

Devising superconvergent HDG methods with symmetric approximate stresses for linear elasticity by M -decompositions

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We propose a new tool, which we call M -decompositions, for devising superconvergent hybridizable discontinuous Galerkin (HDG) methods and hybridized-mixed methods for linear elasticity with strongly symmetric approximate stresses on unstructured polygonal/polyhedral meshes.

We show that for an HDG method, when its local approximation space admits an M -decomposition, optimal convergence of the approximate stress and superconvergence of an element-by-element post-processing of the displacement field are obtained. The resulting methods are locking-free.

Moreover, we explicitly construct approximation spaces that admit M -decompositions on general polygonal elements. We display numerical results on triangular meshes validating our theoretical findings.

Keywords: hybridizable; discontinuous Galerkin; superconvergence; linear elasticity; strong symmetry.

1. Introduction

We present a technique to systematically construct superconvergent hybridizable discontinuous Galerkin (HDG) and mixed methods with *strongly* symmetric approximate stresses on unstructured polygonal/polyhedral meshes for linear elasticity. By a superconvergent method, we mean, in general, a method that provides an approximate displacement converging to certain projection of the exact displacement faster than it converges to the exact displacement itself. It is then possible to obtain, by means of an elementwise and parallelizable computation, a new approximate displacement converging faster than the original one. This property was uncovered back in 1985 in Arnold & Brezzi (1985) in the framework of mixed methods for diffusion problems, and has been extended to various mixed methods for several elliptic problems (see Boffi *et al.*, 2013).

This article is part of a series in which we devise superconvergent HDG and mixed methods for steady-state problems. Indeed, superconvergent HDG (and mixed) methods for second-order diffusion were considered in Cockburn *et al.* (2012a,b), superconvergent HDG methods based on the velocity gradient–velocity–pressure formulation of the Stokes equations of incompressible flow in Cockburn & Shi (2013a), and superconvergent HDG methods with *weakly* symmetric approximate stresses for the equations of linear elasticity in Cockburn & Shi (2013b).

In Cockburn *et al.* (2017b), we refined the work on second-order diffusion carried out in Cockburn *et al.* (2012a,b) and showed that, by using the concept of an M -decomposition (for the divergence and

gradient operators), it is possible to systematically find HDG and mixed methods, which superconverge on unstructured meshes made of polygonal/polyhedral elements of arbitrary shapes. The actual construction of such M -decompositions for arbitrary polygonal elements was carried out in Cockburn & Fu (2017b) and for tetrahedra, prisms, pyramids and hexahedra in three-space dimensions in Cockburn & Fu (2017c). The extension of these results to the heat equation (Chabaud & Cockburn, 2012) and wave equation (Cockburn & Quenneville-Bélaïr, 2014) is straightforward. The extension to the velocity gradient–velocity–pressure formulation of HDG and mixed methods for the Stokes equations is more delicate and was carried out in Cockburn *et al.* (2017a).

Here, we continue this effort and consider the more challenging task of devising superconvergent HDG and mixed methods with strongly symmetric approximate stresses by developing a general theory of M -decompositions for the vector divergence and symmetric gradient operators. We also provide a practical construction for polygonal elements. We do this in the framework of the following model problem:

$$\mathcal{A}\underline{\sigma} - \underline{\epsilon}(\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \tag{1.1a}$$

$$-\nabla \cdot \underline{\sigma} = \mathbf{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.1c}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) is a bounded polyhedral domain, $\partial\Omega$ is the Dirichlet boundary. Here $\mathbf{u} = \{u_i\}_{i=1}^n$ and $\underline{\sigma} = \{\sigma_{ij}\}_{i,j=1}^n$ represent the displacement vector and the Cauchy stress tensor, respectively. The functions $\mathbf{f} = \{f_i\}_{i=1}^n$ and $\mathbf{g} = \{g_i\}_{i=1}^n$ represent the body force vector and the prescribed displacement on $\partial\Omega$, respectively. As usual, $\underline{\epsilon}(\cdot) := \frac{1}{2}(\nabla(\cdot) + \nabla'(\cdot))$ is the symmetric gradient (or strain) operator, and $\mathcal{A} = \{\mathcal{A}_{ijkl}(x)\}_{i,j,k,l=1}^n$ is the so-called compliance tensor, which is bounded and positive definite. In the homogeneous, isotropic case, it is given by

$$\mathcal{A}\underline{\sigma} = \frac{1}{2\mu} \left(\underline{\sigma} - \frac{\lambda}{2\mu + n\lambda} \text{tr}(\underline{\sigma})\mathbf{I} \right), \tag{1.2}$$

where \mathbf{I} is the second-order identity tensor and λ, μ are the Lamé constants.

To better describe our results, let us begin by introducing the general form of the methods we are going to consider. We denote by \mathcal{T}_h a conforming triangulation of Ω made of polygonal/polyhedral elements K . We denote by \mathcal{E}_h the set of faces F of all the elements K in the triangulation \mathcal{T}_h and by $\partial\mathcal{T}_h$ the set of boundaries ∂K of all elements K in \mathcal{T}_h . We set $h_K := \text{diam}(K)$ and $h := \min_{K \in \mathcal{T}_h} h_K$. To each element $K \in \mathcal{T}_h$, we associate (finite-dimensional) spaces of symmetric-matrix-valued functions $\underline{\Sigma}(K)$ and vector-valued functions $\mathbf{V}(K)$. We also consider a (finite-dimensional) space of vector-valued functions $\mathbf{M}(F)$ associated to each $F \in \mathcal{E}_h$. We assume that elements of the above spaces are regular enough, so that all traces belong to $L^2(\partial K)$. To simplify the notation, we denote the normal trace of $\underline{\Sigma}(K)$ on ∂K by

$$\gamma(\underline{\Sigma}(K)) := \{\boldsymbol{\tau}\mathbf{n}|_{\partial K} : \boldsymbol{\tau} \in \underline{\Sigma}(K)\}, \tag{1.3a}$$

and the trace of $\mathbf{V}(K)$ on ∂K by

$$\gamma(\mathbf{V}(K)) := \{\mathbf{v}|_{\partial K} : \mathbf{v} \in \mathbf{V}(K)\}. \tag{1.3b}$$

The methods we are interested in seek an approximation to $(\underline{\sigma}, \mathbf{u}, \mathbf{u}|_{\mathcal{E}_h}), (\underline{\sigma}_h, \mathbf{u}_h, \widehat{\mathbf{u}}_h)$, in the finite element space $\underline{\Sigma}_h \times \mathbf{V}_h \times \mathbf{M}_h$, where

$$\underline{\Sigma}_h := \{\underline{\tau} \in L^2(\mathcal{T}_h; \mathbb{S}) : \underline{\tau}|_K \in \underline{\Sigma}(K) \ \forall K \in \mathcal{T}_h\}, \quad (1.4a)$$

$$\mathbf{V}_h := \{\mathbf{v} \in L^2(\mathcal{T}_h) : \mathbf{v}|_K \in \mathbf{V}(K) \ \forall K \in \mathcal{T}_h\}, \quad (1.4b)$$

$$\mathbf{M}_h := \{\boldsymbol{\mu} \in L^2(\mathcal{E}_h) : \boldsymbol{\mu}|_F \in \mathbf{M}(F) \ \forall F \in \mathcal{E}_h\}, \quad (1.4c)$$

and determine it as the only solution of the following weak formulation:

$$(\mathcal{A} \underline{\sigma}_h, \underline{\tau})_{\mathcal{T}_h} + (\mathbf{u}_h, \nabla \cdot \underline{\tau})_{\mathcal{T}_h} - (\widehat{\mathbf{u}}_h, \underline{\tau} \mathbf{n})_{\partial \mathcal{T}_h} = 0, \quad (1.5a)$$

$$(\underline{\sigma}_h, \nabla \mathbf{v})_{\mathcal{T}_h} - (\widehat{\boldsymbol{\sigma}}_h \mathbf{n}, \mathbf{v})_{\partial \mathcal{T}_h} = (\mathbf{f}, \mathbf{v})_{\mathcal{T}_h}, \quad (1.5b)$$

$$(\widehat{\boldsymbol{\sigma}}_h \mathbf{n}, \boldsymbol{\mu})_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \quad (1.5c)$$

$$(\widehat{\mathbf{u}}_h, \boldsymbol{\mu})_{\partial \Omega} = (\mathbf{g}, \boldsymbol{\mu})_{\partial \Omega} \quad (1.5d)$$

for all $(\underline{\tau}, \mathbf{v}, \boldsymbol{\mu}) \in \underline{\Sigma}_h \times \mathbf{V}_h \times \mathbf{M}_h$. Here, we write $(\eta, \zeta)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (\eta, \zeta)_K$, where $(\eta, \zeta)_D$ denotes the integral of $\eta \zeta$ over the domain $D \subset \mathbb{R}^n$. We also write $\langle \eta, \zeta \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \eta, \zeta \rangle_{\partial K}$, where $\langle \eta, \zeta \rangle_D$ denotes the integral of $\eta \zeta$ over the domain $D \subset \mathbb{R}^{n-1}$ and $\partial \mathcal{T}_h := \{\partial K : K \in \mathcal{T}_h\}$. The definition of the method is completed with the definition of the normal component of the numerical trace:

$$\widehat{\boldsymbol{\sigma}}_h \mathbf{n} = \underline{\boldsymbol{\sigma}}_h \mathbf{n} - \alpha(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \quad \text{on} \quad \partial \mathcal{T}_h, \quad (1.5e)$$

where $\alpha : L^2(\partial K) \rightarrow L^2(\partial K)$ is a suitably chosen *linear local stabilization* operator. By taking particular choices of the local spaces $\underline{\Sigma}(K)$, $\mathbf{V}(K)$ and

$$\mathbf{M}(\partial K) := \{\boldsymbol{\mu} \in L^2(\partial K) : \boldsymbol{\mu}|_F \in \mathbf{M}(F) \ \text{for all} \ F \in \mathcal{F}(K)\}$$

and of the linear local stabilization operator α , all the different HDG methods are obtained; when we can set $\alpha = 0$, we obtain the hybridized version of a mixed method.

Our contributions are two. The first is that we show that, if for every element $K \in \mathcal{T}_h$, the local spaces $\underline{\Sigma}(K)$, $\mathbf{V}(K)$ and $\mathbf{M}(\partial K)$ satisfy certain inclusion conditions, it is possible to define, in an element-by-element fashion, an auxiliary projection

$$\Pi_h = (\underline{\Pi}_\Sigma, \Pi_V) : H^1(K, \mathbb{S}) \times H^1(K) \rightarrow \underline{\Sigma}(K) \times \mathbf{V}(K)$$

such that

$$\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{A}, \mathcal{T}_h} \leq 2 \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{A}, \mathcal{T}_h},$$

$$\|\Pi_V \mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} \leq Ch \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{T}_h},$$

$$\|\mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h\|_{\partial \mathcal{T}_h} \leq Ch^{1/2} \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{T}_h},$$

where $\|\cdot\|_{\mathcal{A}, \mathcal{T}_h}$ denotes the \mathcal{A} -weighted $L^2(\mathcal{T}_h)$ -norm, i.e., $\|\underline{\boldsymbol{\tau}}\|_{\mathcal{A}, \mathcal{T}_h}^2 = (\mathcal{A} \underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\tau}})_{\mathcal{T}_h}$, $\|\cdot\|_D$ denotes the $L^2(D)$ -norm on a domain D , and \mathbf{P}_M is the L^2 -projection onto \mathbf{M}_h . Moreover, if the error $\mathbf{P}_V \mathbf{u} - \mathbf{u}_h$ converges to zero *faster* than the error $\mathbf{u} - \mathbf{u}_h$, this *superconvergence* property can be advantageously exploited to locally obtain a more accurate approximation of the displacement \mathbf{u} (see Arnold & Brezzi, 1985; Brezzi *et al.*, 1985; Stenberg, 1988; Gastaldi & Nochetto, 1989; Stenberg, 1991) for applications to mixed methods and Cockburn *et al.* (2012a,b); Cockburn & Shi (2013a,b) for applications to HDG methods and the references therein.

Our second contribution is a refinement of our previous work (Cockburn *et al.*, 2012a,b; Cockburn & Shi, 2013a,b) along the lines provided in Cockburn *et al.* (2017b): we propose an algorithm that tells us how to systematically obtain the local spaces rendering any given HDG method superconvergent. Indeed, given any local spaces $\underline{\boldsymbol{\Sigma}}_g(K)$, $\mathbf{V}_g(K)$ and $\mathbf{M}(\partial K)$, satisfying very simple inclusion properties, we can give a characterization of the space $\delta \underline{\boldsymbol{\Sigma}}(K)$ of the *smallest* dimension for which the space $(\underline{\boldsymbol{\Sigma}}_g(K) \oplus \delta \underline{\boldsymbol{\Sigma}}(K)) \times \mathbf{V}_g(K)$ admits an $\mathbf{M}(\partial K)$ -decomposition and, hence, defines a superconvergent HDG method. We also show how to obtain (the hybridized version of) two *sandwiching* mixed methods from those spaces. In particular, we show that the HDG method with local spaces $\underline{\boldsymbol{\Sigma}}_g(K) \oplus \delta \underline{\boldsymbol{\Sigma}}(K)$, $\nabla \cdot \underline{\boldsymbol{\Sigma}}_g(K)$ and $\mathbf{M}(\partial K)$ is actually a well-defined hybridizable mixed method. Applying this approach, as done in Cockburn & Fu (2017b,c), we find new superconvergent HDG and mixed methods on general polygonal elements in two-space dimensions. These new spaces are closely related to those of the mixed methods proposed in Arnold *et al.* (1984) for triangles and quadrilaterals, and in Guzmán & Neilan (2014) for triangles.

Let us make a couple of comments on the relevance of these findings. The first concerns the first HDG method for linear elasticity proposed in Soon (2008) (see also Soon *et al.*, 2009). When using polynomial approximations of degree k on triangular meshes, this method was experimentally shown to provide approximations of order $k + 1$ for the stress and displacement for $k \geq 0$, and superconvergence of order $k + 2$ for the displacement. Recently, in Fu *et al.* (2015), it was proven that the stress convergences with order $k + 1/2$ and displacement with $k + 1$ without superconvergence; numerical experiments showed that the orders were actually sharp. It turns out the spaces for that method do not admit \mathbf{M} -decompositions. By using the machinery proposal here, we show that, on triangular meshes, it is enough to add a small space $\delta \underline{\boldsymbol{\Sigma}}(K)$ (of dimension 2 if $k = 1$, and of dimension 3 if $k \geq 2$) to the space of approximate stresses $\underline{\boldsymbol{\Sigma}}(K)$ to obtain convergence of order $k + 1$ for the stress and displacement, as well as superconvergence of order $k + 2$ for the displacement for $k \geq 1$.

The second comment concerns mixed methods (with symmetric approximate stresses) for linear elasticity. A symmetric and conforming mixed method use finite element spaces (for the stress and displacement) $\underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \subset H(\text{div}, \Omega; \mathbb{S}) \times \mathbf{L}^2(\Omega)$. Here $H(\text{div}, \Omega; \mathbb{S})$ denotes the space of $H(\text{div})$ -conforming, symmetric-tensorial fields. It turns out to be quite difficult to design finite elements in $H(\text{div}, \Omega; \mathbb{S})$, which preserve both the symmetry and the $H(\text{div})$ -conformity. The first successful discretization use the so-called composite elements, see the low-order composite elements in Johnson & Mercier (1978) and the high-order composite elements in Arnold *et al.* (1984), both defined on triangular and quadrilateral meshes; see also a low-order composite element on tetrahedral meshes in Ainsworth & Rankin (2011). Although these methods can be efficiently implemented via hybridization, leading to a symmetric-positive-definite linear system, their main drawback is that their basis functions are usually quite hard to construct, especially for the high-order methods in Arnold *et al.* (1984).

In the pioneering work (Arnold & Winther, 2002), the first discretization of $H(\text{div})$ -conforming symmetric-tensorial fields was presented, for triangular meshes, which only uses piecewise polynomial functions. Other piecewise-polynomial discretizations of $H(\text{div}; \mathbb{S})$ on tetrahedral meshes were later obtained in Adams & Cockburn (2005), Arnold *et al.* (2008) and Hu & Zhang (2013), and on rectangular

meshes in Arnold & Awanou (2005) and Yang & Chen (2010). Since all of these polynomial discretizations use vertex degrees of freedom to define the basis functions, the resulting methods cannot be efficiently implemented via hybridization, and a saddle-point linear system needs to be solved. Moreover, as argued in the last paragraph of Section 3 of Arnold & Winther (2002), vertex degrees of freedom cannot be avoided in the construction of piecewise-polynomial $H(\text{div})$ -conforming symmetric-tensorial fields.

For this reason, nonconforming mixed methods (Arnold & Winther, 2003; Yi, 2005, 2006; Hu & Shi, 2007/08; Awanou, 2009; Man *et al.*, 2009; Gopalakrishnan & Guzmán, 2011; Arnold *et al.*, 2014) that violate $H(\text{div})$ -conformity (but preserve symmetry) of the stress field offer an attractive alternative to the conforming methods. These methods also use polynomial basis functions, but do not use vertex degrees of freedom. As a consequence, they can be efficiently implemented via hybridization. See also an interesting nonconforming mixed method (Pechstein & Schöberl, 2011) that uses tangential-continuous displacement field and normal–normal continuous symmetric stress field.

Recently, another high-order conforming discretization of $H(\text{div}, \Omega; \mathbb{S})$ without using vertex degrees of freedom was introduced in Guzmán & Neilan (2014) for triangular meshes. The space enriches the symmetric-tensorial polynomial fields of degree $k \geq 2$ with *three* rational basis functions on each triangle. The resulting mixed methods can be efficiently implemented via hybridization.

In this article, we obtain high-order conforming discretizations of $H(\text{div}, \Omega; \mathbb{S})$ without vertex degrees of freedom on general polygonal meshes. On a general polygon, we enrich the symmetric-tensorial polynomial fields with certain number of composite/rational functions given by explicit formulas. Our spaces on triangular meshes is similar to those in Guzmán & Neilan (2014), whereas our spaces on rectangular meshes enrich the symmetric-tensorial polynomial fields with *four* rational functions, and *two* exponential functions along with a minimal number of polynomial functions.

To conclude, let us point out that there are other HDG methods that superconverge on meshes of arbitrary polyhedral elements which have been recently introduced. A modification of the method (Soon, 2008) which can achieve optimal convergence was introduced in Qiu *et al.* (2017). The spaces, which do *not* admit \mathbf{M} -decompositions, are $\underline{\Sigma}(K) \times \mathbf{V}(K) \times \mathbf{M}(\partial K) := P_k(K; \mathbb{S}) \times P_{k+1}(K) \times P_k(\partial K)$ and the stabilization function is $\alpha(\mathbf{u}_h - \hat{\mathbf{u}}_h) := \frac{1}{h} (\mathbf{P}_M \mathbf{u}_h - \hat{\mathbf{u}}_h)$, where \mathbf{P}_M denotes the L^2 -projection into the space of traces \mathbf{M}_h . These methods were proven to achieve optimal convergence order of $k + 1$ for stress and $k + 2$ for displacement on general polygonal/polyhedral meshes for $k \geq 1$. Another method that can achieve this is the hybrid high-order (HHO) method introduced (in primal form) in Di Pietro & Ern (2015). Typically, our spaces $\underline{\Sigma}(K) \times \mathbf{V}(K)$ are bigger, but the globally coupled system for these and our methods has the same size and sparsity structure, as it only depends on the space of traces $\mathbf{M}(\partial K)$. On the other hand, our mixed methods provide $H(\text{div}, \Omega; \mathbb{S})$ -conforming approximate stresses with optimally convergent divergence.

