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Deviation from weak Banach–Saks property for countable direct sums

Abstract. We introduce a seminorm for bounded linear operators between Banach spaces that shows the deviation from the weak Banach–Saks property. We prove that if (X_ν) is a sequence of Banach spaces and a Banach sequence lattice E has the Banach–Saks property, then the deviation from the weak Banach–Saks property of an operator of a certain class between direct sums $E(X_\nu)$ is equal to the supremum of such deviations attained on the coordinates X_ν. This is a quantitative version for operators of the result for the Köthe–Bochner sequence spaces $E(X)$ that if E has the Banach–Saks property, then $E(X)$ has the weak Banach–Saks property if and only if so has X.

1. Introduction. A Banach space X is said to have the Banach–Saks (BS) property if every bounded sequence in X contains a subsequence (x_n) whose Cesàro means $\sum_{i=1}^{n} x_i/n$ converge in norm. Such a property was proved by Banach and Saks [1] for $L_p[0,1]$ spaces with $1 < p < \infty$. The case $p = 1$ was examined by Szlenk [14] who proved that every weakly convergent sequence in $L_1[0,1]$ contains a subsequence with strongly convergent Cesàro means. This variant of the BS property is considered also for operators (see [2]). A bounded linear operator T between Banach spaces X and Y is said to have the weak Banach–Saks (WBS) property if every weakly null sequence (x_n) in X contains a subsequence (x'_n) such that (Tx'_n) is Cesàro convergent in Y.

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In this note, we focus on weakly null sequences which have no Cesàro convergent subsequences. Some quantitative information on the deviation from summability of such sequences is provided by Rosenthal's dichotomy [13]. Recall that every weakly null sequence in a Banach space X contains a subsequence (x_n) such that either all subsequences of (x_n) are Cesàro convergent in norm to zero or no subsequence of (x_n) is Cesàro convergent and then there is a number $\delta > 0$ such that $\left\|\sum_{n\in A} c_n x_n\right\| \geq \delta \sum_{n\in A} |c_n|$ for all scalars (c_n) and all subsets $A \subset \mathbb{N}$ with $|A| \leq 2^k$, $k \leq \min A$ and $k \in \mathbb{N}$, where $|A|$ is the number of elements of A.

Using Rosenthal's result, Partington [12] proved that a Banach space X has the WBS property if and only if for all $\varepsilon > 0$ and weakly null sequences (x_n) in X there exists a finite subset $A \subset \mathbb{N}$ such that $\left\|\sum_{n\in A} x_n\right\| < \varepsilon\,|A|$. This served to prove that the direct sums of Banach spaces, built on a Banach space with a hyperorthogonal basis and the BS property, preserve the WBS property.

Our generalization of Partington's result for direct sums goes in two directions: it has a quantitative character and concerns operators. We introduce a seminorm for operators which measures the deviation from the WBS property. We consider a certain class of operators acting between direct sums $E(X_\nu)$. In the main result, we show that the deviation from the WBS property of an operator is equal to the supremum of such deviations attained on the coordinates X_ν, providing that a Banach sequence lattice E has the Banach–Saks property. Our main tool in the proofs is a repeated averaging technique elaborated in [7, 8], and based on the spreading models of Brunel and Sucheston [3].

2. Preliminaries. A Banach space E of real-valued functions on $\mathbb{N} = \{1, 2, 3, \ldots\}$ with the natural partial order is called a Banach sequence lattice if, for every finite subset $A \subset \mathbb{N}$, the characteristic function χ_A of A belongs to E, and if $x = (x(\nu)) \in E$ and $|y(\nu)| \leq |x(\nu)|$ for every $\nu \in \mathbb{N}$, then $y = (y(\nu)) \in E$ and $\|y\|_E \leq \|x\|_E$. The lattice E is said to be regular (or σ-order continuous) if, for every sequence (x_n) in E with $x_n \downarrow 0$, it holds $\lim_{n\to\infty} \|x_n\|_E = 0$.

A Banach sequence lattice is a particular case of a Köthe function space with the counting measure space on $\mathbb{N}$ (see [9], [10]). Thus the Köthe dual space E' of E is the space of all real-valued sequences $(y(\nu))$ such that $(x(\nu)y(\nu)) \in l_1$ for every $(x(\nu)) \in E$. The norm in E' is given for every $y = (y(\nu))$ by

$$\|y\|_{E'} = \sup\left\{\sum_{\nu=1}^{\infty} |x(\nu)y(\nu)| :\ \|x\|_E \leq 1,\ x = (x(\nu))\right\}.$$

If E is regular, then the Köthe dual space E' is isometrically isomorphic to the dual space E^* (see [10, p. 29]).

