DG METHOD FOR NUMERICAL PRICING OF MULTI-ASSET ASIAN OPTIONS—THE CASE OF OPTIONS WITH FLOATING STRIKE

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Abstract. Option pricing models are an important part of financial markets worldwide. The PDE formulation of these models leads to analytical solutions only under very strong simplifications. For more general models the option price needs to be evaluated by numerical techniques. First, based on an ideal pure diffusion process for two risky asset prices with an additional path-dependent variable for continuous arithmetic average, we present a general form of PDE for pricing of Asian option contracts on two assets. Further, we focus only on one subclass—Asian options with floating strike—and introduce the concept of the dimensionality reduction with respect to the payoff leading to PDE with two spatial variables. Then the numerical option pricing scheme arising from the discontinuous Galerkin method is developed and some theoretical results are also mentioned. Finally, the aforementioned model is supplemented with numerical results on real market data.

Keywords: option pricing; discontinuous Galerkin method; path-dependent option; basket option; floating strike

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1. Introduction

Mathematics is the science good understanding of which is essential for efficient solving of problems in various fields. One of the most important applications, at least in economics, is pricing of options. While most of the assets at the market get the right price when the supply matches the demand, in case of financial derivatives

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the traders mostly follow the no-arbitrage conditions. This comes from the fact that the value of any financial derivative is derived from the price of its underlying asset. It implies furthermore that by combining the investment into the financial derivative itself and its underlying asset one can find a strategy that should bring riskless return (i.e., we create riskless portfolio), at least if the market is complete.

Among financial derivatives, options are the most interesting contracts, both from the mathematical point of view and the range of financial applications, since the option payoff function is nonlinear—an option is exercised at the maturity time only if it brings the holder positive cashflow; on the other hand, the other contracts, such as forwards, futures or swaps, must be exercised regardless of the will of the holder. At the market we can also find exotic options, which is a term used for a family of options whose payoff conditions are even more complex than in the case of simple options (denoted as plain-vanilla options).

After introducing the seminal papers of option pricing in the 70's, [2] and [23], the problem has been studied extensively by many authors with generally very good results, which is confirmed by the existence of analytical or closed form valuation formulas for plain vanilla options not only under simplifying assumptions of Gaussian distribution, but also for more complex underlying distributions, such as subordinated Lévy processes [6]. The starting point for deriving pricing formulas has been the construction of a system of partial differential equations (PDE) accompanied by boundary and terminal (or initial) conditions, which arise from the aforementioned no-arbitrage conditions.

On the other hand, pricing of many types of exotic options is complicated even under the simplifying assumptions of Gaussian distribution, and solving of the PDE system generally does not lead to analytical formulas. For example, the most common group of exotic options, the path-dependent options, have payoff depending on the path followed by the underlying asset during the option life. Such path is in reality observable only at discrete moments (e.g., closing prices every day or every Friday), which is not feasible to express it by continuous stochastic processes.

If we cannot solve the PDE system analytically, we should apply some numerical approximation technique, such as Monte Carlo simulation (see e.g. [3]), lattices and trees (an approximation technique originally proposed in [7]), finite difference method, finite elements method or discontinuous Galerkin (DG) method formulated for the first time in [24]. The last approach is rather novel, not well studied within option pricing problems and might be of relevance especially for exotic options with very complex conditions.

In this paper we focus on the pricing of Asian basket option using the DG method together with the theoretical justification for the derived numerical scheme. We proceed as follows. In Section 2, we first formulate the PDE system for Asian options on several assets. We proceed with dimensionality reduction and the problem reformulation. Next, in Section 3, the DG approximation is developed. Finally, we show an illustrative example assuming the Asian two-asset basket put option with floating strike.

2. PDE models for multi-asset Asian options

We follow the standard approach to the derivation of a general model for the valuation of path-dependent basket options with a few modifications for the case of Asian options on more assets (see also [4], [26], or [27]). For simpler notation, we consider only two asset prices (the generalization to path-dependent basket options with d assets is straightforward). Here we suppose that the price function $V = V(S_1(t), S_2(t), A(t), t)$ depends on the actual time t, price processes $S_1(t)$, $S_2(t)$ and the newly introduced path-dependent variable

$$A(t) = \frac{1}{t}\int_0^t (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u \tag{2.1}$$

with positive weights α_1 and α_2 satisfying $\alpha_1 + \alpha_2 = 1$.

Let us note that there exist several approaches how to introduce the average A, cf. [10], [22]. The relation (2.1) is often called the continuous arithmetic average of the basket of two asset prices over some prespecified period of time and it does not depend on the current asset prices $S_1(t)$ and $S_2(t)$, in general. Therefore, the option price V is a function of four (generally $d+2$) independent variables.

Next, we contemplate a continuous time trading economy with an infinite horizon. The uncertainty is described by a complete probability space $(\omega, \mathcal{F}, \mathcal{Q})$ with the state space ω; $\mathcal{F}$ is the σ-algebra representing the measurable events, and $\mathcal{Q}$ denotes the risk neutral probability measure, which is assumed to be unique in a complete market with no arbitrage. For a movement of the prices of the underlying assets we assume that they follow a classical geometric Brownian motion (equivalent with the lognormal distribution) and pay dividends at a constant rate continuously, i.e.,

$$\mathrm{d}S_i(t) = (\mu_i - q_i)S_i(t)\,\mathrm{d}t + \sigma_i S_i(t)\,\mathrm{d}W_i(t), \quad i = 1, 2, \tag{2.2}$$

$$\langle \mathrm{d}W_1(t), \mathrm{d}W_2(t)\rangle = \varrho\,\mathrm{d}t, \tag{2.3}$$

where $W_i(t)$ are standard Brownian motions (i.e., Wiener processes), whose increments are correlated.

This is the commonly used stochastic process and implies that the returns of the assets are normally distributed with constant correlation $\varrho \in (-1, 1)$. Since the underlying assets pay dividends, the drift μ_i (i.e., the change of the average value of Brownian motion) of the ith asset is reduced by the amount q_i, denoting the corresponding dividend yield. The variables σ_i are the volatilities of the asset i (i.e., a measure of the standard deviation of the return), and $S_i(t)$ their asset prices at time t. We assume that parameters q_i and σ_i, $i = 1, 2$, are (piecewise) constant functions over the life of the option. The reason why we choose this approach is that it quite well coincides with the way the asset prices move in the real world.

Further, it is necessary to set up the differential equation for the average A. After easy differentiation of (2.1) with respect to the variable t we obtain

$$\begin{aligned}\frac{\mathrm{d}A(t)}{\mathrm{d}t} &= \frac{1}{t}(\alpha_1 S_1(t) + \alpha_2 S_2(t)) - \frac{1}{t^2}\int_0^t (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u \\ &= \frac{1}{t}(\alpha_1 S_1(t) + \alpha_2 S_2(t) - A(t)),\end{aligned} \tag{2.4}$$

which implies that the dynamics of A is driven according to an ordinary differential equation only.