The rest of the article is organized as follows. In Section 2, we present new *a priori* error estimates for HDG methods with spaces admitting \mathbf{M} -decompositions. In Section 3, we present a characterization of \mathbf{M} -decompositions and present three ways to construct them. In Section 4, we give *ready-for-implementation* spaces admitting \mathbf{M} -decompositions on general polygonal elements. In Section 5, we prove the main result of Section 4, Theorem 4.3. In Section 6, we present numerical results on triangular meshes to validate our theoretical results. Finally, in Section 7, we end with some concluding remarks.

2. The error estimates

In this section, we introduce the notion of an \mathbf{M} -decomposition and then present our main results on the *a priori* error estimates of the HDG methods defined with spaces admitting \mathbf{M} -decompositions. The proofs of the error estimates are very similar to those for the diffusion case (Cockburn *et al.*, 2017b).

2.1 Definition of an M -decomposition

To simplify the notation, when there is no possible confusion, we do not indicate the domain on which the functions of a given space are defined. For example, instead of $\underline{\Sigma}(K)$, we simply write $\underline{\Sigma}$.

The notion of an M -decomposition relates the trace of the normal component of the space of approximate stress $\underline{\Sigma} \subset \{\underline{\tau} \in H(\text{div}, K; \mathbb{S}) : \underline{\tau} \mathbf{n}|_{\partial K} \in L^2(\partial K)\}$ and the trace of the space of approximate displacement $V \subset H^1(K)$ with the space of approximate traces $M \subset L^2(\partial K)$. To define it, we need to consider the combined trace operator

$$\begin{aligned} \text{tr} : \underline{\Sigma} \times V &\longrightarrow L^2(\partial K) \\ (\underline{\tau}, v) &\longmapsto (\underline{\tau} \mathbf{n} + v)|_{\partial K}, \end{aligned}$$

where $\mathbf{n} : \partial K \rightarrow \mathbb{R}^n$ is the unit outward pointing normal field on ∂K .

DEFINITION 2.1 (The M -decomposition) We say that $\underline{\Sigma} \times V$ admits an M -decomposition when

(a) $\text{tr}(\underline{\Sigma} \times V) \subset M$,

and there exist subspaces $\tilde{\underline{\Sigma}} \subset \underline{\Sigma}$ and $\tilde{V} \subset V$, satisfying

(b) $\epsilon(V) \subset \tilde{\underline{\Sigma}}, \nabla \cdot \underline{\Sigma} \subset \tilde{V}$,

(c) $\text{tr} : \tilde{\underline{\Sigma}}^\perp \times \tilde{V}^\perp \rightarrow M$ is an isomorphism.

Here, $\tilde{\underline{\Sigma}}^\perp$ and \tilde{V}^\perp are the L^2 -orthogonal complements of $\tilde{\underline{\Sigma}}$ in $\underline{\Sigma}$, and of \tilde{V} in V , respectively.

2.2 The HDG projection

Next, we show an immediate consequence of the fact that the space $\underline{\Sigma} \times V$ admits an M -decomposition, namely, the existence of an auxiliary HDG projection, which is the key to our error analysis.

To state the result, we need to introduce the quantities related to the stabilization operator α :

$$a_{\tilde{V}^\perp} := \begin{cases} \inf_{\mu \in \gamma \tilde{V}^\perp \setminus \{0\}} \langle \alpha(\mu), \mu \rangle_{\partial K} / \|\mu\|_{\partial K}^2 & \text{if } \tilde{V}^\perp \neq \{0\}, \\ \infty & \text{if } \tilde{V}^\perp = \{0\}, \end{cases}$$

and

$$\|\alpha\| := \sup_{\lambda, \mu \in M \setminus \{0\}} \langle \alpha(\lambda), \mu \rangle_{\partial K} / (\|\lambda\|_{\partial K} \|\mu\|_{\partial K}).$$

Throughout this section, we assume that, on each element K , the space $\underline{\Sigma} \times V$ admits an M -decomposition, and that the stabilization operator $\alpha : L^2(\partial K) \rightarrow L^2(\partial K)$ satisfies the following three properties:

(S1) $\langle \alpha(\lambda), \mu \rangle_{\partial K} = \langle \alpha(\mu), \lambda \rangle_{\partial K}$ for all $\lambda, \mu \in L^2(\partial K)$.

(S2) $\langle \alpha(\mu), \mu \rangle_{\partial K} \geq 0$ for all $\mu \in L^2(\partial K)$.

(S3) $a_{\tilde{V}^\perp} > 0$.

Properties (S1) and (S2) mean that α is self-adjoint and non-negative, whereas property (S3) means that α is positive definite on the trace space $\gamma \tilde{V}^\perp$. When $\tilde{V}^\perp = \{\mathbf{0}\}$, we take $\alpha = 0$ so that the resulting method is a (hybridized) mixed method. In this case, $a_{\tilde{V}^\perp} = \infty$ and $\|\alpha\| = 0$. On the other hand, when $\tilde{V}^\perp \neq \{\mathbf{0}\}$, we can take $\alpha = Id$ to be the identity operator. In this case, we have $a_{\tilde{V}^\perp} = 1$ and $\|\alpha\| = 1$. See in Cockburn *et al.* (2011, Section 2.1) for a presentation of different choices of the stabilization operator α .

The HDG projection $\Pi_h(\underline{\sigma}, \mathbf{u}) = (\underline{\Pi}_\Sigma \underline{\sigma}, \Pi_V \mathbf{u}) \in \underline{\Sigma} \times V$ is defined as follows:

$$(\underline{\Pi}_\Sigma \underline{\sigma}, \underline{\boldsymbol{\tau}})_K = (\underline{\sigma}, \underline{\boldsymbol{\tau}})_K \quad \forall \underline{\boldsymbol{\tau}} \in \tilde{\Sigma}, \quad (2.2a)$$

$$(\Pi_V \mathbf{u}, \mathbf{v})_K = (\mathbf{u}, \mathbf{v})_K \quad \forall \mathbf{v} \in \tilde{V}, \quad (2.2b)$$

$$\langle \underline{\Pi}_\Sigma \underline{\sigma} \mathbf{n} - \alpha(\Pi_V \mathbf{u}), \boldsymbol{\mu} \rangle_{\partial K} = \langle \underline{\sigma} \mathbf{n} - \alpha(\mathbf{P}_M \mathbf{u}), \boldsymbol{\mu} \rangle_{\partial K} \quad \forall \boldsymbol{\mu} \in M. \quad (2.2c)$$

We have the following result on the approximation properties of the projection. We refer to our extended paper on arXiv (Cockburn & Fu, 2017a, Appendix B) for a complete proof.

THEOREM 2.2 (Approximation properties of the HDG projection) The auxiliary HDG projection in (2.2) is well defined. Moreover, we have

$$\begin{aligned} \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_K &\leq \|(Id - \underline{\mathbf{P}}_\Sigma) \underline{\sigma}\|_K + C_1 h_K^{1/2} \|(Id - \underline{\mathbf{P}}_\Sigma) \underline{\sigma}\|_K \mathbf{n} \|_{\partial K} \\ &\quad + C_2 h_K \|(Id - \mathbf{P}_{\tilde{V}}) \nabla \cdot \underline{\sigma}\|_K + C_3 h_K^{1/2} \|(Id - \mathbf{P}_V) \mathbf{u}\|_{\partial K}, \\ \|\mathbf{u} - \Pi_V \mathbf{u}\|_K &\leq \|(Id - \mathbf{P}_V) \mathbf{u}\|_K + C_4 h_K^{1/2} \|(Id - \mathbf{P}_V) \mathbf{u}\|_{\partial K} \\ &\quad + C_5 h_K \|(Id - \mathbf{P}_{\tilde{V}}) \nabla \cdot \underline{\sigma}\|_K, \end{aligned}$$

where $C_1 := C_{\tilde{\Sigma}^\perp}$ and

$$C_2 := \frac{C_{\tilde{V}^\perp}}{a_{\tilde{V}^\perp}} C_{\tilde{\Sigma}^\perp} \|\alpha\|, \quad C_3 := \left(1 + \frac{\|\alpha\|}{a_{\tilde{V}^\perp}}\right) C_{\tilde{\Sigma}^\perp} \|\alpha\|, \quad C_4 := \frac{C_{\tilde{V}^\perp}}{a_{\tilde{V}^\perp}} \|\alpha\|, \quad C_5 := \frac{C_{\tilde{V}^\perp}^2}{a_{\tilde{V}^\perp}}.$$

Here,

$$C_{\tilde{\Sigma}^\perp} := \sup_{\underline{\boldsymbol{\tau}} \in \tilde{\Sigma}^\perp \setminus \{\mathbf{0}\}} h_K^{-1/2} \|\underline{\boldsymbol{\tau}}\|_K / \|\underline{\boldsymbol{\tau}} \mathbf{n}\|_{\partial K}, \quad C_{\tilde{V}^\perp} := \sup_{\mathbf{v} \in \tilde{V}^\perp \setminus \{\mathbf{0}\}} h_K^{-1/2} \|\mathbf{v}\|_K / \|\mathbf{v}\|_{\partial K}.$$

Note that, if $V = \tilde{V} = \nabla \cdot \underline{\Sigma}$ then $C_i = 0$ for $i = 2, 3, 4, 5$, since in this case we have $a_{\tilde{V}^\perp} = \infty$ and $\|\alpha\| = 0$. Note also that the above error estimates depend on the choice of the space $\tilde{\Sigma}$ only through the stability constant $C_{\tilde{\Sigma}^\perp}$. The constants $C_{\tilde{\Sigma}^\perp}$ and $C_{\tilde{V}^\perp}$ are optimal bounds for inverse inequalities bounding the $L^2(K)$ -norm of $\underline{\boldsymbol{\tau}} \in \tilde{\Sigma}^\perp$ and $\mathbf{v} \in \tilde{V}^\perp$ by the $L^2(\partial K)$ -norm of their respective traces. They are independent of the mesh size h_K .

Now, if $P_k(K; \mathbb{S}) \times P_k(K) \subset \underline{\Sigma} \times V$, where $P_k(K; \mathbb{S})$ is the symmetric-matrix-valued polynomial space of degree no greater than k and $P_k(K)$ is the vector-valued polynomial space of degree no greater than k , with the choice of stabilization operator $\alpha = Id$, we get the following estimates from Theorem 2.2:

$$\begin{aligned} \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_K &\leq C h_K^{k+1} (\|\underline{\sigma}\|_{k+1, K} + \|\mathbf{u}\|_{k+1, K}), \\ \|\mathbf{u} - \Pi_V \mathbf{u}\|_K &\leq C h_K^{k+1} (\|\underline{\sigma}\|_{k+1, K} + \|\mathbf{u}\|_{k+1, K}), \end{aligned}$$

where $\|\cdot\|_{k+1,K}$ denotes the $H^{k+1}(K)$ -norm. Hence, the HDG projection gives quasi-optimal converge in both variables.

2.3 The error estimates

Now, we are ready to present our main results on the *a priori* error estimates. We display their proofs in the appendix.

2.3.1 Estimate of the stress approximation. We start with the estimation on the projection error $\underline{\Pi}_\Sigma \underline{\sigma} - \underline{\sigma}_h$.

THEOREM 2.3 For the solution of the HDG method given by (1.5), we have

$$\|\underline{\Pi}_\Sigma \underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{A}, \mathcal{T}_h} \leq \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{A}, \mathcal{T}_h}. \tag{2.3a}$$

Moreover, if we have a homogeneous and isotropic material with the compliance tensor given by (1.2), then

$$\|\underline{\Pi}_\Sigma \underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h} \leq C (1 + h^{1/2} \|\alpha\|) \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{T}_h}. \tag{2.3b}$$

Here, the constant C is independent of the mesh size h , the exact solution and the compliance tensor \mathcal{A} .

Note that, since the estimate (2.3b) implies $\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h} \leq C \|\underline{\sigma} - \underline{\Pi}_\Sigma \underline{\sigma}\|_{\mathcal{T}_h}$, the error $\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h}$ only depends on the approximation properties of the first component of the projection Π_h . Note also that the estimate (2.3b) implies that the method is *free from volumetric locking* in the sense that the error $\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h}$ does not grow as the Lamé constant $\lambda \rightarrow \infty$ in the incompressible limit.

2.3.2 Local estimates of the piecewise derivatives and the jump term. Now, we present local stability and error estimates on the piecewise divergence of $\underline{\sigma}_h$, the piecewise symmetric gradient of \mathbf{u}_h , and the jump term $\mathbf{u}_h - \widehat{\mathbf{u}}_h$, similar to the results in Cockburn *et al.* (2017b, Section 4).

THEOREM 2.4 For the solution of (1.5), we have the following local stability and error estimates:

$$\begin{aligned} \|\nabla \cdot \underline{\sigma}_h\|_K &\leq C_1 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma}_h\|_{\mathcal{A},K} + C_2 \|P_V \mathbf{f}\|_K, \\ \|\underline{\epsilon}(\mathbf{u}_h)\|_K &\leq C_3 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma}_h\|_{\mathcal{A},K} + C_4 \|P_{\widehat{V}} \mathbf{f}\|_K, \\ \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{\partial K} &\leq C_5 h_K^{1/2} \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma}_h\|_{\mathcal{A},K} + C_6 h_K^{1/2} \|P_{\widehat{V}} \mathbf{f}\|_K, \\ \|\nabla \cdot (\underline{\Pi}_\Sigma \underline{\sigma} - \underline{\sigma}_h)\|_K &\leq C_1 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{A},K}, \\ \|\underline{\epsilon}(\Pi_V \mathbf{u} - \mathbf{u}_h)\|_K &\leq C_3 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{A},K}, \\ \|\Pi_V \mathbf{u} - \mathbf{u}_h - (P_M \mathbf{u} - \widehat{\mathbf{u}}_h)\|_{\partial K} &\leq C_5 h_K^{1/2} \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{A},K}, \end{aligned}$$

where

$$C_1 := C_{\underline{\Sigma}^\perp} C_{\nabla, \underline{\Sigma}} \|\alpha\| \left(1 + \frac{\|\alpha\|}{a_{\widehat{V}^\perp}} \right), \quad C_2 := 1 + C_{\nabla, \underline{\Sigma}} \|\alpha\| \frac{C_{\widehat{V}^\perp}}{a_{\widehat{V}^\perp}},$$

$$C_3 := 1 + C_{\underline{\Sigma}^\perp} C_{\underline{\epsilon}(V)} \left(1 + \frac{\|\alpha\|}{a_{\tilde{V}^\perp}} \right), \quad C_4 := C_{\underline{\epsilon}(V)} \frac{C_{\tilde{V}^\perp}}{a_{\tilde{V}^\perp}},$$

$$C_5 := C_{\underline{\Sigma}^\perp} \left(1 + \frac{\|\alpha\|}{a_{\tilde{V}^\perp}} \right), \quad C_6 := \frac{C_{\tilde{V}^\perp}}{a_{\tilde{V}^\perp}}.$$

Here,

$$C_{\underline{\epsilon}(V)} := \sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \underline{\epsilon}(V)} h_K^{1/2} \|\underline{\boldsymbol{\tau}} \mathbf{n}\|_{\partial K} / \|\underline{\boldsymbol{\tau}}\|_K, \quad C_{\nabla \cdot \underline{\Sigma}} := \sup_{\mathbf{0} \neq \mathbf{v} \in \nabla \cdot \underline{\Sigma}} h_K^{1/2} \|\mathbf{v}\|_{\partial K} / \|\mathbf{v}\|_K.$$

Note that, summing over all the elements $K \in \mathcal{T}_h$, we easily get that

$$\|\nabla \cdot (\underline{\Pi}_\Sigma \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_{\mathcal{T}_h}, \quad \|\underline{\boldsymbol{\epsilon}}(\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h)\|_{\mathcal{T}_h}, \quad \|\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h - (\mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h)\|_{\partial \mathcal{T}_h}$$

only depend on $\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A}, \mathcal{T}_h}$, which, in turn, depends on the first component of the projection $\underline{\Pi}_h$.

2.3.3 Estimates of the approximation of the displacement. Our next result shows that $\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h$ can also be controlled solely in terms of the approximation error of the auxiliary projection $\underline{\boldsymbol{\sigma}} - \underline{\Pi}_\Sigma \underline{\boldsymbol{\sigma}}$. In addition, an improvement can be achieved under a typical elliptic regularity property we state next. We assume that, for any given $\boldsymbol{\theta} \in L^2(\Omega)$, we have

$$\|\boldsymbol{\phi}\|_{2,\Omega} + \|\underline{\boldsymbol{\psi}}\|_{1,\Omega} \leq C \|\boldsymbol{\theta}\|_{\Omega}, \tag{2.4}$$

where C only depends on the domain Ω , and $(\underline{\boldsymbol{\psi}}, \boldsymbol{\phi})$ is the solution of the *dual* problem:

$$\mathcal{A} \underline{\boldsymbol{\psi}} - \underline{\boldsymbol{\epsilon}}(\boldsymbol{\phi}) = 0 \quad \text{in } \Omega, \tag{2.5a}$$

$$\nabla \cdot \underline{\boldsymbol{\psi}} = \boldsymbol{\theta} \quad \text{in } \Omega, \tag{2.5b}$$

$$\boldsymbol{\phi} = \mathbf{0} \quad \text{on } \partial \Omega. \tag{2.5c}$$

We are now ready to state our result.

THEOREM 2.5 If $\mathbf{P}_1(K) \subset \mathbf{V}(K)$ for every element $K \in \mathcal{T}_h$, and the elliptic regularity property (2.4) holds, then, for the solution of (1.5), we have

$$\|\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} \leq C h \|\underline{\boldsymbol{\sigma}} - \underline{\Pi}_\Sigma \underline{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}.$$

The constant C depends on \mathcal{A} , but is independent of h and the exact solution.

Combining this result with the last estimate in Theorem 2.4 and applying simple triangle, trace and inverse inequalities, we immediately get

$$\|\mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h\|_{\partial \mathcal{T}_h} \leq C h^{1/2} \|\underline{\boldsymbol{\sigma}} - \underline{\Pi}_\Sigma \underline{\boldsymbol{\sigma}}\|_{\mathcal{T}_h},$$

and we see that the quality of the approximation \mathbf{u}_h and $\widehat{\mathbf{u}}_h$ only depends on the approximation error of the auxiliary projection, as claimed.

2.3.4 *Estimates of a postprocessing of the displacement.* Note that if $h \|\underline{\sigma} - \underline{\Pi}_{\mathcal{S}} \underline{\sigma}\|_{\mathcal{T}_h}$ converges faster than $\|\mathbf{u} - \underline{\Pi}_V \mathbf{u}\|_{\mathcal{T}_h}$, the convergence of \mathbf{u}_h to $\underline{\Pi}_V \mathbf{u}$ is faster than that of \mathbf{u}_h to \mathbf{u} . As mentioned before, we can take advantage of this *superconvergence* result to show the existence of a displacement postprocessing \mathbf{u}_h^* converge to \mathbf{u} as fast as \mathbf{u}_h superconverges to $\underline{\Pi}_V \mathbf{u}$. To this end, we associate to each element K a vector space $\mathbf{V}^*(K)$ that contains $\mathbf{V}(K)$. Then, the function \mathbf{u}_h^* is the element of $\mathbf{V}^*(K)$ such that

$$(\nabla \mathbf{u}_h^*, \nabla \mathbf{w})_K = -(\mathbf{u}_h, \Delta \mathbf{w})_K + (\widehat{\mathbf{u}}_h, \nabla \mathbf{w} \mathbf{n})_{\partial K} \quad \forall \mathbf{w} \in \widetilde{\mathbf{V}}^*(K)^\perp, \tag{2.6a}$$

$$(\mathbf{u}_h^*, \mathbf{r})_K = (\mathbf{u}_h, \mathbf{r})_K \quad \forall \mathbf{r} \in \widetilde{\mathbf{V}}^*(K), \tag{2.6b}$$

where $\widetilde{\mathbf{V}}^*(K)^\perp \subset \mathbf{V}^*(K)$ is the L^2 -orthogonal complement of $\widetilde{\mathbf{V}}^*(K)$, and $\widetilde{\mathbf{V}}^*(K)$ is any nontrivial subspace of $\nabla \cdot \underline{\Sigma}(K)$ containing the constant vectors $\mathbf{P}_0(K)$.

