). Yet other correlational evidence comes from positive relations in first graders' linearity of number line estimates and the number of single digit addition problems that the children answered correctly (Booth & Siegler, 2008; Geary, et al., 2007).

In addition to this correlational evidence, linear representations of numerical magnitude also play a causal role in arithmetic learning. Booth and Siegler (2008) pretested first-graders' number-line estimation and retrieval of answers to 13 addition problems, ranging in difficulty from  $1 + 4$  to  $49 + 43$ . Then, children were trained on the easiest 2-digit + 2-digit problems that they had answered incorrectly on the pretest. All children were presented with each of these addition problems three times, with feedback regarding the correct answer being provided after each presentation.

A randomly chosen half of the children were also presented with analog linear representations of the addends and sum. This manipulation was intended to inculcate a linear representation of the numbers in the addition problems and ideally of numbers in the 0–100 range more generally. Children in this experimental condition saw a number line with 0 at one end and 100 at the other, then saw the first addend represented by a red bar just above the line, then the second addend represented by a blue bar just below

the line, and then the sum represented by a purple bar straddling the line. Thus, if the problem was  $43 + 49$ , the red bar would be 43% of the number line's length, the blue bar 49% of its length, and the purple bar 92% of its length. The logic was that seeing the linear representations of the addends and sums along the number line would allow children to encode the numerical magnitudes more accurately and thus help them retrieve the answers to the problems.

Presentation of the analog representations of the addends and sum along the number line was causally related to arithmetic learning; it increased the number of addition problems correctly recalled on the posttest and also improved the linearity of the children's number-line estimates. Moreover, the effect of the experimental manipulation was even stronger for measures of the closeness of addition errors to the correct sum than for the number of correct sums, supporting the view that activating the linear representation was the means through which the experimental manipulation produced its effect. If this mechanism were not involved, why else would children who were presented with the analog representations of the addends and sums increasingly advance incorrect answers that were close to the sum and decreasingly produce answers that were far from it? Similarly, randomly assigning preschoolers to play a linear numerical board game, rather than a color board game, has been found to increase preschoolers' learning of answers to arithmetic problems (Siegler & Ramani, 2009). Thus, linear magnitude representations are both causally and correlationally related to arithmetic learning.

*(h3) Relations between Numerical Magnitude Representations and Overall Mathematics Proficiency.* Both non-symbolic and symbolic numerical magnitude representations have been found to be related to standardized mathematics achievement test performance. Halberda et al. (2008) found that accuracy of non-symbolic number comparison was positively related to achievement test scores in mathematics, but not to other domains of



the school curriculum. Similarly, Holloway and Ansari (2008) found that individual differences in the distance effect during symbolic number comparison among children aged 6 to 8 years is related to mathematics achievement but not reading achievement. Moreover, the linearity of number line, measurement, and numerosity estimation, as well as of numerical categorization, have been related to mathematics achievement test performance (Booth & Siegler, 2006; 2008; Laski & Siegler, 2007); and children with mathematical learning difficulties, as defined by low mathematics achievement test scores, often generate logarithmic patterns of estimates for several years beyond the time when other students have adopted linear representations for the same numerical range (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008). Thus, linear representations of numerical magnitudes seem related to general as well as specific numerical competencies.

*(h3) Theoretically-based Educational Interventions.* Research on the centrality of the mental number line in numerical knowledge suggested an educational intervention that has proved highly effective with low-income preschoolers. The intervention began with the question, “How do children develop an initial linear representation of numerical magnitudes?” Experience with counting likely contributes, but such experience is insufficient for children to construct linear representations of numerical magnitudes, as indicated by the previously described dissociation between counting knowledge and magnitude comparison of the numbers that can be counted.

One common activity that was hypothesized by Siegler and Booth (2004) to help children generate linear representations is playing linear, number board games -- that is, board games with linearly arranged, consecutively numbered, equal-size spaces (e.g., *Chutes and Ladders*.) These board games provide multiple cues to numbers’ magnitudes. The greater the number in a square, the greater: a) the distance that the child has moved the token, b) the number of discrete moves of the token the child has made, c) the

number of number names the child has spoken, d) the number of number names the child has heard, and e) the amount of time since the game began. Thus, children playing the game have the opportunity to relate the number in each square to the time, distance, and number of manual and vocal actions required to reach that number.