Next, we are ready to apply the multidimensional Itô's lemma to the value function V (see [14]), leading to

$$\begin{aligned}\mathrm{d}V &= \sigma_1 S_1 \frac{\partial V}{\partial S_1}\,\mathrm{d}W_1 + \sigma_2 S_2 \frac{\partial V}{\partial S_2}\,\mathrm{d}W_2 \\ &+ \Big(\frac{\partial V}{\partial t} + (\mu_1 - q_1) S_1 \frac{\partial V}{\partial S_1} + (\mu_2 - q_2) S_2 \frac{\partial V}{\partial S_2} + \frac{\alpha_1 S_1 + \alpha_2 S_2 - A}{t}\frac{\partial V}{\partial A} \\ &+ \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \varrho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\Big)\,\mathrm{d}t.\end{aligned} \tag{2.5}$$

Let us now construct a risk-free portfolio Π of one option and $-\Delta_1$ shares of the underlying asset S_1 and $-\Delta_2$ shares of the underlying asset S_2. The value of Π is given by $\Pi = V - \Delta_1 S_1 - \Delta_2 S_2$ and its corresponding increment is

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta_1\,\mathrm{d}S_1 - \Delta_2\,\mathrm{d}S_2 - \Delta_1 q_1 S_1\,\mathrm{d}t - \Delta_2 q_2 S_2\,\mathrm{d}t, \tag{2.6}$$

where the terms $-\Delta_i q_i S_i\,\mathrm{d}t$ arise, since the underlying assets pay dividends continuously, which decreases the value of the portfolio Π by the amount of the dividend. Replacing the term $\mathrm{d}V$ by the values from Itô's lemma and using (2.2) in (2.6) yields

$$\begin{aligned}\mathrm{d}\Pi &= \Big(\sigma_1 S_1 \frac{\partial V}{\partial S_1} - \Delta_1\sigma_1 S_1\Big)\,\mathrm{d}W_1 + \Big(\sigma_2 S_2 \frac{\partial V}{\partial S_2} - \Delta_2\sigma_2 S_2\Big)\,\mathrm{d}W_2 \\ &+ \Big(\frac{\partial V}{\partial t} + (\mu_1 - q_1) S_1 \frac{\partial V}{\partial S_1} - \Delta_1(\mu_1 - q_1) S_1\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&+ (\mu_2 - q_2) S_2 \frac{\partial V}{\partial S_2} - \Delta_2 (\mu_2 - q_2) S_2 + \frac{\alpha_1 S_1 + \alpha_2 S_2 - A}{t} \frac{\partial V}{\partial A} \\
&+ \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \varrho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \\
&- \Delta_1 q_1 S_1 - \Delta_2 q_2 S_2 \Big) \, \mathrm{d}t.
\end{aligned}
$$

To eliminate the stochastic component from our portfolio, we choose $\Delta_i = \partial V / \partial S_i$, $i = 1, 2$. This procedure leads to a portfolio whose increment has only deterministic component, i.e.

$$
\begin{aligned}
\mathrm{d}\Pi = \Big(&\frac{\partial V}{\partial t} + \frac{\alpha_1 S_1 + \alpha_2 S_2 - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} \\
&+ \varrho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} - q_1 S_1 \frac{\partial V}{\partial S_1} - q_2 S_2 \frac{\partial V}{\partial S_2} \Big) \, \mathrm{d}t.
\end{aligned} \tag{2.8}
$$

Further, we apply the assumption of no arbitrage opportunities on the market, in other words, the change of portfolio corresponds to the change of a value of money deposited on the bank account earning risk-free interest rate r (assumed constant or piecewise constant in time). Then we obtain

$$
\mathrm{d}\Pi = r\Pi \, \mathrm{d}t = r(V - \Delta_1 S_1 - \Delta_2 S_2) \, \mathrm{d}t = r\Big(V - S_1 \frac{\partial V}{\partial S_1} - S_2 \frac{\partial V}{\partial S_2} \Big) \, \mathrm{d}t. \tag{2.9}
$$

Putting together (2.8) and (2.9) leads to the modified form of Black-Scholes PDE for pricing Asian option contracts on two assets

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \varrho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \\
&\quad + (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} + \frac{\alpha_1 S_1 + \alpha_2 S_2 - A}{t} \frac{\partial V}{\partial A} - rV = 0
\end{aligned} \tag{2.10}
$$

for $t \in (0, T)$, $S_1 > 0$, $S_2 > 0$, and $A > 0$. Let us note that (2.10) represents a linear parabolic PDE degenerated in variable A, because there is no second order term with respect to A. It is interesting to see that in the system the drifts μ_i vanish. Note, finally, that a singularity exists in equation (2.10) at $t = 0$ (i.e., today). However, this can be avoided due to the following remark.

Remark 2.1. The PDE (2.10) for pricing Asian two-asset options tends to the classical 2D Black-Scholes equation for pricing standard basket options with two underlying assets as $t \to 0+$. Neglecting the term $\partial V / \partial A$ corresponds to setting $\alpha_1 S_1 + \alpha_2 S_2 = A$ at $t = 0$, which is justified by the following simple calculation.

Let us denote $I(t) = \int_0^t (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u$. Then

$$\lim_{t\to 0+} A(t) = \lim_{t\to 0+} \frac{I(t) - I(0)}{t} = \frac{\mathrm{d}I}{\mathrm{d}t}(0) = \alpha_1 S_1(0) + \alpha_2 S_2(0). \tag{2.11}$$

As has been already stated, we focus only on one subclass of path-dependent options, strictly speaking Asian two-asset basket options. The main feature of all Asian options is that their terminal payoff is a function of an average over some time period prior to expiry. Another specific feature is the way in which the average is incorporated into the payoff function. If we denote the strike price by $\mathcal{K}$ and the maturity by T, then according to these two features, we distinguish four basic payoff functions (w.r.t. two-asset basket options):

$$(A(T) - \mathcal{K})^+ \quad \text{average rate call,} \tag{2.12}$$

$$(\mathcal{K} - A(T))^+ \quad \text{average rate put,} \tag{2.13}$$

$$(\alpha_1 S_1(T) + \alpha_2 S_2(T) - A(T))^+ \quad \text{floating strike call,} \tag{2.14}$$

$$(A(T) - \alpha_1 S_1(T) - \alpha_2 S_2(T))^+ \quad \text{floating strike put,} \tag{2.15}$$

where $(\cdot)^+ = \max(\cdot, 0)$. Note that the rate options are sometimes called the fixed strike options and the floating strike options are also called the average strike options.

In our further analysis we consider only Asian options with floating strike. Let $g = g(S_1, S_2, A)$ denote the payoff function defined by (2.14) or (2.15), i.e.

$$V(S_1(T), S_2(T), A(T), T) = g(S_1(T), S_2(T), A(T)). \tag{2.16}$$

Then following the martingale theory (see [13]), we get the value of option as

$$V(S_1(0), S_2(0), A(0), 0) = \mathrm{e}^{-rT}\mathbb{E}(g(S_1(T), S_2(T), A(T)). \tag{2.17}$$

More precisely, the price of an option at time t can be expressed as the expected value of the discounted payoff under the risk neutral probability measure. Moreover, it is well-know that the option price V is also the solution of PDE (2.10).

This PDE has one dimension in time and three spatial variables. It has convection-diffusion character and it is difficult to solve it, since the parabolic operator is degenerated in the variable A. This undesirable feature of (2.10) can be overcome by a suitable dimensionality reduction. Notice that this equation is more general, since the reduction is only possible for specific payoffs.

2.1. Dimensionality reduction. In the past [10], [22], the two-dimensional version of (2.10) (i.e., Asian option restricted only to one underlying asset) was reduced to a PDE with only one spatial dimension. There are several possible ways how to make this procedure, some of them are applicable to the case of average strike options only or to Asian options of European style. In our case, the above mentioned three-dimensional PDE can be reduced to a PDE with only two spatial dimensions, using the inspiring approach from [20] for floating strike options.