We have the following estimate.

THEOREM 2.6 Suppose that

$$\mathbf{P}_0(K) \subset \nabla \cdot \underline{\Sigma}(K), \quad \Delta \mathbf{V}^*(K) \subset \nabla \cdot \underline{\Sigma}(K), \quad (\nabla \mathbf{V}^*) \mathbf{n} |_{\partial K} \subset \mathbf{M}(\partial K).$$

Let \mathbf{u}_h^* be the solution to (2.6) with $(\mathbf{u}_h, \widehat{\mathbf{u}}_h)$ being the solution to (1.5), then

$$\|\mathbf{u} - \mathbf{u}_h^*\|_K \leq C (\|\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h\|_K + h_K^{1/2} \|\mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h\|_{\partial K} + h_K \|\nabla(\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u})\|_K).$$

Here the constant C depends on \mathcal{A} , but is independent of h and the exact solution, and \mathbf{P}_{V^*} is the L^2 -projection onto $\mathbf{V}^*(K)$.

Note again that after summing the estimates over all the elements $K \in \mathcal{T}_h$, we get

$$\|\mathbf{u} - \mathbf{u}_h^*\|_{\mathcal{T}_h} \leq C (\|\underline{\Pi}_V \mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} + h^{1/2} \|\mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h\|_{\partial K} + h \|\nabla(\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u})\|_{\mathcal{T}_h}).$$

2.3.5 *A practical example.* To conclude this section, we apply the error estimates in Theorem 2.2 and the error estimates in Theorems 2.3–2.6 to obtain convergence rates for L^2 -error of $\underline{\sigma}_h$, \mathbf{u}_h and \mathbf{u}_h^* for a special case with the following conditions on the spaces (on each element K) and the stabilization operator:

(C.1) $\mathbf{M} = \mathbf{P}_k(\partial K)$, and $\underline{\Sigma} \times \mathbf{V}$ admits an \mathbf{M} -decomposition with $\mathbf{P}_k(K, \mathbb{S}) \times \mathbf{P}_k(K) \subset \underline{\Sigma} \times \mathbf{V}$.

(C.2) $\mathbf{V}^* = \mathbf{P}_{k+1}(K)$.

(C.3) $\alpha = Id$.

In this case, we get that

$$\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h} \leq C h^{k+1} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+1}), \tag{2.7a}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} \leq C h^{k+1} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+1}), \tag{2.7b}$$

$$\|\mathbf{u} - \mathbf{u}_h^*\|_{\mathcal{T}_h} \leq C h^{k+2} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+2}), \tag{2.7c}$$

where the last estimate require the regularity estimate (2.4) holds.

We remark that, as we will make clear in the next two sections, the natural choice $\underline{\Sigma} \times \mathbf{V} \times \mathbf{M} := \mathbf{P}_k(K, \mathbb{S}) \times \mathbf{P}_k(K) \times \mathbf{P}_k(\partial K)$ proposed in [Soon *et al.* \(2009\)](#) and analysed in [Fu *et al.* \(2015\)](#) does not satisfy condition (C.1) due to the lack of an \mathbf{M} -decomposition for $\underline{\Sigma} \times \mathbf{V}$. Actually, for this choice of spaces, with $\alpha = Id$, it was proven in [Fu *et al.* \(2015\)](#) that

$$\begin{aligned}\|\underline{\sigma} - \underline{\sigma}_h\|_{\mathcal{T}_h} &\leq C h^{k+1/2} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+1}), \\ \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{T}_h} &\leq C h^{k+1} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+1}), \\ \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathcal{T}_h} &\leq C h^{k+1} (\|\underline{\sigma}\|_{k+1} + \|\mathbf{u}\|_{k+1}),\end{aligned}$$

where numerical results suggested that the orders are actually sharp for $k = 1$. We will see in Section 4 that on triangular meshes, we only need to add *two* (rational) basis functions to $\underline{\Sigma}$ for $k = 1$ and *three* for $k \geq 2$ to obtain an \mathbf{M} -decomposition. Then, the desired (superconvergence) error estimates (2.7) follow.

3. The \mathbf{M} -decompositions

In this section, we obtain a characterization of \mathbf{M} -decompositions. We then show how to use it to construct HDG and (hybridized) mixed methods that superconverge on unstructured meshes.

3.1 A characterization of \mathbf{M} -decompositions

We first give a characterization of \mathbf{M} -decompositions expressed solely in terms of the spaces $\underline{\Sigma} \times \mathbf{V}$. In general, it states that $\underline{\Sigma} \times \mathbf{V}$ admits an \mathbf{M} -decomposition if and only if the space \mathbf{M} is the orthogonal sum of the traces of the kernels of $\nabla \cdot$ in $\underline{\Sigma}$ and of $\underline{\epsilon}$ in \mathbf{V} . It is expressed in terms of a special integer we define next.

DEFINITION 3.1 (The \mathbf{M} -index) The \mathbf{M} -index of the space $\underline{\Sigma} \times \mathbf{V}$ is the number

$$\begin{aligned}I_{\mathbf{M}}(\underline{\Sigma} \times \mathbf{V}) &:= \dim \mathbf{M} - \dim\{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}, \nabla \cdot \underline{\tau} = 0\} \\ &\quad - \dim\{\mathbf{v}|_{\partial K} : \mathbf{v} \in \mathbf{V}, \underline{\epsilon}(\mathbf{v}) = 0\}.\end{aligned}$$

THEOREM 3.2 (A characterization of \mathbf{M} -decompositions) For a given space of traces \mathbf{M} , the space $\underline{\Sigma} \times \mathbf{V}$ admits an \mathbf{M} -decomposition if and only if

- (a) $\text{tr}(\underline{\Sigma} \times \mathbf{V}) \subset \mathbf{M}$,
- (b) $\underline{\epsilon}(\mathbf{V}) \subset \underline{\Sigma}$, $\nabla \cdot \underline{\Sigma} \subset \mathbf{V}$,
- (c) $I_{\mathbf{M}}(\underline{\Sigma} \times \mathbf{V}) = 0$.

In this case, we have

$$\mathbf{M} = \{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}, \nabla \cdot \underline{\tau} = 0\} \oplus \{\mathbf{v}|_{\partial K} : \mathbf{v} \in \mathbf{V}, \underline{\epsilon}(\mathbf{v}) = 0\},$$

where the sum is orthogonal.

The proof of the above result is omitted here and documented in our extended paper on arXiv (Cockburn & Fu, 2017a, Appendix C). We remark that the proof is very similar to the diffusion case considered in Cockburn *et al.* (2017b, Section 2.4).

The importance of this result resides in that it allows us to know whether any given space $\underline{\Sigma} \times V$ admits an M -decomposition by just verifying some inclusion properties and by computing a single number, namely, $I_M(\underline{\Sigma} \times V)$ – a natural number, by property (a). Moreover, this result shows explicitly how M can be expressed in terms of traces of the kernels of the divergence in $\underline{\Sigma}$ and the trace of the kernel of the symmetric gradient in V ; we call the identity *the kernels' trace decomposition*. This identity is going to be the guiding principle for the systematic construction of M -decompositions we develop in the next subsection.

3.2 The construction of M -decompositions

Now, we propose three ways of obtaining M -decompositions; we follow Cockburn *et al.* (2017b, Section 5). We show how to modify a given space $\underline{\Sigma}_g \times V_g$, which is assumed to satisfy the first two inclusion properties of an M -decomposition, to obtain a new space $\underline{\Sigma} \times V$ admitting an M -decomposition. By the assumption on the given space $\underline{\Sigma}_g \times V_g$, the indexes $I_M(\underline{\Sigma}_g \times V_g)$ and

$$I_S(\underline{\Sigma} \times V) := \dim V - \dim \nabla \cdot \underline{\Sigma}$$

are non-negative. We propose three different ways of doing this according to whether the indexes are zero or not.

The case $I_M(\underline{\Sigma}_g \times V_g) > 0$. In this case, the space $\underline{\Sigma}_g \times V_g$ does not admit an M -decomposition. By Theorem 3.2, we have that

$$\{\underline{\tau} \cdot \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}_g, \nabla \cdot \underline{\tau} = \mathbf{0}\} \oplus \{\mathbf{v}|_{\partial K} : \mathbf{v} \in V_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} \subsetneq M.$$

To simplify the notation, we set $\underline{\Sigma}_{gs} := \{\underline{\tau} \in \underline{\Sigma}_g : \nabla \cdot \underline{\tau} = \mathbf{0}\}$ to be the divergence-free subspace of $\underline{\Sigma}_g$ (s stands for solenoidal) and $V_{g\text{rm}} := \{\mathbf{v} \in V_g : \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\}$ to be the $\underline{\epsilon}$ -free subspace of V_g (rm stands for rigid motions). We see that, to achieve equality, we have to, in general, *fill* the remaining part of M by adding a space of symmetric-tensorial, solenoidal functions $\delta \underline{\Sigma}_{\text{fill}M}$ of dimension $I_M(\underline{\Sigma}_g \times V_g)$. The precise description of this subspace is in the following result.

PROPOSITION 3.3 (Filling the space of traces M) Let $\underline{\Sigma}_g \times V_g$ satisfy the inclusion properties (a) and (b) of Theorem 3.2. Assume that $\delta \underline{\Sigma}_{\text{fill}M}$ satisfies:

- (a) $\gamma \delta \underline{\Sigma}_{\text{fill}M} \subset M$,
- (b) $\nabla \cdot \delta \underline{\Sigma}_{\text{fill}M} = \{\mathbf{0}\}$,
- (c) $\gamma \underline{\Sigma}_{gs} \cap \gamma \delta \underline{\Sigma}_{\text{fill}M} = \{\mathbf{0}\}$,
- (d) $\dim \delta \underline{\Sigma}_{\text{fill}M} = \dim \gamma \delta \underline{\Sigma}_{\text{fill}M} = I_M(\underline{\Sigma}_g \times V_g)$.

Then $(\underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fill}M}) \times V_g$ admits an M -decomposition. Moreover, at least one space $\delta \underline{\Sigma}_{\text{fill}M}$ can be constructed when $V_{g\text{rm}} = \mathbf{RM}(K)$, where $\mathbf{RM}(K) := \{\mathbf{v} \in H^1(K) : \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\}$ is the space for *rigid motions*.

Proof. Let us just show how to construct one space $\delta \underline{\Sigma}_{\text{fillM}}$. Let \mathcal{B} be a basis for $(\text{tr}(\underline{\Sigma}_{g_s} \times V_{g_{rm}}))^\perp$. Then we can take $\delta \underline{\Sigma}_{\text{fillM}}$ as the span of $\{\underline{\epsilon}(\phi_\mu)\}_{\mu \in \mathcal{B}}$, where

$$\nabla \cdot (\underline{\epsilon}(\phi_\mu)) = \mathbf{0} \text{ in } K, \quad \underline{\epsilon}(\phi_\mu) \mathbf{n} = \mu \text{ on } \partial K,$$

where $\mu \in \mathcal{B}$. Since $V_{g_{rm}} = \mathbf{RM}(K)$, the $L^2(\partial K)$ -projection of μ onto $\gamma \mathbf{RM}(K)$ is zero and so, $\underline{\epsilon}(\phi_\mu)$ is well defined. The boundary condition ensures the satisfaction of conditions (a) and (c), and condition (b) holds by construction. Finally, condition (d) is also satisfied given that the set $\{\underline{\epsilon}(\phi_\mu)\}_{\mu \in \mathcal{B}}$ is linearly independent, and

$$\dim \delta \underline{\Sigma}_{\text{fillM}} = \dim \mathcal{B} = \dim \mathbf{M} - \dim \text{tr}(\underline{\Sigma}_{g_s} \times V_{g_{rm}}) = I_M(\underline{\Sigma}_g \times V_g).$$

This completes the proof. □

The case $I_M(\underline{\Sigma}_g \times V_g) = 0$, but $I_S(\underline{\Sigma}_g \times V_g) > 0$. In this case, the space $\underline{\Sigma}_g \times V_g$ admits an M -decomposition, but $\nabla \cdot \underline{\Sigma}_g$ is a proper subspace of V_g . By the kernels' trace decomposition of Theorem 3.2, we have

$$\{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}_g, \nabla \cdot \underline{\tau} = \mathbf{0}\} \oplus \{\mathbf{v}|_{\partial K} : \mathbf{v} \in V_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} = \mathbf{M},$$

and we then see that, if we seek a modification of $\underline{\Sigma}_g \times V_g$ of the form $\underline{\Sigma}_g \times V$, it must be such that

$$\{\mathbf{v}|_{\partial K} : \mathbf{v} \in V, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} = \{\mathbf{v}|_{\partial K} : \mathbf{v} \in V_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\}.$$

The following result gives a hypothesis under which we are allowed to reduce V_g to $V := \nabla \cdot \underline{\Sigma}_g$.

PROPOSITION 3.4 (Reducing the space V) Assume that $\underline{\Sigma}_g \times V_g$ admits an M -decomposition. Then $\underline{\Sigma}_g \times \nabla \cdot \underline{\Sigma}_g$ admits an M -decomposition provided that $\nabla \cdot \underline{\Sigma}_g$ contains the space of rigid motions $\mathbf{RM}(K)$.

Proof. Since $\mathbf{RM}(K) \subset \nabla \cdot \underline{\Sigma}_g \subset V_g$, we have

$$\{\mathbf{v}|_{\partial K} : \mathbf{v} \in \nabla \cdot \underline{\Sigma}_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} = \{\mathbf{v}|_{\partial K} : \mathbf{v} \in V_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} = \gamma \mathbf{RM}(K).$$

This completes the proof. □

Now, let us seek a modification of $\underline{\Sigma}_g \times V_g$ of the form $\underline{\Sigma} \times V_g$, where $\underline{\Sigma}_g \subset \underline{\Sigma}$. Since

$$\nabla \cdot \underline{\Sigma}_g \subsetneq V_g,$$

we see that to achieve the equality, we have to *fill* the remaining part of V_g by adding a space of symmetric-tensorial, *nonsolenoidal* functions $\delta \underline{\Sigma}_{\text{fillV}}$ of dimension $I_S(\underline{\Sigma}_g \times V_g)$. In this case, we would immediately have that

$$\{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}, \nabla \cdot \underline{\tau} = \mathbf{0}\} = \{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}_g, \nabla \cdot \underline{\tau} = \mathbf{0}\},$$

and, by Theorem 3.2, the resulting space would admit an M -decomposition. The precise way of choosing $\delta \underline{\Sigma}_{\text{fillV}}$ is described in the following result.

PROPOSITION 3.5 (Increasing the space $\underline{\Sigma}_g$) Let the space $\underline{\Sigma}_g \times V_g$ admit an M -decomposition and assume that $\nabla \cdot \underline{\Sigma}_g$ is a proper subspace of V_g . Let $\delta \underline{\Sigma}_{\text{fillV}}$ satisfy the following hypotheses:

- (a) $\gamma \delta \underline{\Sigma}_{\text{fillV}} \subset M$,
- (b) $\nabla \cdot \delta \underline{\Sigma}_{\text{fillV}} \subset V$,
- (c) $\nabla \cdot \underline{\Sigma} \cap \nabla \cdot \delta \underline{\Sigma}_{\text{fillV}} = \{0\}$,
- (d) $\dim \delta \underline{\Sigma}_{\text{fillV}} = \dim \nabla \cdot \delta \underline{\Sigma}_{\text{fillV}} = I_S(\underline{\Sigma} \times V)$.

Then $(\underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillV}}) \times V_g$ admits an M -decomposition with $V_g = \nabla \cdot (\underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillV}})$. Moreover, at least one space $\delta \underline{\Sigma}_{\text{fillV}}$ can be constructed to satisfy all the hypotheses when M contains the space of traces of rigid motions $\gamma \mathbf{RM}(K)$.

Proof. Let us just show how to construct one space $\delta \underline{\Sigma}_{\text{fillV}}$. Let \mathcal{B} be a basis of \tilde{V}^\perp . Then, we can take $\delta \underline{\Sigma}_{\text{fillV}}$ as the span of $\{\underline{\epsilon}(\phi_v)\}_{v \in \mathcal{B}}$, where ϕ_v solves

$$\nabla \cdot (\underline{\epsilon}(\phi_v)) = v \text{ in } K \text{ and } \underline{\epsilon}(\phi_v) \mathbf{n} = c(v) \text{ on } \partial K,$$

where $c(v)$ is the element of $\gamma \mathbf{RM}(K)$ such that

$$\langle c(v), \varphi \rangle_{\partial K} = (v, \varphi)_K \quad \forall \varphi \in \mathbf{RM}(K).$$

Note that since M contains the space $\gamma \mathbf{RM}(K)$, hypothesis (a) is actually satisfied. Finally, it is not difficult to see that hypotheses (b), (c) and (d) are also satisfied by the choice of \mathcal{B} . This completes the proof. \square

3.3 A systematic procedure for obtaining M -decompositions

We can now use these three ways of obtaining M -decompositions, summarized in Table 1, to propose a systematic way for constructing spaces admitting M -decompositions starting from a single, given space $\underline{\Sigma}_g \times V_g$. Let us recall that the space $\underline{\Sigma}_g \times V_g$ is assumed to satisfy the first two inclusion properties of an M -decomposition, and so the indexes $I_M(\underline{\Sigma}_g \times V_g)$ and $I_S(\underline{\Sigma}_g \times V_g)$ are non-negative.

The systematic construction is described in Tables 2 and 3. Note that the construction provides three different spaces admitting M -decompositions. The first is associated to an HDG method. The other two are associated to (hybridized) mixed methods which can be thought of as *sandwiching* the HDG method.

It is now clear that we are left to construct the filling spaces $\delta \underline{\Sigma}_{\text{fillM}}$ and $\delta \underline{\Sigma}_{\text{fillV}}$ that satisfy the properties in Table 3 for a given space $\underline{\Sigma}_g \times V_g$, satisfying the first two inclusions of an M -decomposition. In the next section, we present such spaces defined on general polygonal elements in two-space dimensions.

4. An explicit construction of the spaces $\delta \underline{\Sigma}_{\text{fillM}}$ and $\delta \underline{\Sigma}_{\text{fillV}}$ in two-space dimensions

This section contains the explicit construction of the spaces $\delta \underline{\Sigma}_{\text{fillM}}$ and $\delta \underline{\Sigma}_{\text{fillV}}$, satisfying the properties in Table 3 in two-space dimensions.