To test whether playing number board games promotes number sense, Ramani and Siegler (2008; Siegler & Ramani, 2008; 2009) randomly assigned 4- and 5-year-olds from low-income backgrounds to play either a number board game or a color board game. At the beginning of each session, children in the number board condition were told that on each turn, they would spin a spinner that would point to “1” or “2”, that they should move their token that number of spaces, and that the first player to reach the end would win. Children in the color board condition were told that on each turn, they would spin a spinner that could point to different colors, that they should move their token to the nearest square with the same color as the one to which the spinner pointed, and that the first player to reach the end would win. The experimenter also told children to say the numbers (colors) on the spaces through which they moved. Thus, children in the number board group who were on a 3 and spun a 2 would say, “4, 5” as they moved their token. Children in the color board group who were on green and spun a “blue” would say “purple, blue.” If a child erred or could not name the numbers or colors, the experimenter correctly named them and then had the child repeat the names while moving the token. The preschoolers played the number game or the color game about 20 times over four 15 minute sessions within a two-week period; each game lasted about 3 minutes.

Playing the number board game led to dramatic improvements in the low-income preschoolers’ number line estimates. Accuracy of number line estimation, magnitude comparison, counting, and numeral identification increased from pretest to posttest among children who played the number board game. Gains remained present on a follow-up nine weeks later. In contrast, there was no change in the accuracy of estimates of children who played the color board game (Ramani & Siegler, 2008; Siegler & Ramani,

2009). Playing the game also improved the children's learning of subsequently presented addition problems (Siegler & Ramani, 2009).

## **(H1) CONCLUSIONS AND FUTURE DIRECTIONS**

Quantitative thinking, ranging from the ability to choose the greater of two sets of a few objects to the ability to project arithmetic transformations with very large numbers, is central to the lives of humans and other animals. In this chapter, we have argued that despite quantitative thinking playing important roles in the lives of both humans and other animals, there is also a fundamental distinction present between two kinds of quantitative thinking—non-symbolic and symbolic—and that this distinction is essential to understanding what develops in human quantitative thinking.

On non-symbolic quantitative tasks, similarities between human and non-human quantitative thinking pervade many levels of analysis. In a wide range of species and age groups (1) there is a capacity to mentally represent and compare the approximate number of objects or events in sets; (2) mental representations of non-symbolic numeric quantities are noisy and increase logarithmically with actual quantity, leading to the speed and accuracy of comparison depending on the ratio of the two numbers, and (3) the neural correlates of non-symbolic number representation are distributed in regions of fronto-parietal cortex that overlap greatly across species and age groups from infants to adults. A common cognitive mechanism—a logarithmically-compressed “mental number line”—also appears to underlie thinking about non-symbolic quantities in people and other animals.

Non-symbolic quantitative thinking, however, has important limitations. Non-symbolic representations of quantitative properties of sets greater than three or four objects are inexact, making impossible activities such as economic transactions that require precision (Furlong & Opfer, 2009). Lack of symbols also makes it impossible to

communicate numerical properties to other minds (e.g., that there are 12 sheep in one herd but 18 in another). Further, without symbols, it is impossible to preserve numerical information over time, as when recording tallies of livestock in a ledger, and it is impossible to track small but important changes, for example gradual increments in the size of a herd. Moreover, mentally adding or subtracting non-symbolic quantities of any substantial size yields extremely noisy estimates that preclude many characteristic human activities, such as deciding on a fair exchange involving numerous specific items. Thus, while the capacity to represent and compare non-symbolic quantities is important, and provides a basis for symbolic numerical capabilities, the capacity to represent symbolic quantities offers crucial cognitive advantages.

This theoretical contrast between non-symbolic and symbolic quantitative abilities provides a useful way to approach development of quantitative thinking. Numeric symbols were invented relatively recently in human history, making it implausible that the brain evolved specifically to handle them; they must be learned through prolonged interaction with the environment. Consistent with this perspective, non-symbolic quantitative thinking emerges early in infancy and shows little variation among children of different cultural backgrounds. In contrast, symbolic quantitative thinking emerges much later, typically in preschool among middle and upper class groups in developed societies and even later among less affluent groups in those societies. Indeed, numerical symbols are used only for a few small whole numbers among children in adults in less developed societies, such as some indigenous Amazonian populations (Dehaene, Izard, Spelke, & Pica, 2008). For all of these reasons and many more, the distinction between non-symbolic and symbolic quantitative thinking is fundamental to describing and understanding the development of quantitative thinking.

As we have seen, children's learning of the decimal system and the meaning of the symbols within it occurs over many years through substantial observation of adults and older children, direct instruction from teachers and parents, extensive practice, and surmounting misunderstandings and forming new understandings. Early in the learning process, numerical symbols are meaningless stimuli for young preschoolers. For example, 2- and 3-year-olds who count flawlessly from 1-10 have no idea that  $6 > 4$ , nor do children of these ages know how many pennies to give an adult who asks for 4 or more. As young children gain experience with the symbols in a given numerical range and associate them with non-verbal quantities in that range, they initially map them to a logarithmically-compressed mental number line. Over a period that typically lasts 1-3 years for a given numerical range (0-10, 0-100, or 0-1000), children's mapping of symbolically expressed numbers to non-verbal representations changes from a logarithmically compressed form to a linear form. In the linear representation, subjective and objective numerical values increase in a 1:1 fashion. Use of linear magnitude representations occurs earliest for the numerals that are most frequent in the environment, that is the smallest whole numbers, and is gradually extended to increasingly large numbers.