Since the equation is backward in time, the possible dimensionality reduction for the studied model might be achieved by introducing the new spatial variables $x_1 = S_1/A$ and $x_2 = S_2/A$ together with the reversal time transformation $\hat{t} = T - t$ ($\hat{t}$ is time to maturity) in order to rewrite the Cauchy problem (2.10) with (2.16) to the forward type. Denoting $x = [x_1, x_2]$, the payoff functions (2.14) and (2.15) satisfy

$$g(S_1, S_2, A) = A \cdot u^0(x),\ u^0(x) := \begin{cases} (\alpha_1 x_1 + \alpha_2 x_2 - 1)^+ & \text{for call,} \\ (1 - \alpha_1 x_1 - \alpha_2 x_2)^+ & \text{for put,} \end{cases} \tag{2.18}$$

and the option price transforms to

$$V(S_1, S_2, A, t) = \mathrm{e}^{-r\hat{t}}\mathbb{E}(A \cdot u^0|\mathcal{F}(\hat{t})) = A \cdot \mathrm{e}^{-r\hat{t}}\mathbb{E}(u^0|\mathcal{F}(\hat{t})) = A \cdot u(x, \hat{t}). \tag{2.19}$$

Furthermore, simple usage of the chain rule leads to

$$\begin{gathered} \frac{\partial V}{\partial t} = -\frac{\partial u}{\partial \hat{t}} \cdot A, \quad \frac{\partial V}{\partial A} = u - \Big(\frac{\partial u}{\partial x_1} \cdot x_1 + \frac{\partial u}{\partial x_2} \cdot x_2\Big), \\ \frac{\partial V}{\partial S_1} = \frac{\partial u}{\partial x_1}, \quad \frac{\partial V}{\partial S_2} = \frac{\partial u}{\partial x_2}, \\ \frac{\partial^2 V}{\partial S_1^2} = \frac{\partial^2 u}{\partial x_1^2} \cdot \frac{1}{A}, \quad \frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\partial^2 u}{\partial x_1 \partial x_2} \cdot \frac{1}{A}, \quad \frac{\partial^2 V}{\partial S_2^2} = \frac{\partial^2 u}{\partial x_2^2} \cdot \frac{1}{A}. \end{gathered} \tag{2.20}$$

Next, substituting (2.19) and (2.20) into (2.10) and dividing by A, we obtain the new pricing equation in terms of the value function u with only two spatial variables and forward time $\hat{t}$, written in the divergence-free form as

$$\frac{\partial u}{\partial \hat{t}} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\mathbb{D}(x)\nabla u)_i + \sum_{i=1}^{2} b_i(x, \hat{t})\frac{\partial u}{\partial x_i} + \Big(r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}}\Big)u = 0, \tag{2.21}$$

where $(\mathbb{D}(x)\nabla u)_i$ denotes the ith component of the vector $\mathbb{D}(x)\nabla u$, defined using the symmetric positive semi-definite matrix

$$\mathbb{D}(x) = \frac{1}{2}\begin{pmatrix} \sigma_1^2 x_1^2 & \varrho\sigma_1\sigma_2 x_1 x_2 \\ \varrho\sigma_1\sigma_2 x_1 x_2 & \sigma_2^2 x_2^2 \end{pmatrix}, \tag{2.22}$$

and the vector $(b_1, b_2)^{\mathrm{T}}$ represents the field induced by physical fluxes, written componentwise as

$$b_i(x,\hat{t}) = \Bigl(\sigma_i^2 + \frac{1}{2}\varrho\sigma_1\sigma_2 - r + q_i + \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}}\Bigr) x_i. \tag{2.23}$$

Finally, it is important to note that at $\hat{t} = T$ (i.e., today) a singularity still exists in equation (2.21) due to the presence of the term $(\alpha_1 x_1 + \alpha_2 x_2 - 1)/(T - \hat{t})$ in the convection and reaction terms. This can be also avoided in a way similar to that stated in Remark 2.1.

2.2. Localization to bounded domains. In order to numerically solve the Cauchy problem (2.21)–(2.23) with (2.18), the unbounded domain in the variables x_1 and x_2 has to be truncated at first, and then this initial problem should be supplied with suitable choices of additional boundary conditions on appropriate parts of the boundary of the computational domain $\Omega := (0, x_1^{\max}) \times (0, x_2^{\max})$, where $x_i^{\max}$ stands, in fact, for the maximal price of the scaled ith asset. We distinguish three parts of the rectangular boundary $\partial\Omega$ defined as

$$\Gamma_1 = \{0\} \times (0, x_2^{\max}), \quad \Gamma_2 = (0, x_1^{\max}) \times \{0\}, \quad \Gamma_3 = \partial\Omega \cap \mathbb{R}_+^2. \tag{2.24}$$

In what follows, we consider only put options for simplicity. The generalization of all conditions for call options can be done straightforwardly with the aid of the so-called put-call parity, see [14]. First we mention the asymptotic behaviour of the original price function V at a far-field boundary, i.e.,

$$\lim_{S_i \to \infty} V(S_1, S_2, A, t) = 0, \quad i = 1, 2, \quad \lim_{A \to \infty} \frac{\partial V}{\partial A}(S_1, S_2, A, t) = 1. \tag{2.25}$$

Let us mention that it is sufficient to prescribe only one boundary condition in A-direction, since only the first partial derivative with respect to the variable A appears in equation (2.10). Moreover, on the planes $S_1 = 0$ and $S_2 = 0$, the simple concept of extrapolated boundary condition is used, i.e.

$$\begin{aligned} \frac{\partial V}{\partial S_1}(0, S_2, A, t) &= \lim_{\varepsilon \to 0+} \frac{\partial V}{\partial S_1}(\varepsilon, S_2, A, t), \\ \frac{\partial V}{\partial S_2}(S_1, 0, A, t) &= \lim_{\varepsilon \to 0+} \frac{\partial V}{\partial S_2}(S_1, \varepsilon, A, t). \end{aligned} \tag{2.26}$$

Note that to preserve the homogeneity of the option price V with respect to A, shown in (2.19), all the previously depicted boundary conditions have to be linearly homogeneous in the variable A. Therefore, in terms of the new price function u

and new variables, the boundary conditions become of a mixed type in the following sense for put options:

$$(\mathbb{D}(x)\nabla u(x,\hat{t}))\cdot\vec{n}=0 \quad \text{on } \Gamma_i,\ i=1,2, \quad u(x,\hat{t})=0 \quad \text{on } \Gamma_3, \tag{2.27}$$

where $\vec{n}$ is the outer unit normal to Γ_i. Let us point out that homogeneous Neumann boundary conditions from (2.27) prescribed on axes $x_1=0$ and $x_2=0$ are a priori fulfilled in the variational form and correspond to the so-called do-nothing boundary condition. On the other hand, homogeneous Dirichlet conditions on Γ_3 come from the asymptotic behaviour of put options.

Then, the resulting pricing problem for Asian two-asset basket option with floating strike is formulated as the initial-boundary value problem for an unknown function $u(x,\hat{t})\colon \Omega\times(0,T)\to\mathbb{R}$ satisfying (2.21)–(2.23) with (2.18) and (2.27). In what follows this problem will be denoted as (OPP), i.e., option pricing problem.

2.3. Weak formulation. In the first instance, the standard notation for function spaces should be introduced together with their norms $\|\cdot\|$ and seminorms $|\cdot|$. Let $k\geqslant 0$ be an integer and $p\in[1,\infty]$. We use the well-known Lebesgue, Sobolev spaces $L^p(\Omega)$, $H^k(\Omega)$, and Bochner spaces $L^p(0,T;X)$ of functions defined in $(0,T)$ with values in a Banach space X.

In order to obtain a variational formulation of (OPP) a concept of weighted Sobolev spaces is used. For more details see e.g. [21]. Let us introduce the space

$$V\equiv V(\Omega):=\Big\{v\in L^2(\Omega)\colon\ x_i\frac{\partial v}{\partial x_i}\in L^2(\Omega),\ i=1,2\Big\} \tag{2.28}$$

with a scalar product

$$(u,v)_V=(u,v)+\sum_{i=1}^{2}\Big(x_i\frac{\partial u}{\partial x_i},x_i\frac{\partial v}{\partial x_i}\Big), \tag{2.29}$$

where $(\cdot,\cdot)$ denotes the scalar product of $L^2(\Omega)$ with the induced norm $\|\cdot\|=(\cdot,\cdot)^{1/2}$.