TABLE 1 Three ways of constructing spaces $\underline{\Sigma} \times V$ admitting an M -decomposition. The spaces are obtained by modifying the space $\underline{\Sigma}_g \times V_g$ according to whether it already admits an M -decomposition and according to whether the space $\nabla \cdot \underline{\Sigma}_g$ is a proper subspace of V_g . The space $\underline{\Sigma}_g \times V_g$ is assumed to satisfy the first two inclusion properties of an M -decomposition, namely, $\text{tr}(\underline{\Sigma}_g \times V_g) \subset M$ and $\epsilon(V_g) \times \nabla \cdot \underline{\Sigma}_g \subset \underline{\Sigma}_g \times V_g$

Way no.	Properties of $\underline{\Sigma}_g \times V_g$	$\underline{\Sigma}$	V	Properties of $\underline{\Sigma} \times V$
I (Proposition 3.3)	$I_M(\underline{\Sigma}_g \times V_g) > 0$	$\underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillM}}$	V_g	$I_M(\underline{\Sigma} \times V) = 0$ $I_S(\underline{\Sigma} \times V) = I_S(\underline{\Sigma}_g \times V_g)$ if $RM(K) \subset V_g$
II (Proposition 3.4)	$I_M(\underline{\Sigma}_g \times V_g) = 0$ $I_S(\underline{\Sigma}_g \times V_g) > 0$	$\underline{\Sigma}_g$	$\nabla \cdot \underline{\Sigma}_g$	$I_M(\underline{\Sigma} \times V) = 0$ $I_S(\underline{\Sigma} \times V) = 0$ if $RM(K) \subset \nabla \cdot \underline{\Sigma}_g$
III (Proposition 3.5)	$I_M(\underline{\Sigma}_g \times V_g) = 0$ $I_S(\underline{\Sigma}_g \times V_g) > 0$	$\underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillV}}$	V_g	$I_M(\underline{\Sigma} \times V) = 0$ $I_S(\underline{\Sigma} \times V) = 0$ if $\gamma RM(K) \subset M$

TABLE 2 Spaces $\underline{\Sigma} \times V$ admitting an M -decomposition. They are constructed from the single space $\underline{\Sigma}_g \times V_g$, which is assumed to satisfy the first two inclusion properties of an M -decomposition, namely $\text{tr}(\underline{\Sigma}_g \times V_g) \subset M$ and $\epsilon(V_g) \times \nabla \cdot \underline{\Sigma}_g \subset \underline{\Sigma}_g \times V_g$

Way no.	$\underline{\Sigma}$	V
III	$\underline{\Sigma}^{\text{mix}} := \underline{\Sigma}^{\text{hdg}} \oplus \delta \underline{\Sigma}_{\text{fillV}}$	$V^{\text{mix}} := V_g$
I	$\underline{\Sigma}^{\text{hdg}} := \underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillM}}$	$V^{\text{hdg}} := V_g$
II	$\underline{\Sigma}^{\text{mix}} := \underline{\Sigma}_g \oplus \delta \underline{\Sigma}_{\text{fillM}}$	$V^{\text{mix}} := \nabla \cdot \underline{\Sigma}_g$

TABLE 3 The main properties of the spaces $\delta \underline{\Sigma}$

$\delta \underline{\Sigma}$	$\nabla \cdot \delta \underline{\Sigma}$	$\gamma \delta \underline{\Sigma}$	$\dim \delta \underline{\Sigma}$
$\delta \underline{\Sigma}_{\text{fillM}}$	$\{0\}$	$\subset M, \cap \gamma \underline{\Sigma}_{g_s} = \{0\}$	$I_M(\underline{\Sigma}_g \times V_g) = \dim \gamma \delta \underline{\Sigma}$
$\delta \underline{\Sigma}_{\text{fillV}}$	$\subset V_g, \cap \nabla \cdot \underline{\Sigma}_g = \{0\}$	$\subset M$	$I_S(\underline{\Sigma}_g \times V_g) = \dim \nabla \cdot \delta \underline{\Sigma}$

Here we consider a polygonal element K , fix the trace space $M(\partial K) := P_k(\partial K)$ and study two choices of the initial guess spaces $\underline{\Sigma}_g \times V_g$, namely

$$\underline{\Sigma}_g \times V_g := \underline{P}_k^s \times P_k \quad \text{and} \quad \underline{\Sigma}_g \times V_g := \underline{Q}_k^s \times Q_k,$$

where $\underline{P}_k^s := P_k(K; \mathbb{S})$ and $\underline{Q}_k^s := Q_k(K; \mathbb{S})$ are the symmetric-tensorial polynomial fields.

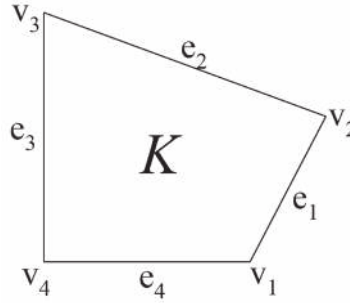


FIG. 1. A quadrilateral element \$K\$.

4.1 Notation, the Airy stress operator and the lifting functions

To state our results, we need to introduce some notation. Let \$\{v_i\}_{i=1}^{ne}\$ be the set of vertices of the polygonal element \$K\$, which we take to be counter-clockwise ordered. Let \$\{e_i\}_{i=1}^{ne}\$ be the set of edges of \$K\$, where the edge \$e_i\$ connects the vertices \$v_i\$ and \$v_{i+1}\$. Here, the subindexes are integers module \$ne\$, for example, \$v_{ne+1} = v_1\$. An illustration for a quadrilateral element \$K\$ is presented in Fig. 1. We also define, for \$1 \le i \le ne\$, \$\lambda_i\$ to be the linear function that vanishes on edge \$e_i\$ and reaches maximum value 1 in the closure of the element \$K\$.

Since \$\delta \underline{\Sigma}_{fillM}\$ is a divergence-free symmetric tensor field, it can be characterized, see, for example, Arnold & Winther (2002), as the Airy stress operator of some \$H^2\$-conforming scalar field, where the Airy stress operator is defined as follows:

$$J := \begin{pmatrix} \frac{\partial^2}{\partial y^2} & -\frac{\partial^2}{\partial x \partial y} \\ -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} \end{pmatrix}.$$

Now, we introduce two functions which we are going to use as tools to define *lifting* of traces on \$\partial K\$ into the inside of the element \$K\$. The first is associated to a vertex. To each vertex \$v_i\$, we associate a function \$\xi_i\$, satisfying the following conditions:

(L.1) \$\xi_i|_{e_j} \in P_1(e_j) \ j = 1, \dots, ne\$,

(L.2) \$\xi_i(v_j) = \delta_{ij} \ j = 1, \dots, ne\$,

(L.3) \$\frac{\partial \xi_i}{\partial n_j}|_{e_j} \in P_0(e_j) \ j = 1, \dots, ne\$.

Here, \$n_j\$ denotes the outward normal to the edge \$e_j\$. The second is associate to an edge. To each edge \$e_i\$, we associate a function \$B_i\$, satisfying the following conditions:

(H.1) \$B_i|_{e_j} = 0 \ j = 1, \dots, ne\$,

(H.2) \$\frac{\partial B_i}{\partial n_j}|_{e_j} = 0 \ j = 1, \dots, ne, j \neq i\$,

(H.3) \$\frac{\partial B_i}{\partial n_i}|_{e_i} = \lambda_{i-1} \lambda_{i+1} \ j = i\$.

Next, we give some examples of these functions. If \$K\$ is a triangle, we simply take \$\xi_i := \lambda_{i+1}\$ and \$B_i = (\prod_{k=1}^3 \lambda_k) \prod_{j \neq i} \frac{\lambda_j}{\lambda_j + \lambda_i}\$. Note that \$B_i\$ is nothing, but the rational bubble related to the edge \$e_i\$ defined in Guzmán & Neilan (2014). If \$K\$ is a star-shaped polygon with respect to an interior node \$v_o\$, we subdivide the element \$K\$ into \$ne\$ triangles \$\{T_i\}_{i=1}^{ne}\$, with \$T_i\$ beginning the triangle with vertices \$v_o, v_{i-1}, v_i\$. We then

take ξ_i to be the piecewise linear function on $\{T_j\}_{j=1}^{ne}$, satisfying condition (L.2) and $\xi_i(v_o) = 0$. We take B_i to be the (composite) function that vanishes on T_j for $j \neq i + 1$ and equals to the rational bubble associated to \mathbf{e}_i on T_{i+1} . This choice is similar to the composite lifting introduced in Cockburn & Fu (2017b).

Let us remark that the conditions (H) on the function B_i , derived from our analysis in the next section (see Lemma 5.8), ensure that $\lim_{x \rightarrow v_{i+1}} (J B_i) \mathbf{n}_i \cdot \mathbf{n}_{i+1}|_{\mathbf{e}_i} \neq \lim_{x \rightarrow v_{i+1}} (J B_i) \mathbf{n}_{i+1} \cdot \mathbf{n}_i|_{\mathbf{e}_{i+1}} = 0$. The importance of this nodal discontinuity, which is not made evident in our construction of \mathcal{M} -decompositions, is well established in the literature; see, for example, the discussion at the last two paragraphs of Section 3 in Arnold & Winther (2002). Indeed, in Arnold & Winther (2002), it is argued that there exist no hybridizable mixed method (which does not use vertex degrees of freedom) that only uses polynomial shape functions. Hence, it comes at no surprise that our space $\delta \underline{\Sigma}_{\text{fillM}}$ consists of nonpolynomial functions.

4.2 The case $\underline{\Sigma}_g \times \mathbf{V}_g := \underline{\mathcal{Q}}_k^s \times \mathcal{Q}_k$

We start by considering K to be a unit square with edges parallel to the axes and $v_1 = (0, 1)$. (This implies $\lambda_1 = x, \lambda_2 = y, \lambda_3 = 1 - x, \lambda_4 = 1 - y$.) We omit the proof due to its similarity with the proof of the more difficult result, Theorem 4.3, in Section 5.

We omit the proof of the next two theorems in this subsection due to its similarity with the proof of the more difficult result, Theorem 4.3, in Section 5. See also a sketch of the proof in our extended paper on arXiv (Cockburn & Fu, 2017a, Appendix D).

THEOREM 4.1 Let K be a unit square with edges parallel to the axes. Then, for $\mathbf{M} = \mathbf{P}_k(\partial K)$ and $\underline{\Sigma}_g \times \mathbf{V}_g = \underline{\mathcal{Q}}_k^s(K) \times \mathcal{Q}_k(K)$, where $k \geq 1$, we have that

$$I_{\mathbf{M}}(\underline{\Sigma}_g \times \mathbf{V}_g) = \begin{cases} 6 & \text{if } k = 1, \\ 9 & \text{if } k = 2, \\ 10 & \text{if } k \geq 3, \end{cases} \quad \text{and} \quad I_S(\underline{\Sigma}_g \times \mathbf{V}_g) = 3.$$

Moreover, the spaces

$$\delta \underline{\Sigma}_{\text{fillM}} := \begin{cases} J \text{span}\{B_2, B_3, B_4, \xi_4^2, \\ \xi_4^2(1-x), \xi_4^2(1-y)\} & \text{if } k = 1, \\ J \text{span}\{B_2, B_3, B_4, B_4x, \\ \xi_4^2(1-x), \xi_4^2(1-y), \\ \xi_4^2(1-x)(1-y), \\ \xi_4^2(1-x)^2, \xi_4^2(1-y)^2\} & \text{if } k = 2, \\ J \text{span}\{B_2, B_3, B_4, B_4x, \\ \xi_4^2(1-x)^{k-1}, \xi_4^2(1-y)^{k-1}, \\ \xi_4^2(1-x)^{k-1}(1-y), \xi_4^2(1-x)(1-y)^{k-1}, \\ \xi_4^2(1-x)^k, \xi_4^2(1-y)^k\} & \text{if } k \geq 3, \end{cases}$$

$$\delta \underline{\Sigma}_{\text{fillIV}} := \text{span} \left\{ \begin{bmatrix} x^{k+1}y^{k-1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x^{k+1}y^k & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & x^k y^{k+1} \end{bmatrix} \right\},$$

satisfy the properties in Table 3. Here, ξ_4 satisfies conditions (L) and B_2, B_3, B_4 satisfy conditions (H).

Let us remark that in practical implementation, we can take ξ_4 to be the composite lifting function presented in the previous subsection and $\{B_i\}$ to be the following rational functions:

$$B_i = \prod_{k=1}^4 \lambda_k \prod_{j \neq i} \frac{\lambda_j}{\lambda_j + \lambda_i}. \tag{4.1}$$

When the polynomial degree $k \geq 2$, we can bypass the use of the composite function ξ_4 in the definition of $\delta \underline{\Sigma}_{\text{fillM}}$ by using an exponential function, as we see in the next result.

THEOREM 4.2 Let K be a unit square with edges parallel to the axes. Then, for $\mathbf{M} = \mathbf{P}_k(\partial K)$ and $\underline{\Sigma}_g \times \mathbf{V}_g = \underline{\mathbf{Q}}_k^s(K) \times \mathbf{Q}_k(K)$, where $k \geq 2$, we have the spaces

$$\delta \underline{\Sigma}_{\text{fillM}} := \begin{cases} J \text{ span}\{B_2, B_3, B_4, B_4x, \\ (xy)^2(1-x), (xy)^2(1-y), \\ (xy)^2(1-x)(1-y), \\ (xe^{1-y})^2(1-x)^2, (e^{1-x}xy)^2(1-y)^2\} & \text{if } k = 2, \\ J \text{ span}\{B_2, B_3, B_4, B_4x, \\ (xy)^2(1-x)^{k-1}, (xy)^2(1-y)^{k-1}, \\ (xy)^2(1-x)^{k-1}(1-y), (xy)^2(1-x)(1-y)^{k-1}, \\ (xe^{1-y})^2(1-x)^k, (e^{1-x}xy)^2(1-y)^k\} & \text{if } k \geq 3, \end{cases}$$

satisfy the properties in Table 3. Here, B_i are defined in (4.1).

4.3 The case $\underline{\Sigma}_g \times \mathbf{V}_g := \underline{\mathbf{P}}_k^s \times \mathbf{P}_k$

Now, we consider K to be a general polygon without hanging nodes.

THEOREM 4.3 Let K be a polygon of ne edges without hanging nodes. Then, for $\mathbf{M} := \mathbf{P}_k(\partial K)$ and $\underline{\Sigma}_g \times \mathbf{V}_g := \underline{\mathbf{P}}_k^s \times \mathbf{P}_k$ with $k \geq 1$, we have that

$$I_M(\underline{\Sigma}_g \times \mathbf{V}_g) = 2(\theta + 1)ne - \frac{1}{2}(\theta + 3)(\theta + 4), \quad \text{and} \quad I_S(\underline{\Sigma}_g \times \mathbf{V}_g) = 2(k + 1),$$

where $\theta := \min\{k, 2ne - 4\}$.

Moreover, the spaces

$$\begin{aligned} \delta \underline{\Sigma}_{\text{fillM}} &:= \bigoplus_{i=1}^{ne} J \Psi_i, \\ \delta \underline{\Sigma}_{\text{fillV}} &:= \{\underline{\phi}_{k+1,a}^1, \underline{\phi}_{k+1,a}^2\}_{a=0}^k \quad (\text{for } k \geq 2) \end{aligned}$$

satisfy the properties in Table 3. Here

$$\Psi_i = \begin{cases} \{0\} & \text{if } i = 1, \\ \text{span} \left\{ (\xi_{i+1})^2 \lambda_{i+1}^b \right\}_{b=\max\{k+5-2i,0\}}^k \\ \oplus \text{span} \left\{ (\xi_{i+1})^2 \lambda_i \lambda_{i+1}^b \right\}_{b=\max\{k+4-2i,0\}}^{k-1} \\ \oplus \text{span} \{B_i\} & \text{if } 2 \leq i \leq ne - 1, \\ \text{span} \{B_i\} & \text{if } i = ne \text{ and } k = 1, \\ \text{span} \left\{ (\xi_{i+1})^2 \lambda_{i+1}^{2+b} \right\}_{b=\max\{k+5-2i,0\}}^{k-2} \\ \oplus \text{span} \left\{ (\xi_{i+1})^2 \lambda_i \lambda_{i+1}^{2+b} \right\}_{b=\max\{k+4-2i,0\}}^{k-3} \\ \oplus \text{span} \{B_i, B_i \lambda_{i+1}\} & \text{if } i = ne \text{ and } k \geq 2, \end{cases}$$

where ξ_{i+1} satisfies conditions (L) and B_i satisfies conditions (H) and

$$\begin{aligned} \underline{\phi}_{k+1,a}^1 &= \begin{bmatrix} x^{k+1-a} y^a & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{ne} J (C_{ai}^1 \xi_{i+1}^2 \lambda_{i+1}^{k+1} + D_{ai}^1 \xi_{i+1}^2 \lambda_i \lambda_{i+1}^k), \\ \underline{\phi}_{k+1,a}^2 &= \begin{bmatrix} 0 & 0 \\ 0 & x^{k+1-a} y^a \end{bmatrix} + \sum_{i=1}^{ne} J (C_{ai}^2 \xi_{i+1}^2 \lambda_{i+1}^{k+1} + D_{ai}^2 \xi_{i+1}^2 \lambda_i \lambda_{i+1}^k), \end{aligned}$$

where the constants $\{C_{ai}^1, D_{ai}^1, C_{ai}^2, D_{ai}^2\}$ are chosen such that $\gamma(\underline{\phi}_{k+1,a}^1), \gamma(\underline{\phi}_{k+1,a}^2) \in P_k(\partial K)$.

Let us give a more compact presentation of the space $\delta \underline{\Sigma}_{\text{fillM}}$ in Theorem 4.3 for two special cases, namely, when K is a triangle and when K is a quadrilateral.

K is a triangle. We have

$$\delta \underline{\Sigma}_{\text{fillM}} = \begin{cases} J \text{span}\{B_2, B_3\} & \text{if } k = 1, \\ J \text{span}\{B_2, B_3, B_3 \lambda_1\} & \text{if } k \geq 2. \end{cases}$$

Here, $B_i = \prod_{i=1}^3 \lambda_i \cdot \prod_{j \neq i} \frac{\lambda_j}{\lambda_j + \lambda_i}$ is the rational bubble defined in Guzmán & Neilan (2014). Notice that the filling space $\delta \underline{\Sigma}_{\text{fillM}}$ on a triangle in Guzmán & Neilan (2014) (defined for $k \geq 2$), in our notation, is

$$\delta \underline{\Sigma}_{\text{fillM}} = J \text{span}\{B_1 \lambda_2, B_2 \lambda_3, B_3 \lambda_1\},$$

which can be easily verified to satisfy the properties in Table 3.

K is a quadrilateral. We have

$$\delta \underline{\Sigma}_{\text{fillM}} = \begin{cases} J \text{ span}\{B_2, B_3, B_4, \xi_4^2, \xi_4^2 \lambda_3, \xi_4^2 \lambda_4\} & \text{if } k = 1, \\ J \text{ span}\{B_2, B_3, B_4, B_4 \lambda_1, \xi_4^2 \lambda_3, \xi_4^2 \lambda_4, \\ \xi_4^2 \lambda_3 \lambda_4, \xi_4^2 \lambda_4^2, \xi_1^2 \lambda_1^2\} & \text{if } k = 2, \\ J \text{ span}\{B_2, B_3, B_4, B_4 \lambda_1, \xi_4^2 \lambda_3 \lambda_4, \xi_4^2 \lambda_4^2, \\ \xi_4^2 \lambda_3 \lambda_4^2, \xi_4^2 \lambda_4^3, \xi_1^2 \lambda_1^2, \xi_1^2 \lambda_1^2 \lambda_4, \xi_1^2 \lambda_1^3\} & \text{if } k = 3, \\ J \text{ span}\{B_2, B_3, B_4, B_4 \lambda_1, \xi_4^2 \lambda_3 \lambda_4^{k-2}, \xi_4^2 \lambda_4^{k-1}, \\ \xi_4^2 \lambda_3 \lambda_4^{k-1}, \xi_4^2 \lambda_4^k, \xi_1^2 \lambda_1^{k-1}, \xi_1^2 \lambda_1^{k-2} \lambda_4, \xi_1^2 \lambda_1^k, \xi_1^2 \lambda_1^{k-1} \lambda_4\} & \text{if } k \geq 4. \end{cases}$$

Now, when K is a square, we can use similar spaces in Theorem 4.2 to bypass the use of composite functions ξ_4 and ξ_1 .