Let E be a Banach sequence lattice and (X_ν) a sequence of Banach spaces. By $E(X_\nu)$ we mean the Banach space of all sequences $x = (x(\nu))$ such that $x(\nu) \in X_\nu$ for every $\nu \in \mathbb{N}$ and $(\|x(\nu)\|_{X_\nu}) \in E$. The norm in $E(X_\nu)$ is given by

$$\|x\|_{E(X_\nu)} = \left\|(\|x(\nu)\|_{X_\nu})\right\|_E .$$

If $X_\nu = X$ for all ν, then $E(X)$ is called a Köthe–Bochner sequence space.

If E is regular, then the dual space $(E(X_\nu))^*$ is isometrically isomorphic to $E^*(X_\nu^*)$ (see [11, Proposition 3.1]). Using this fact, we can prove a counterpart of Lemma 1 of [5] without the separability assumption.

Lemma 1. *Let E be a regular Banach sequence lattice. If $x_n = (x_n(\nu)) \in E(X_\nu)$ for all $n \in \mathbb{N}$ and $x_n \overset{w}{\to} 0$ in $E(X_\nu)$, then $x_n(\nu) \overset{w}{\to} 0$ in X_ν for every $\nu \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$ and let $x^* \in X_k^*$. Put $(f(\nu)) = (0, \dots, 0, x^*, 0, \dots)$ with x^* on kth place. Clearly, $(f(\nu)) \in E^*(X_\nu^*)$. Let τ be the isometric isomorphism between $(E(X_\nu))^*$ and $E^*(X_\nu^*)$ given by Proposition 3.1 of [11] (see also [6]). There exists $f = \tau^{-1}[(f(\nu))]$ in $(E(X_\nu))^*$ such that $f(x) = \sum_{\nu=1}^{\infty} \langle x(\nu), f(\nu)\rangle$ for every $x = (x(\nu)) \in E(X_\nu)$. Then

$$f(x_n) = \sum_{\nu=1}^{\infty} \langle x_n(\nu), f(\nu)\rangle = \langle x_n(k), f(k)\rangle = x^*(x_n(k))$$

Since $\lim_{n\to\infty} f(x_n) = 0$ and $x^* \in X_k^*$ was arbitrary, $x_n(k) \overset{w}{\to} 0$ in X_k. □

3. Results. The space of all bounded linear operators between Banach spaces X and Y we denote by $L(X,Y)$. For a sequence (x_n) in a Banach space, we put

$$\psi(x_n) = \inf\left\{\left\| |A|^{-1} \sum_{n\in A} x_n \right\| : |A| < \infty\right\}.$$

In our quantitative considerations, we will need a certain stability of ψ with respect to repeated averaging of (x_n). This can be achieved through the process of arithmetic averaging of (x_n) on equipollent successive blocks. We say that (y_n) is a sequence of successive arithmetic means (sam) for (x_n) if there exist $m \in \mathbb{N}$ and a sequence of subsets $I_n \subset \mathbb{N}$ with $\max I_n < \min I_{n+1}$ and $|I_n| = m$ such that $y_n = \sum_{i\in I_n} x_i/m$ for all n. Clearly, $\psi(x_n) \le \psi(y_n)$.

The next result is a part of Proposition 2.3 of [7], where the proof based on spreading models was given for a similar characteristics of a sequence related to the alternate signs Banach–Saks property. The proof for ψ runs in much the same way. We include it for completeness.

Proposition 2. *Let (x_n) be a bounded sequence in a Banach space X. Then for every $\varepsilon > 0$ there exists a sequence (y_n) of sam for (x_n) such that for* *all finite subsets* $A \subset \mathbb{N}$,

$$\left\| |A|^{-1} \sum_{n\in A} y_n \right\| \le \psi(y_n) + \varepsilon.$$

Proof. If (x_n) contains a Cauchy subsequence (x'_n), it is enough to ignore a finite number of terms of (x'_n) and put $y_n = x'_n$. Assume now that (x_n) has no Cauchy subsequence. We follow in part the line of the proof of Theorem II.2 of [2]. We extract a subsequence (x'_n) of (x_n) that is the fundamental sequence of the spreading model F built on (x_n). Put

$$K = \inf\left\{ \left\| |A|^{-1} \sum_{n\in A} x'_n \right\|_F \,:\, |A| < \infty \right\}.$$