The logarithmic-to-linear shift in children's representations of symbolic quantities expands children's quantitative thinking profoundly. It improves (1) children's ability to estimate the positions of numbers on number lines, (2) to estimate the measurements of continuous and discrete quantities, (3) to categorize numbers according to size, (4) to remember numbers they have encountered, and (5) to estimate and learn the answers to arithmetic problems. These abilities are crucial for learning mathematics, resulting in the use of linear representations of number being highly correlated with skill at arithmetic and overall mathematics achievement. Especially important, educational interventions aimed at inculcating linear representations have

broad and sustained effects on mathematics performance and learning. In contrast, interventions aimed at improving non-symbolic quantitative abilities (e.g., Räsänen et al., 2009) have proved less effective in leading to positive educational outcomes.

A final contrast between development of non-symbolic and symbolic quantitative thinking is worth highlighting as an important direction for future research. Although much is known about the neural substrate of non-symbolic quantitative abilities, and a reasonable amount is known about its development, little is known about several remaining issues concerning symbolic numerical capabilities: (1) What are the neural substrates for acquiring the ability to think about symbolic numerical quantities? (2) How do the neural correlates of symbolic number representation change with age and experience? (3) How does a limited set of neurons represent the magnitudes of a much larger set of symbolic numeric quantities, including those expressed by integers, fractions, and ordinals? and (4) What mechanisms for representing symbolic number are affected by the various developmental disorders (e.g., William's syndrome) that are marked by difficulties with mathematics?

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Fig. 1

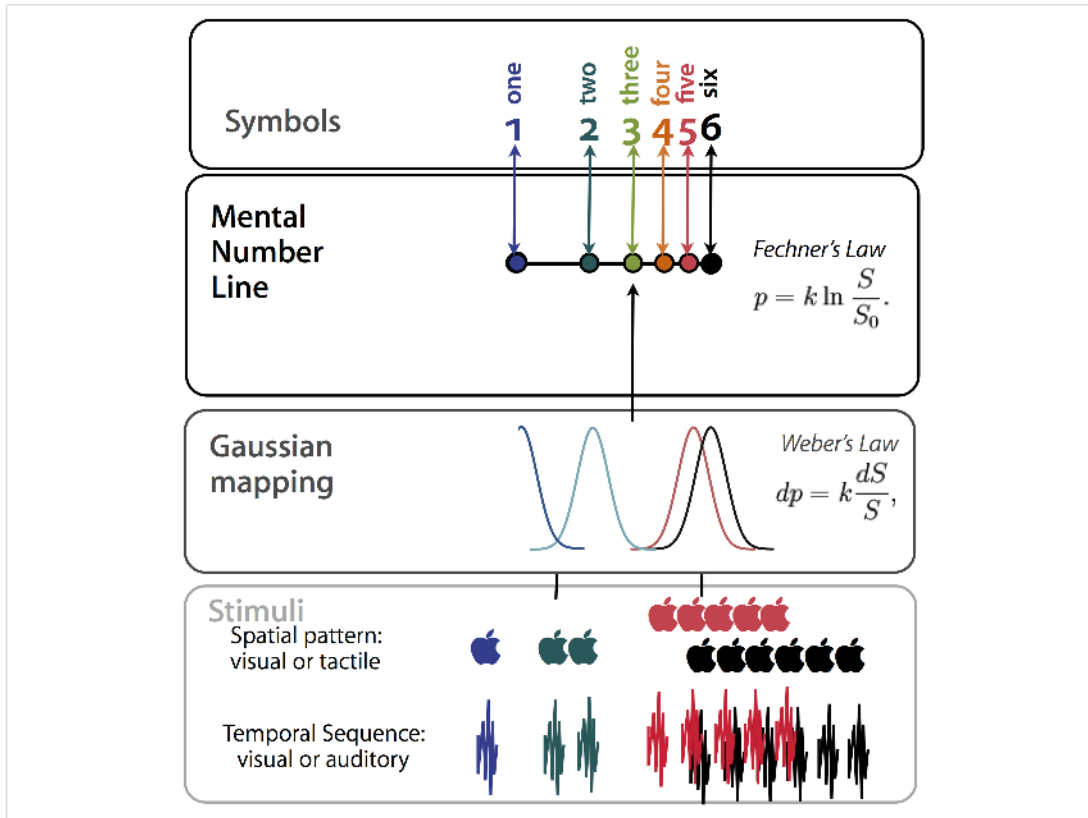


Fig. 2