K is a unit square. We can choose $\delta \underline{\Sigma}_{\text{fillM}}$ as in Theorem 4.2:

$$\delta \underline{\Sigma}_{\text{fillM}} := \begin{cases} J \text{ span}\{B_2, B_3, B_4, B_4 x, \\ (xy)^2(1-x), (xy)^2(1-y), \\ (xy)^2(1-x)(1-y), \\ (x \exp^{1-y} y)^2(1-x)^2, (\exp^{1-x} xy)^2(1-y)^2\} & \text{if } k = 2, \\ J \text{ span}\{B_2, B_3, B_4, B_4 x, \\ (xy)^2(1-x)^2, (xy)^2(1-x)(1-y), (xy)^2(1-y)^2, \\ (xy)^2(1-x)^2(1-y), (xy)^2(1-x)(1-y)^2, \\ (x \exp^{1-y} y)^2(1-x)^3, (\exp^{1-x} xy)^2(1-y)^3\} & \text{if } k = 3, \\ J \text{ span}\{B_2, B_3, B_4, B_4 x, \\ (xy)^2(1-x)^{k-1}, (xy)^2(1-x)^{k-2}(1-y), \\ (xy)^2(1-y)^{k-1}, (xy)^2(1-x)(1-y)^{k-2}, \\ (xy)^2(1-x)^{k-1}(1-y), (xy)^2(1-x)(1-y)^{k-1}, \\ (x \exp^{1-y} y)^2(1-x)^k, (\exp^{1-x} xy)^2(1-y)^k\} & \text{if } k \geq 4. \end{cases}$$

5. Proof of Theorem 4.3

In this section, we prove Theorem 4.3, which is the main result of Section 4. We proceed by carrying out a systematic construction of the spaces $\delta \underline{\Sigma}_{\text{fillM}}$ for the trace space $\mathbf{M} = \mathbf{P}_k(\partial K)$ on a general polygon K . We begin by developing an algorithm that, given a counter-clockwise ordering of the ne edges of K , $\{\mathbf{e}_i\}_{i=1}^{ne}$ and an initial space $\underline{\Sigma}_g \times \mathbf{V}_g$, satisfying the inclusion properties (a) and (b), and $\mathbf{P}_1 \subset \mathbf{V}_g$, provides a space $\delta \underline{\Sigma}_{\text{fillM}}$, satisfying the properties in Table 3. We then apply it to show that the space $\delta \underline{\Sigma}_{\text{fillM}}$ in Theorem 4.3 satisfies the properties in Table 3. We end the proof by showing that the space $\delta \underline{\Sigma}_{\text{fillV}}$ in Theorem 4.3 also satisfies the related properties in Table 3.

5.1 An algorithm to construct the space $\delta \underline{\Sigma}_{\text{fillM}}$

We use the notation introduced in the previous section. For $i = 1, \dots, ne + 1$, we define $\underline{\Sigma}_{g,s,i}$ to be the divergence-free subspace of $\underline{\Sigma}_g$ with vanishing normal traces on the first $i - 1$ edges. In other words, we

set

$$\underline{\Sigma}_{g,s,i} := \{ \underline{\tau} \in \underline{\Sigma}_g : \nabla \cdot \underline{\tau} = \mathbf{0}, \underline{\tau} \mathbf{n}|_{e_j} = \mathbf{0}, 1 \leq j \leq i-1 \}, \text{ for } 1 \leq i \leq ne + 1.$$

The subspace of V_g given by $V_{grm} = \{ \mathbf{v} \in V_g : \underline{\epsilon}(\mathbf{v}) = \mathbf{0} \}$ also plays an important role in the theory of M -decompositions; see the kernels' trace decomposition in Theorem 3.2. Since $P_1 \subset V_g$, we have that $V_{grm} = RM(K)$ is just the space of rigid motions on K , which has dimension 3.

For $i = 1, \dots, ne$, we define $\gamma_i(\underline{\Sigma}) := \{ \underline{\tau} \mathbf{n}|_{e_i} : \underline{\tau} \in \underline{\Sigma} \}$ to be the normal trace of $\underline{\Sigma}$ on e_i , and $\gamma_i(V) := \{ \mathbf{v}|_{e_i} : \mathbf{v} \in V \}$ to be the trace of V on e_i . We have

$$\dim \gamma_i(V_{grm}) = \dim \gamma_i(RM(K)) = 3.$$

Now, we define the M -index for each edge.

DEFINITION 5.1 (The M -index for each edge) The M -index of the space $\underline{\Sigma}_g \times V_g$ for the i th edge e_i is the number

$$I_{M,i}(\underline{\Sigma}_g \times V_g) := \dim M(e_i) - \dim \gamma_i(\underline{\Sigma}_{g,s,i}) - \delta_{i,ne} \dim \gamma_{ne}(V_{grm}),$$

where $\delta_{i,ne}$ is the Kronecker delta.

Since $\underline{\Sigma}_g \times V_g$ satisfies the inclusion properties of an M -decomposition, we have

$$\begin{aligned} \gamma_i(\underline{\Sigma}_{g,s,i}) &\subset M(e_i) \quad \text{for all } 1 \leq i \leq ne - 1, \\ \gamma_{ne}(\underline{\Sigma}_{g,s,ne}) + \gamma_{ne}(V_{grm}) &\subset M(e_{ne}). \end{aligned}$$

Actually, the sum in the last inclusion is an $(L^2(e_{ne}))$ -orthogonal direct sum because, given any $(\underline{\tau}, \mathbf{v}) \in \underline{\Sigma}_{g,s,ne} \times V_{grm}$, we have

$$\langle \gamma_{ne} \underline{\tau}, \gamma_{ne} \mathbf{v} \rangle_{e_{ne}} = \langle \underline{\tau} \mathbf{n}, \mathbf{v} \rangle_{e_{ne}} = \langle \underline{\tau} \mathbf{n}, \mathbf{v} \rangle_{\partial K} = (\underline{\tau}, \underline{\epsilon}(\mathbf{v}))_K + (\nabla \cdot \underline{\tau}, \mathbf{v})_K = 0.$$

Using these facts, we immediately get that $I_{M,i}(\underline{\Sigma}_g \times V_g)$ is a natural number for any $1 \leq i \leq ne$.

We are now ready to state our first result.

THEOREM 5.2 Set $\delta \underline{\Sigma}_{\text{fillM}} := \bigoplus_{i=1}^{ne} \delta \underline{\Sigma}_{\text{fillM}}^i$ where

- (α) $\gamma(\delta \underline{\Sigma}_{\text{fillM}}^i) \subset M$,
- (β) $\nabla \cdot \delta \underline{\Sigma}_{\text{fillM}}^i = \{0\}$,
- ($\gamma.1$) $\gamma_j(\delta \underline{\Sigma}_{\text{fillM}}^i) = \{0\}$, for $1 \leq j \leq i-1$,
- ($\gamma.2$) $\gamma_i(\underline{\Sigma}_{g,s,i}) \cap \gamma_i(\delta \underline{\Sigma}_{\text{fillM}}^i) = \{0\}$,
- (δ) $\dim \delta \underline{\Sigma}_{\text{fillM}}^i = \dim \gamma_i(\delta \underline{\Sigma}_{\text{fillM}}^i) = I_{M,i}(\underline{\Sigma}_g \times V_g)$.

Then $\delta \underline{\Sigma}_{\text{fillM}}$ satisfies the properties in Table 3, that is,

- (a) $\gamma \delta \underline{\Sigma}_{\text{fillM}} \subset \mathbf{M}$,
- (b) $\nabla \cdot \delta \underline{\Sigma}_{\text{fillM}} = \{0\}$,
- (c) $\gamma \underline{\Sigma}_{g,s,1} \cap \gamma \delta \underline{\Sigma}_{\text{fillM}} = \{0\}$,
- (d) $\dim \delta \underline{\Sigma}_{\text{fillM}} = \dim \gamma \delta \underline{\Sigma}_{\text{fillM}} = I_M(\underline{\Sigma}_g \times V_g)$.

Proof. Properties (a), (b) and (c) follow directly from properties (α) , (β) and (γ) , respectively. It remains to prove property (d). But we have

$$\begin{aligned} \dim \delta \underline{\Sigma}_{\text{fillM}} &= \sum_{i=1}^{ne} \dim \delta \underline{\Sigma}_{\text{fillM}}^i = \sum_{i=1}^{ne} I_{M,i}(\underline{\Sigma}_g \times V_g) \\ &= \sum_{i=1}^{ne} \dim \gamma_i \delta \underline{\Sigma}_{\text{fillM}}^i = \dim \gamma \delta \underline{\Sigma}_{\text{fillM}}. \end{aligned}$$

Now, by the definition of $I_{M,i}(\underline{\Sigma}_g \times V_g)$, we get

$$\begin{aligned} \dim \delta \underline{\Sigma}_{\text{fillM}} &= \sum_{i=1}^{ne} (\dim \mathbf{M}(\mathbf{e}_i) - \dim \gamma_i(\underline{\Sigma}_{g,s,i}) - \delta_{i,ne} \dim \gamma_{ne}(V_{g_{rm}})) \\ &= \dim \mathbf{M} - \sum_{i=1}^{ne} \dim \gamma_i(\underline{\Sigma}_{g,s,i}) - \dim \gamma_{ne}(V_{g_{rm}}) \\ &= \dim \mathbf{M} - \sum_{i=1}^{ne} (\dim \underline{\Sigma}_{g,s,i} - \dim \underline{\Sigma}_{g,s,i+1}) - \dim \gamma_{ne}(V_{g_{rm}}) \\ &= \dim \mathbf{M} - (\dim \underline{\Sigma}_{g,s,1} - \dim \underline{\Sigma}_{g,s,ne+1}) - \dim \gamma_{ne}(V_{g_{rm}}). \end{aligned}$$

Finally, since

$$\begin{aligned} \underline{\Sigma}_{g,s,1} &:= \{\underline{\tau} \in \underline{\Sigma}_g : \nabla \cdot \underline{\tau} = \mathbf{0}\}, \\ \underline{\Sigma}_{g,s,ne+1} &:= \{\underline{\tau} \in \underline{\Sigma}_g : \nabla \cdot \underline{\tau} = \mathbf{0}, \underline{\tau} \mathbf{n}|_{\partial K} = \mathbf{0}\}, \\ V_{g_{rm}} &= \{\mathbf{v} \in V_g : \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\}, \end{aligned}$$

we get

$$\begin{aligned} \dim \delta \underline{\Sigma}_{\text{fillM}} &= \dim \mathbf{M} - \dim \{\underline{\tau} \mathbf{n}|_{\partial K} : \underline{\tau} \in \underline{\Sigma}_g, \nabla \cdot \underline{\tau} = \mathbf{0}\} \\ &\quad - \dim \{\mathbf{v}|_{\partial K} : \mathbf{v} \in V_g, \underline{\epsilon}(\mathbf{v}) = \mathbf{0}\} \\ &= I_M(\underline{\Sigma}_g \times V_g). \end{aligned}$$

This completes the proof. \square

On the basis of this result, we can see that the following algorithm provides a practical construction of the filling space $\delta \underline{\Sigma}_{\text{fillM}}$.

Algorithm PC Construction of $\delta \underline{\Sigma}_{\text{fillM}}$, satisfying properties (α) – (δ) of Theorem 5.2.

Input: A counter-clockwise ordering of the ne edges of the polygon K , $\{\mathbf{e}_i\}_{i=1}^{ne}$.

Input: The space of traces \mathbf{M} .

Input: A space $\underline{\Sigma}_g \times \mathbf{V}_g$, satisfying the inclusion properties of an \mathbf{M} -decomposition.

Output: The space $\delta \underline{\Sigma}_{\text{fillM}}$.

For each $i = 1, \dots, ne$,

(1) Find the auxiliary spaces $\underline{\Sigma}_{g,s,i}$.

(2) Find an $I_{M,i}(\underline{\Sigma}_g \times \mathbf{V}_g)$ -dimensional complement space $C_{M,i}$ on edge \mathbf{e}_i :

$$\gamma_i(\underline{\Sigma}_{g,s,i}) \oplus C_{M,i} = \tilde{\mathbf{M}}(\mathbf{e}_i),$$

here $\tilde{\mathbf{M}}(\mathbf{e}_i) = \mathbf{M}(\mathbf{e}_i)$ if $i < ne$, and $\tilde{\mathbf{M}}(\mathbf{e}_{ne}) = \gamma_{ne}(\mathbf{V}_{g_{rm}})^\perp$ is the subspace of $\mathbf{M}(\mathbf{e}_{ne})$ that is $L^2(\mathbf{e}_{ne})$ -orthogonal to $\gamma_{ne}(\mathbf{V}_{g_{rm}})$.

(3) Find an $I_{M,i}(\underline{\Sigma}_g \times \mathbf{V}_g)$ -dimensional, divergence-free filling space $\delta \underline{\Sigma}_{\text{fillM}}^i$ on K :

$$(3.1) \quad \gamma_j(\delta \underline{\Sigma}_{\text{fillM}}^i) = \{\mathbf{0}\}, \quad \text{for } 1 \leq j \leq i - 1,$$

$$(3.2) \quad \gamma_i(\delta \underline{\Sigma}_{\text{fillM}}^i) = C_{M,i},$$

$$(3.3) \quad \gamma_j(\delta \underline{\Sigma}_{\text{fillM}}^i) \subset \mathbf{M}(\mathbf{e}_j), \quad \text{for } i + 1 \leq j \leq ne.$$

return $\delta \underline{\Sigma}_{\text{fillM}} := \bigoplus_{i=1}^{ne} \delta \underline{\Sigma}_{\text{fillM}}^i$.

Now, we apply Algorithm PC to prove the first part of Theorem 4.3, that is, the space $\delta \underline{\Sigma}_{\text{fillM}}$ satisfies the properties in Table 3. Note that in this case, we have $\mathbf{M} = \mathbf{P}_k(\partial K)$ and $\underline{\Sigma}_g \times \mathbf{V}_g = \underline{\mathbf{P}}_k^s \times \mathbf{P}_k$ with $k \geq 1$. We proceed in three steps as follows.

(1). **Finding the spaces $\underline{\Sigma}_{g,s,i}$.** We begin by characterizing the spaces $\underline{\Sigma}_{g,s,i}$.

PROPOSITION 5.3 We have that

$$\underline{\Sigma}_{g,s,i} = J \Phi_i \quad 1 \leq i \leq ne + 1,$$

where $\Phi_i := \{b_{i-1}^2 \phi_i : \phi_i \in P_{k+4-2i}(K)\}$. Here, $b_0 = 1$, and $b_\ell := \prod_{j=1}^\ell \lambda_j$ for $\ell \geq 1$.

To prove this result, we need to characterize the kernel of the operator $\gamma_i J$.

LEMMA 5.4 We have that $\gamma_i(J \phi) = \mathbf{0}$ if and only if $\nabla \phi|_{\mathbf{e}_i} \in \mathbf{P}_0(\mathbf{e}_i)$, for any $\phi \in H^2(K)$ and any edge \mathbf{e}_i of the element K .

Proof. The result follows from the fact that $\gamma_i(J\phi) = \frac{\partial \mathbf{curl} \phi}{\partial \mathbf{t}_i}$, where $\frac{\partial}{\partial \mathbf{t}_i}$ is the tangential derivative on the edge \mathbf{e}_i . □

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. Since $\underline{\Sigma}_g = \underline{P}_k^s$, it is easy to show that

$$J\Phi_i \subset \underline{\Sigma}_{g,s,i} \subset JP_{k+2}.$$

Since $\Phi_1 = P_{k+2}$, the reverse inclusion, $\underline{\Sigma}_{g,s,i} \subset J\Phi_i$, is true for $i = 1$. Let us prove that the reverse inclusion also holds for $i \geq 2$. Let $\underline{\tau} = J\phi \in \underline{\Sigma}_{g,s,i}$ with $\phi \in P_{k+2}$. We have $\gamma_j(J\phi) = \mathbf{0}$ for $1 \leq j \leq i - 1$. By Lemma 5.4, $\nabla \phi|_{\mathbf{e}_j} \in P_0(\mathbf{e}_j)$ for $1 \leq j \leq i - 1$. Since ϕ is defined up to a linear function, we can assume $\nabla \phi|_{\mathbf{e}_1} = \mathbf{0}$, hence λ_1^2 divides ϕ . This immediately implies $\nabla \phi|_{\mathbf{e}_j} = \mathbf{0}$ for $1 \leq j \leq i - 1$, and so b_{i-1}^2 divides ϕ . This completes the proof. □

(2). Finding the complement spaces $C_{M,i}$.

By definition, see Algorithm PC, the space $C_{M,i}$ is any subspace of $\tilde{M}(\mathbf{e}_i)$ such that $\gamma_i(\underline{\Sigma}_{g,s,i}) \oplus C_{M,i} = \tilde{M}(\mathbf{e}_i)$. Thus, to find a choice of $C_{M,i}$, which is not necessarily unique, we first need to characterize $\gamma_i(\underline{\Sigma}_{g,s,i})$. We do that in the following corollary of the previous proposition.

COROLLARY 5.5 We have, for $1 \leq i \leq ne$,

$$\begin{aligned} \gamma_i(\underline{\Sigma}_{g,s,i}) &= \text{span} \left\{ \gamma_i \left(J(b_{i-1}^2 \lambda_{i+1}^a) \right) \right\}_{a=2\delta_{1,i}}^{k+4-2i} \oplus \text{span} \left\{ \gamma_i \left(J(b_{i-1}^2 \lambda_i \lambda_{i+1}^a) \right) \right\}_{a=\delta_{1,i}}^{k+3-2i}, \\ \dim \gamma_i(\underline{\Sigma}_{g,s,i}) &= \dim P_{k+4-2i}(\mathbf{e}_i) + \dim P_{k+3-2i}(\mathbf{e}_i) - 3\delta_{1,i}, \\ I_{M,i}(\underline{\Sigma}_g \times \mathbf{V}_g) &= \min(k+1, 2i-4) + \min(k+1, 2i-3) + 3\delta_{1,i} - 3\delta_{ne,i}. \end{aligned}$$

Here we use the convention that, for any negative integer m , $\dim P_m = 0$.

Proof. The first identity follows from the definition of the auxiliary space $\underline{\Sigma}_{g,s,i}$ and from the fact that $\gamma_i(J(b_{i-1}^2 \lambda_i^b \lambda_{i+1}^a)) = 0$ when $b \geq 2$.

Let us now prove the second identity. By construction,

$$\dim \gamma_i(\underline{\Sigma}_{g,s,i}) = \dim \underline{\Sigma}_{g,s,i} - \dim \underline{\Sigma}_{g,s,i+1},$$

and since, by Proposition 5.3, $\dim \underline{\Sigma}_{g,s,i} = \dim P_{k+4-2i}(K) - 3\delta_{1,i}$, we get that

$$\begin{aligned} \dim \gamma_i(\underline{\Sigma}_{g,s,i}) &= (\dim P_{k+4-2i}(K) - 3\delta_{1,i}) - (\dim P_{k+2-2i}(K) - 3\delta_{1,i+1}) \\ &= \dim P_{k+4-2i}(\mathbf{e}_i) + \dim P_{k+3-2i}(\mathbf{e}_i) - 3\delta_{1,i}. \end{aligned}$$

It remains to prove the last identity. By the definition of $I_{M,i}(\underline{\Sigma}_g \times V_g)$, we have

$$\begin{aligned} I_{M,i}(\underline{\Sigma}_g \times V_g) &= \dim M(\mathbf{e}_i) - \dim \gamma_i(\underline{\Sigma}_{g,s,i}) - \delta_{i,ne} \dim \gamma_{ne}(V_{g_{sm}}) \\ &= 2 \dim P_k(\mathbf{e}_i) - \left(\dim P_{k+4-2i}(\mathbf{e}_i) \right. \\ &\quad \left. + \dim P_{k+3-2i}(\mathbf{e}_i) - 3 \delta_{1,i} \right) - 3 \delta_{i,ne} \\ &= (\dim P_k(\mathbf{e}_i) - \dim P_{k+4-2i}(\mathbf{e}_i)) \\ &\quad + (\dim P_k(\mathbf{e}_i) - \dim P_{k+3-2i}(\mathbf{e}_i)) + 3 \delta_{1,i} - 3 \delta_{i,ne} \end{aligned}$$

and the result follows. This completes the proof. □

We now give a particular choice of the trace space $C_{M,i}$.