Lemma 2.1. *The space V is a Hilbert space with the norm $\|\cdot\|_V:=(\cdot,\cdot)_V^{1/2}$ and has the following properties:*

(P1) *V is separable,*
(P2) *$\mathcal{D}(\overline{\Omega})=\{v\in C_0^\infty(\mathbb{R}_+^2)\colon\ v|_\Omega\}$ is densely embedded in V,*
(P3) *V is densely embedded in $L^2(\Omega)$.*

Proof. The items (P1)–(P3) can be simply proven, see, e.g., [1]. □

Further, in order to fulfill the boundary conditions it is appropriate to define space $V_0 := \{v \in V\colon\ v|_{\Gamma_3} = 0\}$ with seminorm

$$|v|_V = \Bigl(\sum_{i=1}^{2}\Bigl\|x_i\frac{\partial v}{\partial x_i}\Bigr\|^2\Bigr)^{1/2}. \tag{2.30}$$

Lemma 2.2. *If $v \in V_0$, then*

$$\|v\| \leqslant \sqrt{2}|v|_V. \tag{2.31}$$

Moreover, the seminorm $|\cdot|_V$ is in fact a norm on V_0 equivalent to $\|\cdot\|_V$.

P r o o f. Since the space of infinitely differentiable functions with compact support in Ω is dense in V_0, it is enough to prove (2.31) for $v \in C_0^\infty(\Omega)$. We have, for $i = 1, 2$,

$$2\int_\Omega x_i v(x)\frac{\partial v}{\partial x_i}(x)\,\mathrm{d}x = \int_{\partial\Omega} x_i v^2(x) n_i\,\mathrm{d}S - \int_\Omega v^2(x)\,\mathrm{d}x = -\int_\Omega v^2(x)\,\mathrm{d}x. \tag{2.32}$$

Using the Cauchy-Schwarz inequality on the left-hand side of (2.32), we deduce that

$$\|v\|^2 \leqslant 2\|v\|\Bigl\|x_i\frac{\partial v}{\partial x_i}\Bigr\|, \quad i = 1, 2. \tag{2.33}$$

Then, we can write

$$2\|v\|^2 = \|v\|^2 + \|v\|^2 \leqslant 4\Bigl(\Bigl\|x_1\frac{\partial v}{\partial x_1}\Bigr\|^2 + \Bigl\|x_2\frac{\partial v}{\partial x_2}\Bigr\|^2\Bigr) \tag{2.34}$$

and the square root of (2.34) yields the estimate (2.31). Finally, the equivalence of norms comes from (2.31) and the fact that norm $\|\cdot\|_V$ is by definition stronger then the L^2-norm. □

In order to simplify the notation, we define the linear partial differential operator $\mathcal{L}\colon\ V \to V'$ as

$$\mathcal{L}u = -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\bigl(\mathbb{D}(x)\cdot\nabla u\bigr)_i + \sum_{i=1}^{2} b_i(x,t)\frac{\partial u}{\partial x_i} + \Bigl(r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}}\Bigr)u. \tag{2.35}$$

Then using standard approach—multiplication of (2.21) by a test function $v \in V_0$, integration over the whole domain Ω and application of Green's theorem on the diffusion term—leads to

$$\Bigl(\frac{\partial u}{\partial \hat{t}}, v\Bigr) + (\mathcal{L}u, v) = 0 \quad \forall\, v \in V_0,\ \text{a.e. } \hat{t} \in (0, T), \tag{2.36}$$

with the bilinear form

$$(\mathcal{L}u, v) = \int_\Omega \mathbb{D}(x)\nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{i=1}^{2} \int_\Omega b_i(x, \hat{t}) \frac{\partial u}{\partial x_i} v \,\mathrm{d}x + \int_\Omega \Big(r - \frac{\alpha_1 x_1 + \alpha_2 x_2 - 1}{T - \hat{t}}\Big) uv \,\mathrm{d}x. \tag{2.37}$$

Definition 2.1. Assuming that $u^0 \in L^2(\Omega)$, the initial-boundary problem (OPP) has the following variational (weak) formulation: Find $u \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; V_0)$ such that $\partial u/\partial \hat{t} \in L^2(0,T; V')$ satisfies

$$(u(0), v) = (u^0, v) \quad \forall\, v \in V_0, \tag{2.38}$$

$$\Big(\frac{\partial u}{\partial \hat{t}}(\hat{t}), v\Big) + (\mathcal{L}u\,(\hat{t}), v) = 0 \quad \forall\, v \in V_0,\ \text{a.e. } \hat{t} \in (0,T). \tag{2.39}$$

where $u(\hat{t})$ denotes the function on Ω such that $u(\hat{t})(x)$, $x \in \Omega$.

In order to investigate the properties of the operator $\mathcal{L}$ we recall the assumptions on the model parameters. We assume that the coefficients σ_i, q_i, and r are constant or piecewise constant, i.e., there exist four constants, $0 < \sigma_{\min} \leqslant \sigma_{\max}$, $r_{\max} \geqslant 0$, and $q_{\max} \geqslant 0$ such that for all $\hat{t} \in [0,T]$ and $x \in \Omega$,

$$\begin{gathered} 0 < \sigma_{\min} \leqslant \sigma_i(x, \hat{t}) \leqslant \sigma_{\max},\ i = 1,2, \quad 0 \leqslant r(x, \hat{t}) \leqslant r_{\max}, \\ 0 \leqslant q_i(x, \hat{t}) \leqslant q_{\max},\ i = 1,2. \end{gathered} \tag{2.40}$$

Lemma 2.3. *Under the assumptions* (2.40), *the operator* $\mathcal{L}$ *given by* (2.35) *is bounded, i.e., there exists a positive constant* $C_B(\hat{t})$ *such that*

$$|(\mathcal{L}u, v)| \leqslant C_B(\hat{t}) |u|_V |v|_V \quad \forall\, u, v \in V_0,\ \hat{t} \in [0, T^*], \tag{2.41}$$

where $0 < T^* < T$.

P r o o f. We split the term $(\mathcal{L}u, v)$ into several parts and estimate them separately. First, for the diffusion term, from (2.40), $|\varrho| < 1$, and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big|\int_\Omega \mathbb{D}(x)\nabla u \cdot \nabla v \,\mathrm{d}x\Big| &\leqslant \frac{\sigma_{\max}^2}{2} \sum_{i,j=1}^{2} \int_\Omega \Big|x_i \frac{\partial u}{\partial x_i}\Big| \Big|x_j \frac{\partial v}{\partial x_j}\Big| \,\mathrm{d}x \\ &\leqslant \frac{\sigma_{\max}^2}{2} \Big(\Big\|x_1 \frac{\partial u}{\partial x_1}\Big\| + \Big\|x_2 \frac{\partial u}{\partial x_2}\Big\|\Big)\Big(\Big\|x_1 \frac{\partial v}{\partial x_1}\Big\| + \Big\|x_2 \frac{\partial v}{\partial x_2}\Big\|\Big) \\ &\leqslant \sigma_{\max}^2 |u|_V |v|_V, \end{aligned} \tag{2.42}$$

where the last inequality in (2.42) comes from the relation $(a+b)^2 \leqslant 2(a^2+b^2)$, $a,b \geqslant 0$.