There exist a finite subset $I \subset \mathbb{N}$ and $z = \sum_{i\in I} x'_i / |I|$ such that $K \le \|z\|_F \le K + \varepsilon/4$. Let (I_n) be a sequence of subsets $I_n \subset \mathbb{N}$ with $\max I_n < \min I_{n+1}$ and $|I_n| = |I|$ for all n. Put $z_n = \sum_{i\in I_n} x'_i / |I_n|$. Since the norm of F is invariant under spreading, $\|z_n\|_F = \|z\|_F$ for all n. Consequently, $K \le \left\|\sum_{n\in A} z_n / |A|\right\|_F \le K + \varepsilon/4$ for all finite subsets $A \subset \mathbb{N}$.

By [2, Proposition I.1], for every $k \in \mathbb{N}$, we can choose n_k so that for all $A \subset \mathbb{N}$ with $|A| \le 2^k$ and $n_k \le \min A$,

$$\left| \left\| |A|^{-1} \sum_{n\in A} z_n \right\| - \left\| |A|^{-1} \sum_{n\in A} z_n \right\|_F \right| < \varepsilon/4.$$

We may assume that $n_k < n_{k+1}$. Let $z'_k = z_{n_k}$. Then for all $A \subset \mathbb{N}$ with $|A| \le 2^k$ and $k \le \min A$,

$$K - \varepsilon/4 \le \left\| |A|^{-1} \sum_{n\in A} z'_n \right\| \le K + \varepsilon/2.$$

Passing to a sequence of the arithmetic means of (z'_n) built on long enough successive blocks, we show now similar estimates for all finite $A \subset \mathbb{N}$. Let $|A| < \infty$ and $A_0 = \{n \in A\colon n < \log_2 |A|\}$. Then

$$\left\| \sum_{n\in A_0} z'_n \right\| \le |A_0|\,(K + \varepsilon/2), \qquad \left\| \sum_{n\in A\setminus A_0} z'_n \right\| \ge (|A| - |A_0|)\,(K - \varepsilon/4).$$

It follows that

$$\begin{aligned} \left\| |A|^{-1} \sum_{n\in A} z'_n \right\| &\ge |A|^{-1} \left(\left\| \sum_{n\in A\setminus A_0} z'_n \right\| - \left\| \sum_{n\in A_0} z'_n \right\| \right) \\ &\ge K - \varepsilon/4 - |A_0|\,|A|^{-1}\,(2K + \varepsilon/4). \end{aligned}$$

There exists $m \in \mathbb{N}$ such that if $|A| \geq m$, then $|A_0|\,|A|^{-1}(2K + \varepsilon/4) \leq \varepsilon/4$ and, consequently,

$$K - \varepsilon/2 \leq \left\| |A|^{-1} \sum_{n \in A} z_n' \right\| < K + \varepsilon/2.$$

Put $y_n = \sum_{i \in J_n} z_i'/|J_n|$, where (J_n) is a sequence of subsets $J_n \subset \mathbb{N}$ with $\max J_n < \min J_{n+1}$ and $|J_n| = m$ for all n. Then

$$\left\| |A|^{-1} \sum_{n \in A} y_n \right\| \leq \psi(y_n) + \varepsilon$$

for every finite $A \subset \mathbb{N}$. Clearly, (y_n) is a sequence of sam for (x_n). □

Definition 3. Let X, Y be Banach spaces and $T \in L(X, Y)$. Define

$$\Psi(T) = \sup \left\{ \psi(Tx_n) \colon x_n \xrightarrow{w} 0,\ \|x_n\| \leq 1 \right\}.$$

Clearly, if $T \in L(X, Y)$ and $x_n \xrightarrow{w} 0$ in X, then $Tx_n \xrightarrow{w} 0$ in Y. Thus, by [12, Theorem 2], $\Psi(T) = 0$ if and only if T has the WBS property. Applying Proposition 2, we can show, as in the proof of Proposition 2.5 of [7], that Ψ is a seminorm in $L(X, Y)$. The procedure of stabilization of ψ plays a key role also in the next result. The arguments of the proof are similar to those used in the proofs of Theorem 3 of [12] and Theorem 3.2 of [7].