PROPOSITION 5.6 Set, for $i = 1, \dots, ne$,

$$C_{M,i} = \begin{cases} \{0\} & \text{if } i = 1, \\ \text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_{i+1}^b) \right) \right\}_{b=\max\{k+5-2i,0\}}^k \\ \quad \oplus \text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_i \lambda_{i+1}^b) \right) \right\}_{b=\max\{k+4-2i,0\}}^{k-1} \\ \quad \oplus \text{span} \left\{ \gamma_i \left(J(\eta_i \lambda_i \lambda_{i+1}) \right) \right\} & \text{if } 2 \leq i \leq ne - 1, \\ \text{span} \left\{ \gamma_i \left(J(\eta_i \lambda_i \lambda_{i+1}) \right) \right\} & \text{if } i = ne \text{ and } k = 1, \\ \text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_{i+1}^{2+b}) \right) \right\}_{b=\max\{k+5-2i,0\}}^{k-2} \\ \quad \oplus \text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_i \lambda_{i+1}^{2+b}) \right) \right\}_{b=\max\{k+4-2i,0\}}^{k-3} \\ \quad \oplus \text{span} \left\{ \gamma_i \left(J(\eta_i \lambda_i \lambda_{i+1}), \gamma_i \left(J(\eta_i \lambda_i \lambda_{i+1}^2) \right) \right) \right\} & \text{if } i = ne \text{ and } k \geq 2, \end{cases}$$

where η_i is any linear function on \mathbb{R}^2 such that $\eta_i(v_i) = 0$ and $\eta_i(v_{i+1}) \neq 0$.

Then, for $i = 1, \dots, ne$, the space $C_{M,i}$ of functions defined on the edge \mathbf{e}_i has dimension $I_{M,i}(\underline{\Sigma}_g \times V_g)$ and satisfies the identity

$$\gamma_i(\underline{\Sigma}_{g,s,i}) \oplus C_{M,i} = \tilde{M}(\mathbf{e}_i).$$

Proof. Since η_i is a linear function, it is easy to check that $\dim C_{M,i} = I_{M,i}(\underline{\Sigma}_g \times V_g)$ and $C_{M,i} \subset \tilde{M}(\mathbf{e}_i)$. We are left to show that $\gamma_i(\underline{\Sigma}_{g,s,i}) \cap C_{M,i} = \{0\}$. We prove this result for the case $2 \leq i \leq ne - 1$ and $k \geq 2i - 3$. The other cases are similar and simpler.

To show $\gamma_i(\underline{\Sigma}_{g,s,i}) \cap C_{M,i} = \{0\}$, we only need to prove the linear independence of the following five sets

$$\begin{aligned} &\text{span} \left\{ \gamma_i \left(J(b_{i-1}^2 \lambda_{i+1}^a) \right) \right\}_{a=0}^{k+4-2i}, & \text{span} \left\{ \gamma_i \left(J(b_{i-1}^2 \lambda_i \lambda_{i+1}^a) \right) \right\}_{a=0}^{k+3-2i}, \\ &\text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_{i+1}^b) \right) \right\}_{b=k+5-2i}^k, & \text{span} \left\{ \gamma_i \left(J(\eta_i^2 \lambda_i \lambda_{i+1}^b) \right) \right\}_{b=k+4-2i}^{k-1}, \\ &\text{span} \left\{ \gamma_i \left(J(\eta_i \lambda_i \lambda_{i+1}) \right) \right\}. \end{aligned}$$

Note that the first two sets span a set of bases for $\gamma_i(\underline{\Sigma}_{g,s,i})$ and the last three sets span a set of bases for $C_{M,i}$. Let us assume that there exists constants $\{C_a\}_{a=0}^{k+4-2i}$, $\{D_a\}_{a=0}^{k+3-2i}$, $\{E_b\}_{b=k+5-2i}^k$, $\{F_b\}_{b=k+4-2i}^{k-1}$, and G such that

$$\gamma_i(J\phi) = 0,$$

where

$$\begin{aligned} \phi := & \sum_{a=0}^{k+4-2i} C_a b_{i-1}^2 \lambda_{i+1}^a + \sum_{a=0}^{k+3-2i} D_a b_{i-1}^2 \lambda_i \lambda_{i+1}^a \\ & + \sum_{b=k+5-2i}^k E_b \eta_i^2 \lambda_{i+1}^b + \sum_{b=k+4-2i}^{k-1} F_b \eta_i^2 \lambda_i \lambda_{i+1}^b + G \eta_i \lambda_i \lambda_{i+1}. \end{aligned}$$

By Lemma 5.4, this implies that $\nabla \phi|_{\mathbf{e}_i} \in \mathbf{P}_0(\mathbf{e}_i)$ and so that $\phi|_{\mathbf{e}_i} \in P_1(\mathbf{e}_i)$. As a consequence,

$$\left(\sum_{a=0}^{k+4-2i} C_a b_{i-1}^2 \lambda_{i+1}^a + \sum_{b=k+5-2i}^k E_b \eta_i^2 \lambda_{i+1}^b \right) \Big|_{\mathbf{e}_i} \in P_1(\mathbf{e}_i),$$

because $\lambda_i = 0$ on \mathbf{e}_i . Since $b_{i-1}(v_i) = 0$ (because $i \geq 2$) and since $\eta_i(v_i) = 0$, we have $\eta_i|_{\mathbf{e}_i}$ proportional to $\lambda_{i-1}|_{\mathbf{e}_i}$ and we get that

$$\left(\sum_{a=0}^{k+4-2i} C_a b_{i-1}^2 \lambda_{i+1}^a + \sum_{b=k+5-2i}^k E_b \eta_i^2 \lambda_{i+1}^b \right) \Big|_{\mathbf{e}_i} = 0.$$

Now, evaluating the expression at the node $v_{i+1} = \mathbf{e}_i \cap \mathbf{e}_{i+1}$, we get $C_0 = 0$ since $b_{i-1}(v_{i+1}) \neq 0$, $\eta_i(v_{i+1}) \neq 0$ and $\lambda_{i+1}(v_{i+1}) = 0$. Then, dividing it by λ_{i+1} and evaluating the resulting expression again at $v_{i+1} = \mathbf{e}_i \cap \mathbf{e}_{i+1}$, we get $C_1 = 0$. Similarly, we get $C_a = 0$ for $a = 2, \dots, k+4-2i$, and $E_b = 0$ for $b = k+5-2i, \dots, k$. This implies that

$$\phi = \sum_{a=0}^{k+3-2i} D_a b_{i-1}^2 \lambda_i \lambda_{i+1}^a + \sum_{b=k+4-2i}^{k-1} F_b \eta_i^2 \lambda_i \lambda_{i+1}^b + G \eta_i \lambda_i \lambda_{i+1},$$

and so that $\nabla \phi|_{\mathbf{e}_i} = \varphi \nabla \lambda_i$, where

$$\varphi := \sum_{a=0}^{k+3-2i} D_a b_{i-1}^2 \lambda_{i+1}^a + \sum_{b=k+4-2i}^{k-1} F_b \eta_i^2 \lambda_{i+1}^b + G \eta_i \lambda_{i+1}.$$

Since $\varphi|_{\mathbf{e}_i} \in P_0(\mathbf{e}_i)$ and $\varphi(v_i) = 0$, because $b_{i-1}(v_i) = 0$ and $\eta_i(v_{i+1}) = 0$, we conclude that $\varphi|_{\mathbf{e}_i} = 0$, that is,

$$\left(\sum_{a=0}^{k+3-2i} D_a b_{i-1}^2 \lambda_{i+1}^a + \sum_{b=k+4-2i}^{k-1} F_b \eta_i^2 \lambda_{i+1}^b + G \eta_i \lambda_{i+1} \right) \Big|_{\mathbf{e}_i} = 0.$$

Since $\eta_i(v_{i+1}) = 0$, $\eta_i = \alpha \lambda_{i-1}|_{\mathbf{e}_i}$ for some number α . Then, dividing the above expression λ_{i-1} and evaluating the resulting expression at v_i , we obtain that $G = 0$. Finally, we can get that $D_a = 0$ and $F_b = 0$ by consecutively evaluating the expression at v_{i+1} and dividing it by λ_{i+1} . This completes the proof. \square

(3). Finding the filling spaces $\delta \underline{\Sigma}_{\text{fillM}}^i$.

Note that the definition of Ψ_i in Theorem 4.3 is obtained from our choice of the space $C_{M,i}$ by formally replacing, in the definition of the basis of $C_{M,i}$, η_i^2 by ξ_{i+1}^2 and then $\eta_i \lambda_i \lambda_{i+1}$ by B_i . The fact that the choice $\delta \underline{\Sigma}_{\text{fillM}}^i := J \Psi_i$ does satisfy the conditions (3.1), (3.2) and (3.3) of Algorithm PC follows immediately from the following results.

LEMMA 5.7 Let ψ be any function in $P_k(K)$. Then, we have that

- (i) $\gamma_j(J(\xi_{i+1}^2 \psi)) = \mathbf{0}$ for $j = 1, \dots, i - 1$ provided $2 \leq i < ne$, or $i = ne$ and ψ is divisible by λ_1^2 .
- (ii) $\gamma_i(J(\xi_{i+1}^2 \psi)) = \gamma_j(J(\eta_i^2 \psi))$ for some linear function η_i such that $\eta_i(v_i) = 0$.
- (iii) $\gamma_j(J(\xi_{i+1}^2 \psi)) \in P_k(\mathbf{e}_j)$ for $j = i + 1, \dots, ne$.

LEMMA 5.8 Let ψ be any function in $H^2(K)$. Then we have that

- (i) $\gamma_i(J(B_i \psi)) = \gamma_j(J(\alpha \eta_i \lambda_i \lambda_{i+1} \psi))$ for some constant α and some linear function η_i such that $\eta_i(v_i) = 0$.
- (ii) $\gamma_j(J(B_i \psi)) = \mathbf{0}$ for $1 \leq j \leq ne$ and $j \neq i$.

Proof of Lemma 5.7. Let us begin by proving (i). By properties (L.1) and (L.2) in Section 4.1, $\xi_{i+1} = 0$ on \mathbf{e}_j for $j = 1, \dots, i - 1$ if $i < ne$. Since this implies that $\nabla(\xi_{i+1}^2 S)|_{\mathbf{e}_j} = 0$, property (i) follows from Lemma 5.4 for $i < ne$. If $i = ne$, we have, by properties (L.1) and (L.2), that $\xi_{i+1} = 0$ on \mathbf{e}_j for $j = 2, \dots, i - 1$ and property (i) follows in the same manner. It remains to consider the case $i = ne$ and $j = 1$. In this case, on \mathbf{e}_1 , ξ_{ne+1} is different from zero. As a consequence, property (i) holds if S is divisible by λ_1^2 . This proves property (i).

Let us now prove property (ii). We can take η_i such that $\eta_i|_{\mathbf{e}_i} = \xi_{i+1}|_{\mathbf{e}_i}$ and $\frac{\partial}{\partial n_i} \eta_i = \frac{\partial}{\partial n_i} \xi_{i+1}$. This is possible by properties (L). Then, we have

$$\left(\nabla((\xi_{i+1}^2 - \eta_i^2) \psi) \right) \Big|_{\mathbf{e}_i} = \left(\psi (\xi_{i+1} + \eta_i) \nabla(\xi_{i+1} - \eta_i) \right) \Big|_{\mathbf{e}_i} = 0.$$

Finally, property (iii) follows by simple manipulations and using properties (L.1) and (L.3). This completes the proof of Lemma 5.7. \square

Proof of Lemma 5.8. We first prove property (i). Since $\lambda_i = 0$ on \mathbf{e}_i and $B_i = 0$ on \mathbf{e}_i , by property (H.1) in Section 4.1, we have that

$$\begin{aligned} \nabla(B_i\psi - \alpha\eta_i\lambda_i\lambda_{i+1})|_{\mathbf{e}_i} &= \mathbf{n}_i \left(\psi \left(\frac{\partial}{\partial \mathbf{n}_i} B_i - \alpha \eta_i \lambda_{i+1} \frac{\partial}{\partial \mathbf{n}_i} \lambda_i \right) \right) \Big|_{\mathbf{e}_i} \\ &= \mathbf{n}_i \left(\psi \left(\lambda_{i-1} \lambda_{i+1} - \alpha \eta_i \lambda_{i+1} \frac{\partial}{\partial \mathbf{n}_i} \lambda_i \right) \right) \Big|_{\mathbf{e}_i} \quad \text{by (H.3),} \\ &= \mathbf{n}_i \left(\psi \lambda_{i-1} \lambda_{i+1} \left(1 - \alpha \eta_i \nu_{i+1} \frac{\partial}{\partial \mathbf{n}_i} \lambda_i \right) \right) \Big|_{\mathbf{e}_i} \\ &= 0, \end{aligned}$$

if we take α as the unique solution of the equation $\alpha \eta_i \nu_{i+1} \frac{\partial}{\partial \mathbf{n}_i} \lambda_i = 1$ on the edge \mathbf{e}_i . Property (i) now follows from Lemma 5.4.

It remains to prove property (ii). We have, by property (H.1) of B_i , that

$$\nabla(B_i\psi)|_{\mathbf{e}_j} = \mathbf{n}_j \left(\psi \frac{\partial}{\partial \mathbf{n}_i} B_i \right) \Big|_{\mathbf{e}_j} = 0$$

by property (H.2). Property (i) now follows from Lemma 5.4. This completes the proof of Lemma 5.8. \square

With these results, we conclude that the choice $\delta \underline{\Sigma}_{\text{fillM}}$ indeed satisfies the related properties in Table 3. **The computation of the dimension of $\delta \underline{\Sigma}_{\text{fillM}}$.** Now, we compute the dimension of $\delta \underline{\Sigma}_{\text{fillM}}$. We have

$$\begin{aligned} \dim \delta \underline{\Sigma}_{\text{fillM}} &= \sum_{i=1}^{ne} \dim \delta \underline{\Sigma}_{\text{fillM}}^i = \sum_{i=1}^{ne} I_{M,i}(\underline{\Sigma}_g \times V_g) \\ &= \sum_{i=1}^{ne} (\min\{k + 1, 2i - 4\} + \min\{k + 1, 2i - 3\} + 3 \delta_{1,i} - 3 \delta_{ne,i}) \\ &= \sum_{i=1}^{ne} (\min\{k + 1, 2i - 4\} + \min\{k + 1, 2i - 3\} + 3 \delta_{1,i} - 3 \delta_{ne,i}), \end{aligned}$$

by Corollary 5.5. Finally, simple algebraic manipulations give that

$$\begin{aligned} \dim \delta \underline{\Sigma}_{\text{fillM}} &= \begin{cases} 2(k + 1) ne - \frac{(k+3)(k+4)}{2} & \text{if } k \leq 2ne - 5, \\ (2ne - 5) ne & \text{if } k \geq 2ne - 4, \end{cases} \\ &= 2(\theta + 1)ne - \frac{1}{2}(\theta + 3)(\theta + 4), \end{aligned}$$

where $\theta := \min\{k, 2ne - 4\}$.

Finally, we conclude the proof of Theorem 4.3 by proving that the choices of $\delta \underline{\Sigma}_{\text{fillV}}$ also satisfy the related properties in Table 3. Since $\underline{\Sigma}_g \times \mathbf{V}_g = \underline{\mathbf{P}}_k^s \times \mathbf{P}_k$, we have $\nabla \cdot \underline{\Sigma}_g = \mathbf{P}_{k-1}$. Hence, we have

$$I_S(\underline{\Sigma}_g \times \mathbf{V}_g) = 2(k+1).$$

It is then elementary to prove that the choice of $\delta \underline{\Sigma}_{\text{fillV}}$ satisfies the properties in Table 3. This completes the proof of Theorem 4.3.

6. Numerical results

In this section, we present numerical results validating the theory in the case of triangular elements. For simplicity, the material is chosen to be isotropic (1.2). Recall that the Lamé modules λ and μ have the following form in terms of Young's modules E and Poisson's ratio ν :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

For comparison, we also present the numerical results with the HDG method in Soon *et al.* (2009) and Fu *et al.* (2015). The method in Soon *et al.* (2009), see also Fu *et al.* (2015), uses the following local spaces:

$$\underline{\Sigma}(K) \times \mathbf{V}(K) \times \mathbf{M}(\partial K) = \mathbf{P}_k(K; \mathbb{S}) \times \mathbf{P}_k(K) \times \mathbf{P}_k(\partial K).$$

This space $\underline{\Sigma}(K) \times \mathbf{V}(K)$ does *not* admit an $\mathbf{M}(\partial K)$ -decomposition. We denote this method by HDG_k.

Our method on triangles enriches the local stress space on each element with a rational function space $\delta \underline{\Sigma}_{\text{fillM}}$ that has dimension 2 if $k = 1$ and dimension 3 if $k \geq 2$; see the discussion following Theorem 4.3. We denote this method by HDG_k-M.

For the postprocessing \mathbf{u}_h^* , we take $\mathbf{V}^*(K) := \mathbf{P}_{k+1}(K)$ and $\tilde{\mathbf{V}}^*(K) := \mathbf{P}_0(K)$:

$$\begin{aligned} (\nabla \mathbf{u}_h^*, \nabla \mathbf{w})_K &= -(\mathbf{u}_h, \Delta \mathbf{w})_K + (\widehat{\mathbf{u}}_h, \nabla \mathbf{w} \mathbf{n})_{\partial K} \quad \forall \mathbf{w} \in \mathbf{V}^*(K), \\ (\mathbf{u}_h^*, \mathbf{r})_K &= (\mathbf{u}_h, \mathbf{r})_K \quad \forall \mathbf{r} \in \mathbf{P}_0(K). \end{aligned}$$

We present the same two test problems considered in Fu *et al.* (2015). The first test problem is obtained by taking $E = 1$, $\nu = 0.3$, and choosing data so that the exact solution for the displacement is $u_1(x, y) = 10(y - y^2)\sin(\pi x)(1 - x)(1 - \frac{y}{2})$ and $u_2(x, y) = 0$ on the domain Ω . The second test problem is obtained by taking $E = 3$ and choosing data so that the exact solution is $u_1(x, y) = -x^2(x - 1)^2y(y - 1)(2y - 1)$ and $u_2(x, y) = -u_1(y, x)$. For the second problem, we also vary the Poisson ratio ν from 0.3 to 0.499999 to show that the methods are free of volumetric locking.

We carry out our experiments on uniform triangular meshes obtained by discretizing the domain $\Omega = (0, 1) \times (0, 1)$ with triangles of side 2^{-l} as depicted in Fig. 2. And we fix the polynomial degree to be either $k = 1$ or $k = 2$.

For both methods, we choose the stabilization function $\alpha = Id$.

The history of convergence for the first test is displayed in Table 4, and the one for the second test in Table 5. The orders of convergence of the HDG_k-M method match the theory developed in Section 2 very well. In particular, we get the optimal orders of convergence in the L^2 -error for \mathbf{u}_h , $\underline{\sigma}_h$ and \mathbf{u}_h^* , that is, $k + 1$, $k + 1$ and $k + 2$, respectively. We also see clearly the superior performance of HDG_k-M over

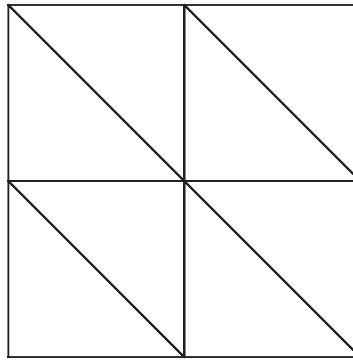


FIG. 2. Example of meshes with $h = 2^{-1}$.