Secondly, for the convection part, by an approach similar to that in (2.42) and using the inequality (2.31), we deduce that

$$\tag{2.43} \begin{aligned} \Big|\sum_{i=1}^{2}\int_\Omega b_i(x,\hat t)\frac{\partial u}{\partial x_i}v\,\mathrm{d}x\Big| &\leqslant \sum_{i=1}^{2}\int_\Omega |b_i(x,\hat t)|\Big|\frac{\partial u}{\partial x_i}\Big||v|\,\mathrm{d}x \\ &\leqslant \underbrace{\Big(\frac{3}{2}\sigma_{\max}^2 + r_{\max} + q_{\max} + \frac{\alpha_1 x_1^{\max} + \alpha_2 x_2^{\max} + 1}{T-\hat t}\Big)}_{:=c_1(\hat t)>0}\int_\Omega \sum_{i=1}^{2}\Big|x_i\frac{\partial u}{\partial x_i}\Big||v|\,\mathrm{d}x \\ &\leqslant c_1(\hat t)\sum_{i=1}^{2}\Big\|x_i\frac{\partial u}{\partial x_i}\Big\|\|v\| \leqslant c_1(\hat t)\sqrt{2}\Big(\sum_{i=1}^{2}\Big\|x_i\frac{\partial u}{\partial x_i}\Big\|^2\Big)^{1/2}\|v\| \\ &\leqslant 2c_1(\hat t)|u|_V|v|_V. \end{aligned}$$

Next, for the reaction term, using (2.40), the Cauchy-Schwarz inequality, and (2.31), we find that

$$\tag{2.44} \begin{aligned} \Big|\int_\Omega\Big(r - \frac{\alpha_1x_1+\alpha_2x_2-1}{T-\hat t}\Big)uv\,\mathrm{d}x\Big| &\leqslant \underbrace{\Big(r_{\max} + \frac{\alpha_1 x_1^{\max} + \alpha_2 x_2^{\max} + 1}{T-\hat t}\Big)}_{:=c_2(\hat t)>0}\int_\Omega |u||v|\,\mathrm{d}x \\ &\leqslant c_2(\hat t)\|u\|\|v\| \leqslant 2c_2(\hat t)|u|_V|v|_V. \end{aligned}$$

Finally, putting (2.42), (2.43), and (2.44) together and setting $C_B(\hat t) = \sigma_{\max}^2 + 2c_1(\hat t) + 2c_2(\hat t)$, we obtain the desired estimate (2.41). □

Lemma 2.4. *Under the assumptions* (2.40), *the operator* $\mathcal{L}$ *given by* (2.35) *satisfies the so-called Gårding inequality, i.e., there exists a nonnegative constant* $c_g(\hat t)$ *such that*

$$\tag{2.45} (\mathcal{L}u,u) \geqslant \frac{\sigma_{\min}^2}{4}(1-|\varrho|)|u|_V^2 - c_g(\hat t)\,\|u\|^2 \quad \forall\, u \in V_0,\ \hat t \in [0,T^*],$$

where $0 < T^* < T$.

P r o o f. We again split the estimation of $(\mathcal{L}u,u)$ into several parts. From (2.40) and the relation $ab \leqslant a^2/2 + b^2/2$, $a,b \geqslant 0$, we have

$$\tag{2.46} \begin{aligned} \Big|\int_\Omega \mathbb{D}(x)\nabla u\cdot\nabla u\,\mathrm{d}x\Big| &\geqslant \frac{\sigma_{\min}^2}{2}\int_\Omega\Big(x_1\frac{\partial u}{\partial x_1}\Big)^2\mathrm{d}x + \frac{\sigma_{\min}^2}{2}\int_\Omega\Big(x_2\frac{\partial u}{\partial x_2}\Big)^2\mathrm{d}x \\ &\quad - |\varrho|\sigma_{\min}^2\int_\Omega\Big|x_1\frac{\partial u}{\partial x_1}\Big|\Big|x_2\frac{\partial u}{\partial x_2}\Big|\,\mathrm{d}x \geqslant \frac{\sigma_{\min}^2}{2}(1-|\varrho|)|u|_V^2. \end{aligned}$$

Further, an approach similar to that in (2.43) with $v := u$ leads to

$$\left|\sum_{i=1}^{2}\int_\Omega b_i(x,\hat t)\frac{\partial u}{\partial x_i}u\,\mathrm{d}x\right| \leqslant c_1(\hat t)\sqrt{2}|u|_V\|u\| \leqslant c_1(\hat t)\sqrt{2}\Big(\varepsilon|u|_V^2+\frac{1}{4\varepsilon}||u||^2\Big) \tag{2.47}$$
$$= \frac{\sigma_{\min}^2}{4}(1-|\varrho|)\,|u|_V^2+\frac{2c_1^2(\hat t)}{\sigma_{\min}^2(1-|\varrho|)}\|u\|^2,$$

where the last inequality in (2.47) comes from the Young inequality

$$ab \leqslant \varepsilon a^2+\frac{1}{4\varepsilon}b^2, \quad a,b\geqslant 0,\ \varepsilon>0 \tag{2.48}$$

by setting $\varepsilon := \sigma_{\min}^2(1-|\varrho|)/(4\sqrt{2}c_1(\hat t))$. Then the estimates (2.46), (2.47), and (2.44) with $v := u$ complete the proof with $c_g(\hat t) = 2c_1^2(\hat t)/(\sigma_{\min}^2(1-|\varrho|))+c_2(\hat t)$. □

R e m a r k 2.2. Due to the singularity of (2.21) at $\hat t = T$ the boundedness (2.41) and the Gårding inequality (2.45) hold only on every compact interval $[0,T^*]\subset[0,T)$.

R e m a r k 2.3. The Gårding inequality is a sufficient condition for ellipticity (i.e., strict positivity). Using easy transformation $u=\mathrm{e}^{\lambda\hat t}w$, $\lambda=\max_{\hat t\in[0,T^*]}c_g(\hat t)$, in equation (2.39) and setting $\mathcal{A}_e(w,v) := (\mathcal{L}w,v)+\lambda(w,v)$ leads to

$$\Big(\frac{\partial w}{\partial\hat t},v\Big)+\mathcal{A}_e(w,v)=0 \quad \forall\, v\in V_0,\ \text{a.e. } \hat t\in(0,T^*), \tag{2.49}$$
$$w(0)=u^0\in L^2(\Omega) \tag{2.50}$$

with the strictly positive bilinear form $\mathcal{A}_e(\cdot,\cdot)$.

Theorem 2.1. *Problem* (2.38)–(2.39) *has a unique weak solution.*

P r o o f. The proof is based on the main theorem on first-order linear evolution equations, see [28]. Therefore, it is enough to prove the following assumptions:

(A1) $V\subseteq L^2(\Omega)\subseteq V'$ form a Gelfand triple.

(A2) The mapping $(\mathcal{L}\cdot,\cdot)\colon V\times V\to\mathbb{R}$ is bilinear, bounded and strictly positive.

The fulfilment of assumption (A1) follows directly from Lemma 2.1. Since $(\mathcal{L}\cdot,\cdot)$ is bilinear by definition (2.37), bounded, cf. Lemma 2.3, and the Gårding inequality is applicable, see Lemma 2.4, (A2) is also satisfied. With these facts, we can apply the abstract theory of variational parabolic problems, which guarantees the existence of a unique weak solution of problem (OPP), i.e., it concludes the proof. □

3. Discontinuous Galerkin approximations

Since the governing equation of the problem (OPP) cannot be transformed into a standard heat equation with constant coefficients, it has no closed-form solution of Black-Scholes type and we need to use a numerical approach. Here we select an approach based on the DG framework. The standard DG method uses piecewise polynomial, generally discontinuous, approximation of the pth order describing a global solution on the whole domain, for a survey see [9], [25].

We proceed as follows. First, we start with a triangulation of a computational domain and define the finite dimensional space S_h^p, which approximates the weighted Sobolev space V in some reasonable sense. Secondly, we derive the space semidiscrete DG formulation of the problem, followed by the fully time-space discrete one and finally, we construct the corresponding linear algebraic problem and end up with the resulting numerical scheme.

3.1. Triangulation. Let $\mathcal{T}_h$ $(h > 0)$ represent a partition of the closure of the computational domain $\overline{\Omega}$ into a finite number of closed elements K (i.e., polygons) with mutually disjoint interiors. We set $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$ and call $\mathcal{T}_h = \{K\}_{K\in\mathcal{T}_h}$ a triangulation of Ω; we do not require the conforming properties from the finite element method, see, e.g., [5].