Theorem 4. *Let (X_ν) and (Y_ν) be sequences of Banach spaces and let (T_ν) be a sequence of operators such that $T_\nu \in \mathcal{L}(X_\nu, Y_\nu)$ for every $\nu \in \mathbb{N}$ and $\sup_{\nu \in \mathbb{N}} \|T_\nu\| < \infty$. If a Banach sequence lattice E has the BS property and $T \in L(E(X_\nu), E(Y_\nu))$ is given by $Tx = (T_\nu x(\nu))$ for every $x = (x(\nu)) \in E(X_\nu)$, then $\Psi(T) = \sup_{\nu \in \mathbb{N}} \Psi(T_\nu)$.*

Proof. It is enough to prove that $\Psi(T) \leq \sup_{\nu \in \mathbb{N}} \Psi(T_\nu)$, since $E(X_\nu)$ and $E(Y_\nu)$ contain isometric copies respectively of X_ν and Y_ν. Let us fix $\varepsilon > 0$ and choose a weakly null sequence (x_n) in the unit ball of $E(X_\nu)$ so that $\Psi(T) - \varepsilon \leq \psi(Tx_n)$.

First, we show that we can focus on a finite number of coordinates of the direct sums. Let $t_n = (\|T_\nu x_n(\nu)\|_{Y_\nu})$ for every $x_n = (x_n(\nu))$. Since E has the BS property, passing to a subsequence, we may assume that the Cesàro means of all subsequences of $(t_n) \subset E$ converge to the same limit $t \in E$ (see [4]). Then $\psi(t_n^0 - t) = 0$ for every sequence (t_n^0) of sam for (t_n) and, by Proposition 2, (t_n^0) can be taken so that for every finite $A \subset \mathbb{N}$,

$$\left\| |A|^{-1} \sum_{n \in A} t_n^0 - t \right\|_E < \frac{\varepsilon}{2}.$$

Let (I_n) be a sequence of finite subsets of $\mathbb{N}$ with $|I_n| = m$ and $\max I_n < \min I_{n+1}$ for all n such that $t_n^0 = m^{-1} \sum_{i \in I_n} t_i$. Put $x_n^0 = m^{-1} \sum_{i \in I_n} x_i$.

For every $r \in \mathbb{N}$ and $z = (z(\nu))$, we will write $P_r z = (z(1), \ldots, z(r), 0, 0, \ldots)$ and $Q_r z = z - P_r z$. Since the reflexive lattice E is σ-order continuous, there is $r \in \mathbb{N}$ such that $\|Q_r t\|_E < \varepsilon/2$. It follows that

$$\left\| Q_r \left(|A|^{-1} \sum_{n \in A} t_n^0 \right) \right\|_E < \frac{\varepsilon}{2} + \|Q_r t\|_E < \varepsilon.$$

Thus, for every finite $A \subset \mathbb{N}$,

$$\begin{aligned}
\varepsilon > \left\| Q_r \left(|A|^{-1} \sum_{n \in A} t_n^0 \right) \right\|_E &= \left\| Q_r \left(|A|^{-1} \sum_{n \in A} \frac{1}{m} \sum_{i \in I_n} \|T_\nu x_i(\nu)\|_{Y_\nu} \right) \right\|_E \\
\geq \left\| Q_r \left(|A|^{-1} \sum_{n \in A} \|T_\nu x_n^0(\nu)\|_{Y_\nu} \right) \right\|_E &\geq \left\| Q_r \left(\left\| |A|^{-1} \sum_{n \in A} T_\nu x_n^0(\nu) \right\|_{Y_\nu} \right) \right\|_E \\
= \left\| |A|^{-1} \sum_{n \in A} Q_r T x_n^0 \right\|_{E(Y_\nu)}.
\end{aligned}$$

Passing to a subsequence of (x_n^0), we may assume that for each coordinate $1 \leq \nu \leq r$ the limit $\lambda_\nu = \lim_n \|x_n^0(\nu)\|$ exists and $\|x_n^0(\nu)\| < \lambda_\nu + \varepsilon / \|P_r e\|_E$ for every n, where $e = (1, 1, \ldots)$. Put $\alpha_\nu = \lambda_\nu + \varepsilon / \|P_r e\|_E$. By the equipollence of blocks, all sequences of sam for (x_n) are weakly null and, by Lemma 1, so are all sequences restricted to coordinates. Now we stabilize ψ consecutively on coordinates $k = 1, 2, \ldots, r$. Write $y_n^0(\nu) = T_\nu x_n^0(\nu)/\alpha_\nu$.