TABLE 4 History of convergence for the first test

Mesh k	l	$\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{T}_h}$		$\ \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h^*\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{T}_h}$		$\ \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h^*\ _{\mathcal{T}_h}$	
		Error	Order	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
HDG $_k$													
	3	2.10E-2	—	6.00E-2	—	4.25E-3	—	2.06E-2	—	5.35E-2	—	1.89E-3	—
	4	5.30E-3	1.99	1.59E-2	1.91	1.20E-3	1.83	5.21E-3	1.98	1.40E-2	1.94	3.48E-4	2.44
1	5	1.33E-3	2.00	4.22E-3	1.91	3.27E-4	1.88	1.31E-3	1.99	3.61E-3	1.95	5.49E-5	2.67
	6	3.32E-4	2.00	1.13E-3	1.90	8.68E-5	1.91	3.29E-4	1.99	9.20E-4	1.97	7.73E-6	2.83
	7	8.31E-5	2.00	3.07E-4	1.88	2.26E-5	1.94	8.26E-5	2.00	2.32E-4	1.98	1.03E-6	2.91
HDG $_k$ -M													
	3	1.25E-3	—	3.65E-3	—	1.59E-4	—	1.25E-3	—	3.30E-3	—	6.22E-5	—
	4	1.57E-4	2.99	4.71E-4	2.95	2.30E-5	2.79	1.58E-4	2.98	4.17E-4	2.99	4.52E-6	3.78
2	5	1.97E-5	3.00	6.06E-5	2.96	3.19E-6	2.85	1.99E-5	2.99	5.24E-5	2.99	3.07E-7	3.88
	6	2.46E-6	3.00	7.82E-6	2.95	4.25E-7	2.91	2.49E-6	3.00	6.57E-6	3.00	2.01E-8	3.94
	7	3.08E-7	3.00	1.02E-6	2.94	5.53E-8	2.94	3.12E-7	3.00	8.22E-7	3.00	1.28E-9	3.97

HDG $_k$ for the stress error $\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathcal{T}_h}$ as well as for the postprocessed displacement error $\|\mathbf{u} - \mathbf{u}_h^*\|_{\mathcal{T}_h}$. Finally, note that, since the global equation for both methods have *exactly* the same dimension and sparsity structure, the HDG methods whose spaces admit \mathbf{M} -decompositions perform significantly better.

7. Concluding remarks

We extended the use of \mathbf{M} -decomposition for the devising of new superconvergent methods for the pure diffusion problems (Cockburn *et al.*, 2017b) to the linear elasticity with symmetric approximate stresses. It provides a *simple* a priori *error analysis* of HDG methods for linear elasticity with strong symmetry and gives us guidelines for the devising of new superconvergent methods.

We applied the concept of an \mathbf{M} -decomposition to construct new HDG and (hybridized) mixed methods with symmetric approximate stresses for linear elasticity in two-space dimensions. Numerical results on triangular meshes confirm the theoretical convergence properties.

Let us emphasize the fact that it is not necessary to use the compliance matrix for formulating the methods. The same results obtained here do hold for methods formulated in terms of the standard constitutive tensor \mathcal{A}^{-1} , like the one in Soon (2008), Soon *et al.* (2009) and Fu *et al.* (2015), for example.

TABLE 5 *History of convergence for the second test*

k	Mesh l	$\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{T}_h}$		$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h^*\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h\ _{\mathcal{T}_h}$		$\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ _{\mathcal{T}_h}$		$\ \mathbf{u} - \mathbf{u}_h^*\ _{\mathcal{T}_h}$	
		Error	Order	Error	Order	Error	Order	Error	Order	Error	Order	Error	Order
HDG $_k$, $\nu = 0.3$													
HDG $_k$ -M, $\nu = 0.3$													
1	3	5.23E-4	—	3.62E-3	—	1.32E-4	—	4.81E-4	—	1.98E-3	—	2.53E-5	—
	4	1.39E-4	1.91	1.28E-3	1.51	4.46E-5	1.57	1.22E-4	1.98	5.31E-4	1.90	4.66E-6	2.44
	5	3.66E-5	1.93	4.49E-4	1.51	1.47E-5	1.60	3.06E-5	2.00	1.39E-4	1.94	7.82E-7	2.57
	6	9.58E-6	1.93	1.56E-4	1.53	4.62E-6	1.67	7.66E-6	2.00	3.57E-5	1.96	1.16E-7	2.75
	7	2.49E-6	1.94	5.27E-5	1.57	1.39E-6	1.73	1.91E-6	2.00	9.06E-6	1.98	1.59E-8	2.87
2	3	3.46E-5	—	3.33E-4	—	8.84E-6	—	3.38E-5	—	2.02E-4	—	1.77E-6	—
	4	4.58E-6	2.92	5.01E-5	2.72	1.47E-6	2.59	4.37E-6	2.95	2.68E-5	2.91	1.43E-7	3.63
	5	5.94E-7	2.95	7.63E-6	2.72	2.36E-7	2.64	5.51E-7	2.99	3.42E-6	2.97	1.02E-8	3.81
	6	7.64E-8	2.96	1.17E-6	2.70	3.54E-8	2.74	6.92E-8	2.99	4.31E-7	2.99	6.82E-10	3.90
	7	9.76E-9	2.97	1.80E-7	2.71	5.02E-9	2.82	8.66E-9	3.00	5.40E-8	3.00	4.41E-11	3.95
HDG $_k$, $\nu = 0.499$													
HDG $_k$ -M, $\nu = 0.499$													
1	3	4.49E-4	—	3.30E-3	—	1.01E-4	—	4.14E-4	—	2.75E-3	—	2.73E-5	—
	4	1.20E-4	1.91	1.18E-3	1.49	3.51E-5	1.52	1.06E-4	1.97	5.53E-4	2.31	2.59E-6	3.40
	5	3.15E-5	1.92	4.21E-4	1.48	1.18E-5	1.57	2.67E-5	1.99	1.15E-4	2.27	2.64E-7	3.29
	6	8.25E-6	1.93	1.48E-4	1.50	3.77E-6	1.64	6.69E-6	2.00	2.79E-5	2.04	3.31E-7	3.00
	7	2.15E-6	1.94	5.08E-5	1.55	1.15E-6	1.71	1.68E-6	2.00	7.30E-6	1.93	4.66E-9	2.83
2	3	3.02E-5	—	2.98E-4	—	7.03E-6	—	2.96E-5	—	3.39E-4	—	1.63E-6	—
	4	3.99E-6	2.92	4.54E-5	2.71	1.19E-6	2.57	3.83E-6	2.95	3.70E-5	3.20	9.52E-8	4.10
	5	5.18E-7	2.95	7.08E-6	2.68	1.94E-7	2.62	4.85E-7	2.98	4.02E-6	3.20	5.77E-9	4.04
	6	6.66E-8	2.96	1.11E-6	2.67	2.95E-8	2.72	6.08E-8	2.99	4.53E-7	3.15	3.56E-10	4.02
	7	8.52E-9	2.97	1.74E-7	2.68	4.23E-9	2.80	7.62E-9	3.00	5.31E-8	3.09	2.21E-11	4.01
HDG $_k$, $\nu = 0.49999$													
HDG $_k$ -M, $\nu = 0.49999$													
1	3	4.49E-4	—	3.30E-3	—	1.01E-4	—	4.13E-4	—	3.62E-3	—	3.44E-5	—
	4	1.19E-4	1.91	1.18E-3	1.49	3.50E-5	1.52	1.06E-4	1.97	9.32E-4	1.96	4.07E-6	3.08
	5	3.15E-5	1.92	4.21E-4	1.48	1.18E-5	1.57	2.66E-5	1.99	2.28E-4	2.03	4.81E-7	3.08
	6	8.25E-6	1.93	1.48E-4	1.50	3.77E-6	1.64	6.68E-6	2.00	5.14E-5	2.15	5.37E-8	3.16
	7	2.15E-6	1.94	5.08E-5	1.55	1.15E-6	1.71	1.67E-6	2.00	1.01E-5	2.34	5.35E-9	3.33
2	3	3.02E-5	—	2.98E-4	—	7.02E-6	—	2.95E-5	—	3.65E-4	—	1.72E-6	—
	4	3.99E-6	2.92	4.54E-5	2.71	1.19E-6	2.57	3.83E-6	2.95	3.90E-5	3.22	9.80E-8	4.13
	5	5.17E-7	2.95	7.08E-6	2.68	1.93E-7	2.62	4.84E-7	2.98	4.18E-6	3.22	5.85E-9	4.07
	6	6.66E-8	2.96	1.11E-6	2.67	2.95E-8	2.71	6.08E-8	2.99	4.65E-7	3.17	3.57E-10	4.03
	7	8.51E-9	2.97	1.74E-7	2.68	4.23E-9	2.80	7.61E-9	3.00	5.39E-8	3.11	2.32E-11	3.95

The practical construction of \mathbf{M} -decompositions in the three-dimensional case constitutes the subject of ongoing work.

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Appendix. Proofs of the error estimates of Section 2

In this appendix, we provide proofs of our *a priori* error estimates in Section 2, namely Theorems 2.3–2.6. The main idea is to work with the following projection of the errors:

$$\begin{aligned}\boldsymbol{\varepsilon}_\sigma &:= \boldsymbol{\Pi}_{\Sigma} \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, & \boldsymbol{\varepsilon}_u &:= \boldsymbol{\Pi}_V \mathbf{u} - \mathbf{u}_h, \\ \widehat{\boldsymbol{\varepsilon}}_u &:= \mathbf{P}_M \mathbf{u} - \widehat{\mathbf{u}}_h, & \widehat{\boldsymbol{\varepsilon}}_\sigma &:= \boldsymbol{\varepsilon}_\sigma \mathbf{n} - \alpha(\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u).\end{aligned}$$

We also use $e_\sigma := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ to simplify notation.

We begin by obtaining the equations satisfied by these projections. We then use an energy argument to obtain an estimate of $\boldsymbol{\varepsilon}_\sigma$; this would prove first part of Theorem 2.3. We prove the second part of Theorem 2.3 following the idea in Fu *et al.* (2015, Appendix A.1) for treating the incompressible limit.

Then, we prove the local error estimates in Theorem 2.4 for the piecewise divergence $\nabla \cdot \boldsymbol{\varepsilon}_\sigma$, the piecewise symmetric gradient $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}_u)$ and the jump term $\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u$, using an adjoint HDG projection similar to Cockburn *et al.* (2017b). Next, we obtain an estimate of $\boldsymbol{\varepsilon}_u$ with an elliptic duality. After that, we obtain the estimate for the displacement postprocessing in Theorem 2.6.

Step 1: The equations for the projection of the errors

We begin our error analysis with the following auxiliary result.

LEMMA A.1 Suppose that for every $K \in \mathcal{T}_h$, the space $\underline{\boldsymbol{\Sigma}}(K) \times \boldsymbol{V}(K)$ admits an $\boldsymbol{M}(\partial K)$ -decomposition and that the stabilization function α satisfies $a_{\nabla^\perp} > 0$. Then, we have

$$(\mathcal{A} \boldsymbol{\varepsilon}_\sigma, \underline{\boldsymbol{\tau}})_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}_u, \nabla \cdot \underline{\boldsymbol{\tau}})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}}_u, \underline{\boldsymbol{\tau}} \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = - (\mathcal{A}(\boldsymbol{\sigma} - \underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma}), \underline{\boldsymbol{\tau}})_{\mathcal{T}_h} \tag{A.1a}$$

$$(\boldsymbol{\varepsilon}_\sigma, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = 0, \tag{A.1b}$$

$$\langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \cap \partial \Omega} = 0, \tag{A.1c}$$

$$\langle \widehat{\boldsymbol{\varepsilon}}_u, \boldsymbol{\mu} \rangle_{\partial \Omega} = 0, \tag{A.1d}$$

for all $(\underline{\boldsymbol{\tau}}, \boldsymbol{v}, \boldsymbol{\mu}) \in \underline{\boldsymbol{\Sigma}}_h \times \boldsymbol{V}_h \times \boldsymbol{M}_h$.

Proof. The proof follows directly from the consistency of the HDG method (1.5) and the definition of the HDG projection (2.2). For example, to prove the second equation (A.1b), we proceed as follows. We have that

$$\begin{aligned} (\boldsymbol{\varepsilon}_\sigma, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} &= (\underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma} \boldsymbol{n} - \alpha(\boldsymbol{\Pi}_V \boldsymbol{u} - \boldsymbol{P}_M \boldsymbol{u}), \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} \\ &\quad - (\boldsymbol{\sigma}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} \\ &= (\boldsymbol{\sigma}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \boldsymbol{\sigma} \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} - (\boldsymbol{\sigma}_h, \nabla \boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} \end{aligned}$$

by equations (2.2a) and (2.2c). Finally, by equation (1.5b),

$$(\boldsymbol{\varepsilon}_\sigma, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = (\boldsymbol{\sigma}, \nabla \boldsymbol{v})_{\mathcal{T}_h} - \langle \boldsymbol{\sigma} \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} - (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} = (-\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h} = 0.$$

The other equations can be proven in a similar way. □

Step 2: The proof of Theorem 2.3

We begin with an energy argument to prove the first result (2.3a) in Theorem 2.3. We proceed as follows. Taking $\underline{\boldsymbol{\tau}} := \boldsymbol{\varepsilon}_\sigma$ in the error equation (A.1a), $\boldsymbol{v} := \boldsymbol{\varepsilon}_u$ in the error equation (A.1b), $\boldsymbol{\mu} := \widehat{\boldsymbol{\varepsilon}}_u$ in the error equation (A.1c) and $\boldsymbol{\mu} := \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}$ in the error equation (A.1d) and adding the resulting equations up, we obtain

$$(\mathcal{A} \boldsymbol{\varepsilon}_\sigma, \boldsymbol{\varepsilon}_\sigma)_{\mathcal{T}_h} + \Theta_h = - (\mathcal{A}(\boldsymbol{\sigma} - \underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma}), \boldsymbol{\varepsilon}_\sigma)_{\mathcal{T}_h},$$

where

$$\Theta_h := (\boldsymbol{\varepsilon}_u, \nabla \cdot \boldsymbol{\varepsilon}_\sigma)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}}_u, \boldsymbol{\varepsilon}_\sigma \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} + (\boldsymbol{\varepsilon}_\sigma, \nabla \boldsymbol{\varepsilon}_u) - \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \boldsymbol{\varepsilon}_u \rangle_{\partial \mathcal{T}_h} + \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \boldsymbol{n}, \widehat{\boldsymbol{\varepsilon}}_u \rangle_{\partial \mathcal{T}_h}$$

$$\begin{aligned} &= \langle \boldsymbol{\varepsilon}_{\underline{\sigma}} \mathbf{n} - \widehat{\boldsymbol{\varepsilon}}_{\underline{\sigma}} \mathbf{n}, \boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u \rangle_{\partial \mathcal{T}_h} \\ &= \langle \alpha(\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u), \boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

So we have that

$$\|\boldsymbol{\varepsilon}_{\underline{\sigma}}\|_{\mathcal{A}, \mathcal{T}_h}^2 + \|\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u\|_{\alpha, \partial \mathcal{T}_h}^2 \leq -(\mathcal{A}(\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}), \boldsymbol{\varepsilon}_{\underline{\sigma}})_{\mathcal{T}_h} \leq \|\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}\|_{\mathcal{A}, \mathcal{T}_h} \|\boldsymbol{\varepsilon}_{\underline{\sigma}}\|_{\mathcal{A}, \mathcal{T}_h},$$

and the result follows. This completes the proof of the first result (2.3a) of Theorem 2.3.

Now, let us prove the second result (2.3b). We define the deviatoric part of a tensor by $\underline{\boldsymbol{\tau}}^D := \underline{\boldsymbol{\tau}} - \frac{1}{n} \text{tr}(\underline{\boldsymbol{\tau}}) \mathbf{I}$. Hence, we have

$$(\mathcal{A} \underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}})_{\mathcal{T}_h} = \frac{1}{2\mu} (\underline{\boldsymbol{\sigma}}^D, \underline{\boldsymbol{\tau}}^D)_{\mathcal{T}_h} + \frac{1}{n(2\mu + n\lambda)} (\text{tr}(\underline{\boldsymbol{\sigma}}), \text{tr}(\underline{\boldsymbol{\tau}}))_{\mathcal{T}_h}.$$

Then, the first result (2.3a) implies that

$$\|\boldsymbol{\varepsilon}_{\underline{\sigma}}^D\|_{\mathcal{T}_h} \leq 2\mu \|\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}\|_{\mathcal{A}, \mathcal{T}_h} \leq \|\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}\|_{\mathcal{T}_h}. \quad (\text{A.2})$$

Let $\boldsymbol{\varepsilon}_p := \frac{1}{n} \text{tr}(\boldsymbol{\varepsilon}_{\underline{\sigma}})$, then $\boldsymbol{\varepsilon}_{\underline{\sigma}} = \boldsymbol{\varepsilon}_{\underline{\sigma}}^D + \boldsymbol{\varepsilon}_p \mathbf{I}$. To prove (2.3b), we are left to bound the L^2 -norm of $\boldsymbol{\varepsilon}_p$. Now, taking $\underline{\boldsymbol{\tau}}$ to be the identity tensor in (A.1a) and by the fact that $\widehat{\boldsymbol{\varepsilon}}_u = \mathbf{0}$ on $\partial \Omega$, we obtain

$$n(\boldsymbol{\varepsilon}_p, 1)_{\mathcal{T}_h} = (\text{tr}(\boldsymbol{\varepsilon}_{\underline{\sigma}}), 1)_{\mathcal{T}_h} = -(\text{tr}(\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}), 1)_{\mathcal{T}_h} = -(\boldsymbol{\sigma} - \underline{\Pi}_{\Sigma} \boldsymbol{\sigma}, \mathbf{I})_{\mathcal{T}_h} = 0.$$

Hence $(\boldsymbol{\varepsilon}_p, 1)_{\mathcal{T}_h} = 0$. It is well known (Témam, 1979) that for any function $q \in L^2(\Omega)$ such that $(q, 1)_{\Omega} = 0$, we have

$$\|q\|_{\Omega} \leq \theta \sup_{\mathbf{0} \neq \mathbf{w} \in H_0^1(\Omega)} \frac{(q, \nabla \cdot \mathbf{w})}{\|\mathbf{w}\|_{1, \Omega}}$$

for some constant θ independent of q . Now, we take $q := \boldsymbol{\varepsilon}_p$ and work with the numerator in the above expression. We have

$$(\boldsymbol{\varepsilon}_p, \nabla \cdot \mathbf{w})_{\Omega} = -(\nabla \boldsymbol{\varepsilon}_p, \mathbf{w})_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_p \mathbf{n}, \mathbf{w} \rangle_{\partial \mathcal{T}_h} = T_1 + T_2,$$

where

$$\begin{aligned} T_1 &= -(\nabla \boldsymbol{\varepsilon}_p, \mathbf{w} - \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h}, \\ T_2 &= -(\nabla \boldsymbol{\varepsilon}_p, \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_p \mathbf{n}, \mathbf{w} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