By $\mathcal{F}_h$ we denote the set of all open edges of all elements $K \in \mathcal{T}_h$. Further, the symbol $\mathcal{F}_h^I$ stands for the set of all $\Gamma \in \mathcal{F}_h$ that are contained in Ω (inner edges), the symbol $\mathcal{F}_h^D$ for the set of all $\Gamma \in \mathcal{F}_h$ such that $\Gamma \subset \Gamma_3$ (Dirichlet edges) and the symbol $\mathcal{F}_h^N$ for the set of all $\Gamma \in \mathcal{F}_h$ such that $\Gamma \subset \Gamma_1 \cup \Gamma_2$ (Neumann edges). Obviously, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N$. For a shorter notation we put $\mathcal{F}_h^{ID} \equiv \mathcal{F}_h^I \cup \mathcal{F}_h^D$ and $\mathcal{F}_h^{DN} \equiv \mathcal{F}_h^D \cup \mathcal{F}_h^N$.

Next, for each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\vec{n}_\Gamma$. We assume that $\vec{n}_\Gamma$, $\Gamma \subset \partial\Omega$, has the same orientation as the outer normal of $\partial\Omega$. For $\vec{n}_\Gamma$, $\Gamma \in \mathcal{F}_h^I$, the orientation is arbitrary but fixed for each edge. For each $\Gamma \in \mathcal{F}_h^I$ there exist two neighboring elements K_+ and K_-. We use a convention that K_- lies in the direction of $\vec{n}_\Gamma$ and K_+ in the opposite direction of $\vec{n}_\Gamma$, see Figure 1 (left).

Over the triangulation $\mathcal{T}_h$ we define the finite dimensional space of discontinuous piecewise polynomial functions

$$S_h^p \equiv S_h^p(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega);\ v|_K \in P_p(K) \quad \forall\, K \in \mathcal{T}_h\}, \tag{3.1}$$

where $P_p(K)$ denotes the space of all polynomials of order less than or equal to p defined on K. Then we seek an approximate solution of the problem (OPP) u_h in the space S_h^p, see the simple example of a such function in Figure 1 (right).

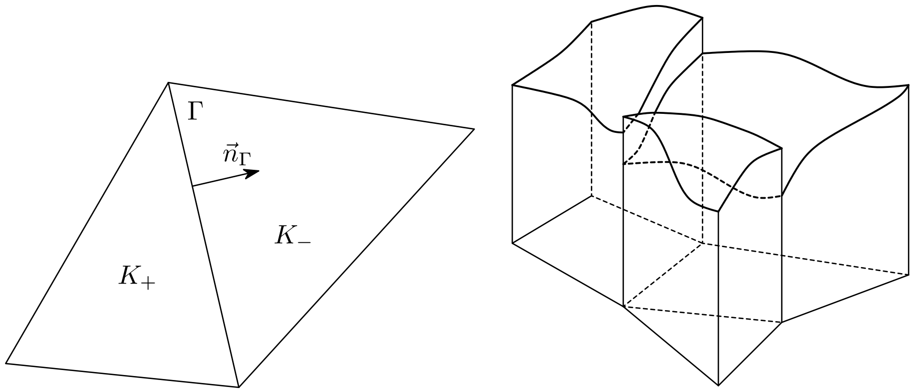

Figure 1. Orientation of a normal to the common edge of two neighbouring elements (left) and an example of a discontinuous piecewise polynomial function (right).

Since we deal with the discontinuous functions along edges Γ, it is suitable to introduce operators of the jump $[\cdot]$ and of the mean value $\langle\cdot\rangle$, defined in the following way:

$$(3.2)\qquad [v] := v|_\Gamma^+ - v|_\Gamma^-, \quad \langle v\rangle := \frac{1}{2}(v|_\Gamma^+ + v|_\Gamma^-),$$

where $v|_\Gamma^+$ is the trace of $v|_{K_+}$ on Γ and $v|_\Gamma^-$ is the trace of $v|_{K_-}$ on Γ, which are different in general. For $\Gamma \in \partial\Omega$, we simply put $\langle v\rangle = [v] = v|_\Gamma^+$.

3.2. Space semidiscrete solution. Now, we recall the space semidiscrete DG scheme presented in [16] with a slight modification for Asian options, cf. [19]. First, we multiply (2.21) by a test function $v_h \in S_h^p$, integrate over an element $K \in \mathcal{T}_h$ and use integration by parts in the diffusion as well as the convection part of (2.21) subsequently. Further, we sum over all $K \in \mathcal{T}_h$ and add some auxiliary terms vanishing for the exact solution such as penalty and stabilization terms, which replace the inter-element discontinuities and guarantee the stability of the resulting numerical scheme, respectively.

Then we employ a concept of an upwind numerical flux for the discretization of the convection term and end up with the following DG formulation for the semidiscrete solution $u_h(\hat{t})$, introduced in [17] as a system of ordinary differential equations:

$$(3.3)\qquad \Big(\frac{\partial u_h(\hat{t})}{\partial \hat{t}}, v_h\Big) + a_h(u_h(\hat{t}), v_h) + b_h(u_h(\hat{t}), v_h) + J_h(u_h(\hat{t}), v_h) + (\gamma(x,\hat{t})u_h(\hat{t}), v_h) = 0 \quad \forall\, v_h \in S_h^p, \quad \forall\, \hat{t} \in (0,T)$$

where

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h}\int_K \mathbb{D}(x)\,\nabla u\cdot\nabla v\,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h^{ID}}\int_\Gamma \langle\mathbb{D}(x)\,\nabla u\cdot\vec{n}_\Gamma\rangle[v]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^{ID}}\int_\Gamma \langle\mathbb{D}(x)\,\nabla v\cdot\vec{n}_\Gamma\rangle[u]\,\mathrm{d}S, \tag{3.4}$$

$$b_h(u,v) = -\sum_{K\in\mathcal{T}_h}\int_K (b_1(x,\hat t), b_2(x,\hat t))u\cdot\nabla v\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^{I}}\int_\Gamma H(u|_\Gamma^+, u|_\Gamma^-, \vec{n}_\Gamma)[v]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^{DN}}\int_\Gamma H(u|_\Gamma^+, u^*|_\Gamma, \vec{n}_\Gamma)[v]_\Gamma\,\mathrm{d}S, \tag{3.5}$$

$$\gamma(x,\hat t) = 3r - q_1 - q_2 - \frac{4\alpha_1x_1 + 4\alpha_2x_2 - 3}{T-\hat t} - \sigma_1^2 - \varrho\sigma_1\sigma_2 - \sigma_2^2. \tag{3.6}$$

The crucial item of the DG formulation of the model problem is the treatment of the convection part. We proceed analogously to [17], where the convection terms are approximated with the aid of the numerical flux $H(\cdot,\cdot)$ through Γ in the direction $\vec{n}_\Gamma = (n_1,n_2)$, i.e.

$$H(u|_\Gamma^+, u|_\Gamma^-, \vec{n}_\Gamma) = \begin{cases} \sum\limits_{i=1}^{2} b_i(x,\hat t)n_i\cdot u\big|_\Gamma^+ & \text{if } A>0,\\ \sum\limits_{i=1}^{2} b_i(x,\hat t)n_i\cdot u\big|_\Gamma^- & \text{if } A\leqslant 0,\end{cases} \tag{3.7}$$

where $A = \sum\limits_{i=1}^{2} b_i(x,\hat t)n_i$ and the function u^* on the boundary edges $\Gamma\in\mathcal{F}_h^{DN}$ has to be chosen according to the prescribed boundary conditions. Here we use

$$u^*|_\Gamma = \begin{cases} 0 & \text{if } \Gamma\in\mathcal{F}_h^D \text{ (homogeneous b.c.)},\\ u|_\Gamma^+ & \text{if } \Gamma\in\mathcal{F}_h^N \text{ (extrapolation)}.\end{cases} \tag{3.8}$$

The numerical flux $H\colon \mathbb{R}^2\to\mathbb{R}$ given by (3.7) is based on the concept of upwinding and one can easily see that it is Lipschitz continuous on any bounded subset of $\mathbb{R}^2$, consistent and conservative, for more details see [11].