In the first step, we apply Proposition 2 to $(y_n^0(1))$. There is a sequence (x_n^1) of sam for (x_n^0) such that for the sequence $(y_n^1(1))$ of sam for $(y_n^0(1))$, where $y_n^1(1) = T_1 x_n^1(1)/\alpha_1$, we have

$$\left\| |A|^{-1} \sum_{n \in A} y_n^1(1) \right\|_{Y_1} \leq \psi\left(y_n^1(1)\right) + \varepsilon$$

for all finite $A \subset \mathbb{N}$. We put $y_n^1(\nu) = T_\nu x_n^1(\nu)/\alpha_\nu$ for $\nu \neq 1$.

Let $k > 1$. By Proposition 2 applied to $(y_n^{k-1}(k))$, we obtain a sequence (x_n^k) of sam for (x_n^{k-1}) such that for the sequence $(y_n^k(k))$ of sam for $(y_n^{k-1}(k))$, where $y_n^k(k) = T_k x_n^k(k)/\alpha_k$, we have

$$\left\| |A|^{-1} \sum_{n \in A} y_n^k(k) \right\|_{Y_k} \leq \psi\left(y_n^k(k)\right) + \varepsilon$$

for all finite $A \subset \mathbb{N}$. Again we put $y_n^k(\nu) = T_\nu x_n^k(\nu)/\alpha_\nu$ for $\nu \neq k$. Since the relation sam is transitive, all sequences $(y_n^r(\nu))$, $1 \leq \nu \leq r$, are built on the common sequence (x_n^r) of sam for (x_n^ν). Consequently,

$$\left\| |A|^{-1} \sum_{n \in A} y_n^r(\nu) \right\|_{Y_\nu} \leq \psi\left(y_n^\nu(\nu)\right) + \varepsilon \leq \psi\left(y_n^{\nu+1}(\nu)\right) + \varepsilon \leq \cdots \leq \psi\left(y_n^r(\nu)\right) + \varepsilon$$

for all finite $A \subset \mathbb{N}$ and every $1 \leq \nu \leq r$. Clearly, $\|x_n^r(\nu)/\alpha_\nu\|_{X_\nu} \leq 1$ for all n. It follows that

$$\left\| |A|^{-1} \sum_{n \in A} P_r T x_n^r \right\|_{E(Y_\nu)} = \left\| P_r \left(\alpha_\nu \left\| |A|^{-1} \sum_{n \in A} y_n^r(\nu) \right\|_{Y_\nu} \right) \right\|_E$$

$$\leq \left\| P_r \left(\lambda_\nu + \varepsilon / \|P_r e\|_E \right) \right\|_E \max_{1 \leq \nu \leq r} \left\| |A|^{-1} \sum_{n \in A} y_n^r(\nu) \right\|_{Y_\nu}$$

$$\leq (1+\varepsilon) \left(\max_{1 \leq \nu \leq r} \psi\left(y_n^r(\nu)\right) + \varepsilon \right).$$

Assume that $\max_{1 \leq \nu \leq r} \psi\left(y_n^r(\nu)\right)$ is attained for j, $1 \leq j \leq r$. By transitivity of the relation sam, (x_n^r) is a sequence of sam for (x_n). It follows that

$$\Psi(T) - \varepsilon \leq \psi(Tx_n) \leq \psi(Tx_n^r) \leq \left\| |A|^{-1} \sum_{n \in A} T x_n^r \right\|_{E(Y_\nu)}$$

$$\leq \left\| |A|^{-1} \sum_{n \in A} P_r T x_n^r \right\|_{E(Y_\nu)} + \left\| |A|^{-1} \sum_{n \in A} Q_r T x_n^r \right\|_{E(Y_\nu)}$$

$$\leq (1+\varepsilon)\left(\psi\left(y_n^r(j)\right) + \varepsilon\right) + \varepsilon \leq (1+\varepsilon)\left(\Psi(T_j) + \varepsilon\right) + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrary, $\Psi(T) \leq \sup_{\nu \in \mathbb{N}} \Psi(T_\nu)$. □

Considering the identity operator on $E(X_\nu)$, we obtain the following corollary which includes Partington's [12] qualitative result. By an example of [12], the BS property of E cannot be replaced here by the WBS property.

Corollary 5. *Let E have the BS property. Then $E(X_\nu)$ has the WBS property if and only if every X_ν has the WBS property.*

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