Let us bound the above terms individually. We have

$$\begin{aligned} T_1 &= -(\nabla \boldsymbol{\varepsilon}_p, \mathbf{w} - \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} \\ &= -(\nabla \cdot \boldsymbol{\varepsilon}_{\underline{\sigma}} - \nabla \cdot \boldsymbol{\varepsilon}_{\underline{\sigma}}^D, \mathbf{w} - \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} \\ &= (\nabla \cdot \boldsymbol{\varepsilon}_{\underline{\sigma}}^D, \mathbf{w} - \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} \quad \text{since } \nabla \cdot \underline{\Sigma}(K) \subset V(K), \\ &\leq Ch \|\nabla \cdot \boldsymbol{\varepsilon}_{\underline{\sigma}}^D\|_{\mathcal{T}_h} \|\mathbf{w}\|_{1, \mathcal{T}_h}, \end{aligned}$$

and

$$\begin{aligned} T_2 &= (\boldsymbol{\varepsilon}_p, \nabla \cdot \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}_p \mathbf{n}, \mathbf{P}_V \mathbf{w} - \mathbf{w} \rangle_{\partial \mathcal{T}_h} \\ &= -(\boldsymbol{\varepsilon}_\sigma^D, \nabla \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \mathbf{n}, \mathbf{P}_V \mathbf{w} \rangle_{\partial \mathcal{T}_h} - \langle \boldsymbol{\varepsilon}_p \mathbf{n}, \mathbf{P}_V \mathbf{w} - \mathbf{w} \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$

by the error equation (A.1b) with $\mathbf{v} := \mathbf{P}_V \mathbf{w}$. Now, since $\widehat{\boldsymbol{\varepsilon}}_\sigma \mathbf{n}$ is single valued and $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, we have that $\langle \widehat{\boldsymbol{\varepsilon}}_\sigma \mathbf{n}, \mathbf{w} \rangle_{\partial \mathcal{T}_h} = 0$, and so

$$\begin{aligned} T_2 &= -(\boldsymbol{\varepsilon}_\sigma^D, \nabla \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\varepsilon}}_\sigma \mathbf{n} - \boldsymbol{\varepsilon}_p \mathbf{n}, \mathbf{P}_V \mathbf{w} - \mathbf{w} \rangle_{\partial \mathcal{T}_h} \\ &= -(\boldsymbol{\varepsilon}_\sigma^D, \nabla \mathbf{P}_V \mathbf{w})_{\mathcal{T}_h} \\ &\quad + \langle \boldsymbol{\varepsilon}_\sigma^D \mathbf{n} - \alpha(\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u), \mathbf{P}_V \mathbf{w} - \mathbf{w} \rangle_{\partial \mathcal{T}_h} \quad \text{by the definition of } \widehat{\boldsymbol{\varepsilon}}_\sigma \mathbf{n}, \\ &= -(\boldsymbol{\varepsilon}_\sigma^D, \nabla \mathbf{w})_{\mathcal{T}_h} - \langle \alpha(\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u), \mathbf{P}_V \mathbf{w} - \mathbf{w} \rangle_{\partial \mathcal{T}_h} \quad \text{by integration by parts,} \\ &\leq (\|\boldsymbol{\varepsilon}_\sigma^D\|_{\mathcal{T}_h} + Ch^{1/2}\|\alpha\| \|\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u\|_{\alpha, \partial \mathcal{T}_h}) \|\mathbf{w}\|_{1, \Omega}, \end{aligned}$$

by the Cauchy–Schwarz inequality. Hence,

$$\|\boldsymbol{\varepsilon}_p\|_{\mathcal{T}_h} \leq \theta (\|\boldsymbol{\varepsilon}_\sigma^D\|_{\mathcal{T}_h} + Ch \|\nabla \cdot \boldsymbol{\varepsilon}_\sigma^D\|_{\mathcal{T}_h} + Ch^{1/2}\|\alpha\| \|\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u\|_{\alpha, \partial \mathcal{T}_h}).$$

Combining this result with (A.2) and the estimate of $\boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u$ in Theorem 2.4, we obtain

$$\|\boldsymbol{\varepsilon}_\sigma\|_{\mathcal{T}_h} \leq \|\boldsymbol{\varepsilon}_\sigma^D\|_{\mathcal{T}_h} + n\|\boldsymbol{\varepsilon}_p\|_{\mathcal{T}_h} \leq C(1 + h^{1/2}\|\alpha\|)\|\boldsymbol{\sigma} - \underline{\Pi}_\Sigma \boldsymbol{\sigma}\|_{\mathcal{T}_h},$$

with C independent of h, α and \mathcal{A} . This completes the proof of Theorem 2.3. □

Step 3: The proof of Theorem 2.4

Following Cockburn *et al.* (2017b), we first introduce an auxiliary adjoint HDG projection onto $\underline{\Sigma} \times \mathbf{V}$ for functions in the finite element space $\underline{\Sigma} \times \mathbf{V} \times \mathbf{M}$, and then choose test functions in the local equations to be the adjoint HDG projection of certain finite element data to prove Theorem 2.4. The adjoint HDG projection is defined as follows.

DEFINITION A.2 (The auxiliary adjoint HDG projection) Let $\underline{\Sigma} \times \mathbf{V}$ admit an \mathbf{M} -decomposition. Let $d := (\underline{\mathbf{d}}_\tau, \mathbf{d}_v, \mathbf{d}_\mu) \in \underline{\Sigma} \times \mathbf{V} \times \mathbf{M}$. Then, $\Pi_h^* d := (\underline{\Pi}_\Sigma^* d, \Pi_V^* d) \in \underline{\Sigma} \times \mathbf{V}$ defined by the equations

$$\begin{aligned} (\Pi_V^* d, \boldsymbol{\tau})_K &= (\mathbf{d}_v, \boldsymbol{\tau})_K \quad \forall \boldsymbol{\tau} \in \widetilde{\mathbf{V}}, \\ (\underline{\Pi}_\Sigma^* d, \mathbf{v})_K &= (\underline{\mathbf{d}}_\tau, \mathbf{v})_K \quad \forall \mathbf{v} \in \widetilde{\underline{\Sigma}}, \\ \langle \underline{\Pi}_\Sigma^* d \mathbf{n} + \alpha(\Pi_V^* d), \boldsymbol{\mu} \rangle_{\partial K} &= \langle \mathbf{d}_\mu, \boldsymbol{\mu} \rangle_{\partial K} \quad \forall \boldsymbol{\mu} \in \mathbf{M} \end{aligned}$$

is the auxiliary adjoint HDG projection associated to the \mathbf{M} -decomposition.

It is easy to see that this adjoint is well defined whenever the HDG projection is. In fact, a glance to the definition of the auxiliary HDG projection in Theorem 2.2 allows us to see that $(\underline{\Pi}_\Sigma \boldsymbol{\sigma}, \Pi_V \mathbf{u}) = (-\underline{\Pi}_\Sigma^* d, \Pi_V^* d)$ for $d = (-\boldsymbol{\sigma}, \mathbf{u}, -\boldsymbol{\sigma} \mathbf{n} - \alpha(\mathbf{P}_M \mathbf{u}))$.

We have the following bounds for the adjoint projection whose proof is omitted since it is very similar to the one for the case of M -decompositions for diffusion (see Cockburn *et al.*, 2017b, Appendix). See also a sketch of the proof in our extended paper on arXiv (Cockburn & Fu, 2017a, Appendix B).

LEMMA A.3 (Stability of the adjoint HDG projection) Let $\underline{\Sigma} \times V$ admit an M -decomposition and let the stabilization function α satisfy Property (S). Then, we have, for data $d := (\underline{\mathbf{d}}_\tau, \mathbf{d}_v, \mathbf{d}_\mu) \in \underline{\epsilon}(V) \times \nabla \cdot \underline{\Sigma} \times M$,

$$\begin{aligned} \|\Pi_{\underline{\Sigma}}^* d\|_K &\leq C_1 \|\mathbf{d}_v\|_K + C_3 \|\underline{\mathbf{d}}_\tau\|_K + C_5 h_K^{1/2} \|\mathbf{d}_\mu\|_{\partial K}, \\ \|\Pi_V^* d\|_K &\leq C_2 \|\mathbf{d}_v\|_K + C_4 \|\underline{\mathbf{d}}_\tau\|_K + C_6 h_K^{1/2} \|\mathbf{d}_\mu\|_{\partial K}, \end{aligned}$$

where $\{C_i\}_{i=1}^6$ are the constants defined in Theorem 2.4.

Now, we are ready to prove the stability estimates in Theorem 2.4. To do that, we begin by noting that we can rewrite the first two equations defining the HDG methods (1.5) on each element K as

$$\begin{aligned} (\mathcal{A} \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}})_K - (\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h), \underline{\boldsymbol{\tau}})_K - (\nabla \cdot \underline{\boldsymbol{\sigma}}_h, \mathbf{v})_K \\ + (\mathbf{u}_h - \widehat{\mathbf{u}}_h, \underline{\boldsymbol{\tau}} \mathbf{n} + \alpha(\mathbf{v}))_{\partial K} = (\mathbf{f}, \mathbf{v})_K, \end{aligned} \quad (\text{A.3})$$

for all $(\underline{\boldsymbol{\tau}}, \mathbf{v}) \in \underline{\Sigma}(K) \times V(K)$. Therefore, testing with $(\underline{\boldsymbol{\tau}}, \mathbf{v}) := (\Pi_{\underline{\Sigma}}^* d, \Pi_V^* d)$ and using the equations that define the adjoint HDG projection, it follows that

$$(\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h), \underline{\mathbf{d}}_\tau)_K + (\nabla \cdot \underline{\boldsymbol{\sigma}}_h, \mathbf{d}_v)_K - (\mathbf{u}_h - \widehat{\mathbf{u}}_h, \mathbf{d}_\mu)_{\partial K} = -(\mathbf{f}, \Pi_V^* d)_K + (\mathcal{A} \underline{\boldsymbol{\sigma}}_h, \Pi_{\underline{\Sigma}}^* d)_K \quad (\text{A.4})$$

for an arbitrary $d = (\underline{\mathbf{d}}_\tau, \mathbf{d}_v, \mathbf{d}_\mu)$.

To prove the first stability estimate, we take $d := (\mathbf{0}, \nabla \cdot \underline{\boldsymbol{\sigma}}_h, \mathbf{0})$ in (A.4) and use that $\Pi_V^* d \in V$, so that

$$\begin{aligned} \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h\|_K^2 &= -(\mathbf{P}_V \mathbf{f}, \Pi_V^* d)_K + (\mathcal{A} \underline{\boldsymbol{\sigma}}_h, \Pi_{\underline{\Sigma}}^* d)_K \\ &\leq \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A},K} \|\Pi_{\underline{\Sigma}}^* d\|_K + \|\mathbf{P}_V \mathbf{f}\|_K \|\Pi_V^* d\|_K \\ &\leq \left(C_1 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A},K} + C_2 \|\mathbf{P}_V \mathbf{f}\|_K \right) \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h\|_K, \end{aligned}$$

by the stability properties of the adjoint projection in Proposition A.3. To prove the second estimate, we take $d := (\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h), \mathbf{0}, \mathbf{0})$ in (A.4) and note that $\Pi_V^* d \in \widetilde{V}^\perp$ because $\mathbf{d}_v = \mathbf{0}$. Then,

$$\begin{aligned} \|\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h)\|_K^2 &= -(\mathbf{P}_{\widetilde{V}^\perp} \mathbf{f}, \Pi_V^* d)_K + (\mathcal{A} \underline{\boldsymbol{\sigma}}_h, \Pi_{\underline{\Sigma}}^* d)_K \\ &\leq \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A},K} \|\Pi_{\underline{\Sigma}}^* d\|_K + \|\mathbf{P}_{\widetilde{V}^\perp} \mathbf{f}\|_K \|\Pi_V^* d\|_K \\ &\leq \left(C_3 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A},K} + C_4 \|\mathbf{P}_{\widetilde{V}^\perp} \mathbf{f}\|_K \right) \|\underline{\boldsymbol{\epsilon}}(\mathbf{u}_h)\|_K, \end{aligned}$$

by Proposition A.3. To prove the third estimate, we take $d := -(\mathbf{0}, \mathbf{0}, \mathbf{u}_h - \widehat{\mathbf{u}}_h)$ in (A.4)

$$\begin{aligned} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{\partial K}^2 &= -(\mathbf{P}_{\widetilde{V}^\perp} \mathbf{f}, \Pi_V^* d)_K + (\mathcal{A} \underline{\boldsymbol{\sigma}}_h, \Pi_{\underline{\Sigma}}^* d)_K \\ &\leq \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\boldsymbol{\sigma}}_h\|_{\mathcal{A},K} \|\Pi_{\underline{\Sigma}}^* d\|_K + \|\mathbf{P}_{\widetilde{V}^\perp} \mathbf{f}\|_K \|\Pi_V^* d\|_K \end{aligned}$$

$$\leq \left(C_5 \|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\underline{\sigma}_h\|_{\mathcal{A},K} + C_6 \|\mathbf{P}_{\tilde{V}^\perp} \mathbf{f}\|_K \right) h_K^{1/2} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{\partial K},$$

by Proposition A.3.

The remaining estimates can be proven in exactly the same way given that we have

$$\langle \underline{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}_u), \underline{\mathbf{d}}_\tau \rangle_K + \langle \nabla \cdot \boldsymbol{\varepsilon}_\sigma, \mathbf{d}_v \rangle_K - \langle \boldsymbol{\varepsilon}_u - \widehat{\boldsymbol{\varepsilon}}_u, \mathbf{d}_\mu \rangle_{\partial K} = (\mathcal{A} \mathbf{e}_\sigma, \underline{\boldsymbol{\Pi}}_\Sigma^* d)_K$$

for an arbitrary $d = (\underline{\mathbf{d}}_\tau, \mathbf{d}_v, \mathbf{d}_\mu)$. This completes the proof of Theorem 2.4. □

Step 4: The proof of Theorem 2.5

The estimate of $\boldsymbol{\varepsilon}_u$ in Theorem 2.5 will follow from the following identity, whose proof is a standard duality argument, hence is omitted. We refer to Fu *et al.* (2015, Lemma 3) for details of a similar proof.

LEMMA A.4 Suppose that the assumptions of Theorem 2.5 are satisfied. Then, we have

$$\langle \boldsymbol{\varepsilon}_u, \boldsymbol{\theta} \rangle_{\mathcal{T}_h} = (\mathcal{A} \mathbf{e}_\sigma, \underline{\boldsymbol{\psi}} - \underline{\boldsymbol{\Pi}}_\Sigma \underline{\boldsymbol{\psi}})_{\mathcal{T}_h} + (\boldsymbol{\sigma} - \underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma}, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h))_{\mathcal{T}_h} \quad \forall \boldsymbol{\phi}_h \in \mathbf{V}_h,$$

where $(\underline{\boldsymbol{\psi}}, \boldsymbol{\phi})$ is the solution of the dual problem (2.5).

From Lemma A.4, we get that

$$\|\boldsymbol{\varepsilon}_u\|_{\mathcal{T}_h} \leq H(\boldsymbol{\theta}) \left(\|\mathcal{A}\|_{L^\infty(K)}^{1/2} \|\mathbf{e}_\sigma\|_{\mathcal{A},\mathcal{T}_h} + \|\boldsymbol{\sigma} - \underline{\boldsymbol{\Pi}}_\Sigma \boldsymbol{\sigma}\|_{\mathcal{T}_h} \right),$$

where

$$H(\boldsymbol{\theta}) := \sup_{\mathbf{0} \neq \boldsymbol{\theta} \in L^2(\Omega)} \frac{\|\underline{\boldsymbol{\psi}} - \underline{\boldsymbol{\Pi}}_\Sigma \underline{\boldsymbol{\psi}}\|_{\mathcal{T}_h}}{\|\boldsymbol{\theta}\|_{\mathcal{T}_h}} + \sup_{\mathbf{0} \neq \boldsymbol{\theta} \in L^2(\Omega)} \inf_{\boldsymbol{\phi}_h \in \mathbf{V}_h} \frac{\|\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_{\partial \mathcal{T}_h}}{\|\boldsymbol{\theta}\|_{\mathcal{T}_h}}.$$

If the elliptic regularity estimate (2.4) holds, we have

$$H(\boldsymbol{\theta}) \lesssim h \sup_{\mathbf{0} \neq \boldsymbol{\theta} \in L^2(\Omega)} \frac{\|\underline{\boldsymbol{\psi}}\|_{H^1(\mathcal{T}_h)} + \|\boldsymbol{\phi}\|_{H^2(\mathcal{T}_h)}}{\|\boldsymbol{\theta}\|_{\mathcal{T}_h}} \lesssim h.$$

This completes the proof of Theorem 2.5. □

Step 5: The proof of Theorem 2.6

We conclude this section by sketching the proof of Theorem 2.6 on the postprocessing of the displacement. We denote \mathbf{P}_{V^*} , $\mathbf{P}_{\tilde{V}^*}$ and $\mathbf{P}_{V^*,\perp}$ as the corresponding L^2 -projections onto the spaces $V^*(K)$, $\tilde{V}^*(K)$ and $V^*(K)^\perp$, respectively.

First, the inclusion $\mathbf{P}_0(K) \subset \tilde{V}^*(K)$ ensures the well posedness of the postprocessing in (2.6). Next, using $\tilde{V}^*(K) \subset \nabla \cdot \underline{\boldsymbol{\Sigma}}(K)$, equation (2.6b) and the definition of $\underline{\boldsymbol{\Pi}}_V \mathbf{u}$, we have

$$\mathbf{P}_{\tilde{V}^*}(\mathbf{u} - \mathbf{u}_h^*) = \mathbf{P}_{\tilde{V}^*}(\underline{\boldsymbol{\Pi}}_V \mathbf{u} - \mathbf{u}_h) = \mathbf{P}_{\tilde{V}^*} \boldsymbol{\varepsilon}_u.$$

Hence,

$$\|\mathbf{P}_{\tilde{V}^*}(\mathbf{u} - \mathbf{u}_h^*)\|_K \leq \|\boldsymbol{\varepsilon}_u\|_K. \quad (\text{A.5})$$

Then, by the assumptions $\Delta \mathbf{V}^* \subset \nabla \cdot \underline{\boldsymbol{\Sigma}}$ and $(\nabla \mathbf{V}^* \mathbf{n})|_{\partial K} \subset \mathbf{M}(\partial K)$, we have the following identity

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{w})_K &= -(\mathbf{u}, \Delta \mathbf{w})_K + \langle \mathbf{u}, \nabla \mathbf{w} \mathbf{n} \rangle_{\partial K} \\ &= -(\boldsymbol{\Pi}_V \mathbf{u}, \Delta \mathbf{w})_K + \langle \mathbf{P}_M \mathbf{u}, \nabla \mathbf{w} \mathbf{n} \rangle_{\partial K}. \end{aligned}$$

Combining the above equality with equation (2.6a), we get

$$(\nabla(\mathbf{u} - \mathbf{u}_h^*), \nabla \mathbf{w})_K = -(\boldsymbol{\varepsilon}_u, \Delta \mathbf{w})_K + \langle \widehat{\boldsymbol{\varepsilon}}_u, \nabla \mathbf{w} \mathbf{n} \rangle_{\partial K}.$$

Now, taking $\mathbf{w} = \mathbf{P}_{\tilde{V}^*, \perp}(\mathbf{u} - \mathbf{u}_h^*)$ in the above equality, we get

$$\begin{aligned} \|\mathbf{P}_{\tilde{V}^*, \perp}(\mathbf{u} - \mathbf{u}_h^*)\|_K &\leq C h_K \|\nabla \mathbf{P}_{\tilde{V}^*, \perp}(\mathbf{u} - \mathbf{u}_h^*)\|_K \\ &\leq C h_K \left(\|\nabla(\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u})\|_K + \|\nabla \mathbf{P}_{\tilde{V}^*}(\mathbf{u} - \mathbf{u}_h^*)\|_K \right. \\ &\quad \left. + h_K^{-1} \|\boldsymbol{\varepsilon}_u\|_K + h_K^{-1/2} \|\widehat{\boldsymbol{\varepsilon}}_u\|_{\partial K} \right) \\ &\leq C (h_K \|\nabla(\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u})\|_K + \|\boldsymbol{\varepsilon}_u\|_K + h_K^{1/2} \|\widehat{\boldsymbol{\varepsilon}}_u\|_{\partial K}). \end{aligned} \quad (\text{A.6})$$

Combining the estimates in (A.5) and (A.6), and the fact that $\|\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u}\|_K \leq C h_K \|\nabla(\mathbf{u} - \mathbf{P}_{V^*} \mathbf{u})\|_K$, we conclude the proof of Theorem 2.6. \square