A particular attention should be also paid to the treatment of the diffusion terms, which include an artificially added stabilization $\sum\limits_{\Gamma\in\mathcal{F}_h^{ID}}\int_\Gamma\langle\mathbb{D}(x)\nabla v\cdot\vec{n}_\Gamma\rangle[u]\,\mathrm{d}S$, in order to guarantee stability of the numerical scheme. In our case, where this stabilization is added with positive sign, we speak of the nonsymmetric interior penalty Galerkin method.

In the end, the semi-discrete DG scheme is completed with the weighted penalty

$$J_h(u,v) = \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \frac{\sigma_{\min}^2}{2|\Gamma|}[u]\,[v]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \frac{\sigma_{\min}^2}{2|\Gamma|} u\,v\,\mathrm{d}S, \tag{3.9}$$

where $|\Gamma|$ is the length of the edge Γ. The term (3.9) replaces the inter-element discontinuities and guarantees the fulfilment of the prescribed boundary conditions.

In order to simplify the notation we introduce the bilinear form

$$\mathcal{C}_h(u,v) := a_h(u,v) + b_h(u,v) + J_h(u,v) + (\gamma(x,\hat t)u,v) \tag{3.10}$$

for all $u,v \in S_h^p$, $\hat t \in (0,T)$. Consequently, we can define here the semidiscrete solution u_h of the problem (2.38)–(2.39).

Definition 3.1. We say that u_h is a semidiscrete solution of problem (OPP), if $u_h \in C^1(0,T;S_h^p)$ and the following conditions are satisfied:

$$(u_h(0),v_h) = (u^0,v_h) \quad \forall\, v_h \in S_h^p, \tag{3.11}$$

$$\Big(\frac{\partial u_h(\hat t)}{\partial \hat t}, v_h\Big) + \mathcal{C}_h(u_h(\hat t),v) = 0 \quad \forall\, v_h \in S_h^p,\ \forall\, \hat t \in (0,T). \tag{3.12}$$

Let us comment that the problem (3.12) represents a system of ordinary differential equations (ODEs) for the unknown function $u_h(\hat t)$ supplemented with the initial condition (3.11).

Theorem 3.1. *Problem* (3.11)–(3.12) *has a unique semidiscrete solution.*

Proof. Let us introduce $B = \{\varphi_j;\ \mathrm{supp}(\varphi_j) \subset K, K \in \mathcal{T}_h\}_{j=1}^{\mathrm{DOF}} \subset S_h^p$, a standard basis of the space S_h^p, i.e.

$$u_h(\hat t)(x) = \sum_{j=1}^{\mathrm{DOF}} \xi_j(\hat t)\varphi_j(x), \quad x \in \Omega, \tag{3.13}$$

where DOF denotes the number of degrees of freedom. Then we substitute (3.13) into (3.12) choosing $v_h = \varphi_j(x)$. Setting

$$U(\hat t) = (\xi_1(\hat t),\dots,\xi_{\mathrm{DOF}}(\hat t))^{\mathrm{T}}, \quad \mathbf{M} = (\varphi_j,\varphi_i)_{\mathrm{DOF}\times\mathrm{DOF}}, \\ \mathbf{C}(\hat t) = (\mathcal{C}_h(\varphi_j,\varphi_i))_{\mathrm{DOF}\times\mathrm{DOF}}, \tag{3.14}$$

we rewrite (3.12) as follows: Find $U(\hat{t})$ for all $\hat{t} \in (0, T)$ such that

$$\mathbf{M}U'(\hat{t}) = -\mathbf{C}(\hat{t})U(\hat{t}), \tag{3.15}$$

$$U(0) = (\xi_1(0), \ldots, \xi_{\mathrm{DOF}}(0))^{\mathrm{T}}, \tag{3.16}$$

where $U(0)$ is determined by $u_h(0)$ in (3.11). Since the numerical flux H is Lipschitz continuous and the form $\mathcal{C}_h(\cdot, \cdot)$ is bilinear, we prove that the right-hand side of (3.15) is Lipschitz continuous. Thus, by the theory of differential equations, (3.15)–(3.16) has a unique solution (see [12]), and equivalently (OPP) has a unique semidiscrete solution. □

The detailed numerical analysis of the introduced DG method applied to the option pricing problem is available in weighted function spaces under very strong simplifications only, see [1]. On the other hand, there is a lot of results on the convergence of this method in non-weighted function spaces, for survey see [9], [25] and references cited therein.

The basic error estimate for the discontinuous Galerkin method (with nonsymmetric discretization of diffusion terms and with the interior and boundary penalty) for convection-diffusion equations indicates an $O(h^p)$ convergence rate in the L^2-norm and the H^1-seminorm for the pth degree polynomial approximation over a polygonal mesh of size h. However, the optimal $O(h^{p+1})$ rate in the L^2-norm is frequently seen in practise for odd polynomial orders, see [8]. Since (OPP) belongs to the class of convection-diffusion problems, similar results about the asymptotic order of the convergence can be expected for sufficiently regular data.

3.3. Fully time-space discrete solution. In order to obtain the fully time-space discrete DG formulation, we have to discretize also in the temporal variable. There exists a wide range of approaches for the time discretization of ODE systems. In practical computations, the simplest time discretization is via an explicit scheme, which suffers from a limitation on the length of the time step due to a CFL-stability condition. On the other hand, to avoid this time step restriction, it is advantageous to use a fully implicit time discretization, which does not require any additional linearization in our case, cf. [16], [18].

Therefore, the fully discrete solution of problem (3.11)–(3.12) via the backward Euler method is defined in the following way.

Definition 3.2. Let $0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_s = T^* < T$ be a partition of the interval $[0, T^*]$ with the constant time step $\tau = T^*/s$. We define the discrete solution of problem (OPP) as functions $u_h^k \approx u_h(\hat{t}_k)$, $\hat{t}_k \in [0, T^*]$, $k = 0, \ldots, s-1$, satisfying the conditions

$$u_h^0 \quad \text{is } S_h^p\text{-approximation of } u^0, \tag{3.17}$$

$$\frac{1}{\tau}(u_h^{k+1}, v_h) + \mathcal{C}_h(u_h^{k+1}, v_h) = \frac{1}{\tau}(u_h^k, v_h) \quad \forall\, v_h \in S_h^p. \tag{3.18}$$

The discrete problem (3.18) is equivalent to a system of linear algebraic equations at each time level $\hat{t}_k$, which can be solved by a suitable solver.

Theorem 3.2. *Problem* (3.17)–(3.18) *has a unique discrete solution.*

P r o o f. Suppose $u_h^k = 0$. Then the relation (3.18) is equivalent to

$$\begin{aligned}\frac{1}{\tau}(u_h^{k+1}, v_h) &+ a_h(u_h^{k+1}, v_h) + b_h(u_h^{k+1}, v_h) \\ &+ J_h(u_h^{k+1}, v_h) + (\gamma(x, \hat{t}_{k+1})u_h^{k+1}, v_h) = 0 \quad \forall\, v_h \in S_h^p.\end{aligned} \tag{3.19}$$

Since the problem (3.18) is a linear algebraic system, the existence of the solution is implied by the uniqueness. Taking $v_h = u_h^{k+1}$ in (3.19) and neglecting the positive terms $a_h(u_h^{k+1}, u_h^{k+1})$ and $J_h(u_h^{k+1}, u_h^{k+1})$, we obtain

$$\frac{1}{\tau}\|u_h^{k+1}\|^2 \leqslant |b_h(u_h^{k+1}, u_h^{k+1})| + \overline{\gamma_{k+1}}\|u_h^{k+1}\|^2, \tag{3.20}$$

where $\overline{\gamma_{k+1}} = \max_{x\in\overline{\Omega}} |\gamma(x, \hat{t}_{k+1})|$. Since $u_h^{k+1} \in S_h^p$, we use the result from [8], Lemma 4 with a slight modification for the broken weighted spaces and obtain the estimate

$$|b_h(u_h^{k+1}, u_h^{k+1})| \leqslant C\biggl(J_h(u_h^{k+1}, u_h^{k+1})^{1/2} + \Bigl(\sum_{K\in\mathcal{T}_h} |u_h^{k+1}|^2_{V(K)}\Bigr)^{1/2}\biggr)\|u_h^{k+1}\| \tag{3.21}$$

with a constant $C > 0$ independent of u_h^{k+1} and h. Using (3.21) in (3.20), we have

$$\|u_h^{k+1}\|^2 \leqslant \tau C\biggl(J_h(u_h^{k+1}, u_h^{k+1})^{1/2} + \Bigl(\sum_{K\in\mathcal{T}_h} |u_h^{k+1}|^2_{V(K)}\Bigr)^{1/2}\biggr)\|u_h^{k+1}\| + \tau\overline{\gamma_{k+1}}\|u_h^{k+1}\|^2.$$

Further, putting

$$\delta := \tau C\biggl(J_h(u_h^{k+1}, u_h^{k+1})^{1/2} + \Bigl(\sum_{K\in\mathcal{T}_h} |u_h^{k+1}|^2_{V(K)}\Bigr)^{1/2}\biggr) + \tau\overline{\gamma_{k+1}}\|u_h^{k+1}\|, \tag{3.22}$$

then for sufficiently small τ we have

$$\|u_h^{k+1}\| \leqslant \delta\|u_h^{k+1}\| \quad \text{with } \delta \in (0,1), \tag{3.23}$$

which implies $\|u_h^{k+1}\| = 0$, i.e., $u_h^{k+1} \equiv 0$. We prove that for the homogeneous linear algebraic system (3.19) there exists only the trivial solution, i.e., we complete the proof of the existence and uniqueness of the discrete solution of (OPP). □

3.4. Linear algebraic representation. We proceed similarly to [18]. More precisely, according to (3.13), rewriting the discrete DG solution as a linear combination of basis functions, i.e.

$$u_h^k(x) = \sum_{j=1}^{\mathrm{DOF}} \xi_j^k v_j(x), \quad x \in \Omega \tag{3.24}$$

and setting the vector of real coefficients $U_k = \{\xi_j^k\}_{j=1}^{\mathrm{DOF}} \in \mathbb{R}^{\mathrm{DOF}}$, we obtain the sparse matrix equation

$$(\mathbf{M} + \tau\mathbf{C}(\hat{t}_k))U_{k+1} = \mathbf{M}U_k, \tag{3.25}$$

where $\mathbf{M}$ is the symmetric, positive definite block diagonal mass matrix related to L^2-scalar product of basis functions. Using (3.14), the matrix $\mathbf{C}$ corresponds to the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ and has also a sparse pattern. The solvability of linear algebraic problem (3.25) follows from Theorem 3.2.

4. Numerical example

We illustrate the usage of the DG method to the pricing of Asian basket put option assuming a risk management problem of corporate finance. For this purpose we utilize real market data, though with some simplification (see below). All computations are carried out with an algorithm implemented in the solver Freefem++, i.e., a mesh generation/adaptation, the DG discretization, assembly of a linear algebraic problem and its solving. A detailed description can be found in [15].

The computational domain Ω is set large enough to suppress the undesirable effect of the treatment of boundary conditions on the far-field boundary Γ_3 by asymptotic values, usually $x_i^{\max} \approx 2/\alpha_i$. Further, in order to obtain the approximate solution well resolved in the whole computational domain, the mesh is adaptively refined according to the orientation of the vector field induced by physical fluxes (2.23), see Figure 2 (left). Moreover, to eliminate the influence of the approximation of the payoff on the option value, the starting mesh allows to construct the initial condition exactly. For the purpose of comparison the piecewise linear and quadratic polynomial approximations are employed. For simplicity and in line with the option type, we use a constant time step proportional to one day and GMRES as a sparse solver for (3.25) during all simulations.

We consider an Asian basket put option written on two underlying assets, exchange rates of EUR and USD, both with respect to GBP. We assume an air-service firm from the UK, which has fixed prices of the outputs (60% EUR and 40% USD) in the respective currencies for the next month, but receives them on a daily basis. The basket put option allows the firm to transfer the cashflow from local currencies into the home currency (GBP) at favourable conditions (i.e., using either the market price or the strike price) while the Asian feature enables the firm to average the value over the whole month.

The parameter setting comes from real market observations of both currency pairs over last 15 years (2001 to 2015) and fits the option maturity (one month). For the basic setting we use the most commonly observed values. On the other hand, the sensitivity analysis provided in Table 1 and Figure 3 should cover the observed variability of the parameters (standard deviations, correlation).

In particular, we set up the input as follows: maturity time is one month ($T = 1/12$), risk-free interest rates and dividend yields in all currencies are set to zero for a given horizon, volatilities of the exchange rate returns are $\sigma_1 = 0.1$ and $\sigma_2 = 0.15$, respectively, the coefficient of linear correlation of the exchange rate returns is $\varrho = 0.45$ and the reference node given by the closing values of both underlying assets $S_1^{\mathrm{ref}} = 0.83$ and $S_2^{\mathrm{ref}} = 0.75$, respectively, and the average $A^{\mathrm{ref}} = \alpha_1 S_1^{\mathrm{ref}} + \alpha_2 S_2^{\mathrm{ref}}$. The approximate solution is depicted in Figure 2 (right). Since we deal with (Asian basket) put option, the surface of solutions attains the highest value at the point $[0, 0]$. On the other hand, the surface of solutions approaches zero if any of the variables (exchange rates) rises above 2.5.

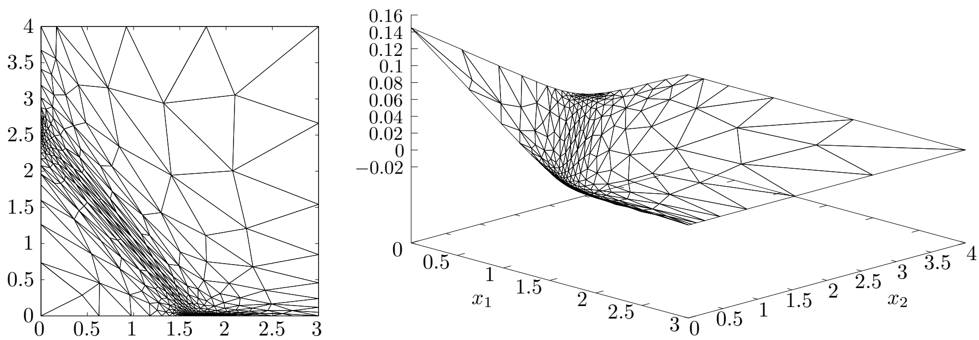

Figure 2. The adaptively refined triangulation with $\#\mathcal{T}_h \approx 1\,500$ (left) and the corresponding piecewise linear DG discrete solution after 30 days (right).

For efficient risk management it is important to know not only the price, but also the sensitivity to input parameters. First, we investigate the sensitivity of the option prices to the volatilities of both the underlying asset returns with a fixed correlation. We generate the option prices using both polynomial orders for five values of $\sigma_i \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ so that we can capture various market conditions (from sleeping market with low rate of information arrival up to busy trading coming from the arrival of many important information). The comparative results evaluated at reference node are presented in Table 1 along with the option values in thousands after 30 days (i.e. $T^* = 30/365$). Obviously, since the weight of the first underlying asset (EUR) is higher (60%), the impact of its volatility on the option price is higher as well. The option prices for linear and quadratic approximations are very close to each other. However, one can observe that the difference between these approximations is decreasing with increasing magnitude of volatilities. This is caused by the character of the transport term (i.e. physical fluxes) in the vicinity of the point $[S_1^{\rm ref}/A^{\rm ref},\ S_2^{\rm ref}/A^{\rm ref}]$.

σ_1 \ σ_2	0.05	0.10	0.15	0.20	0.25
0.05	2.35934 2.25641	3.14940 3.06311	4.01797 3.95030	4.92643 4.87346	5.85842 5.81509
0.10	3.86766 3.79658	4.55077 4.49091	5.32706 5.27593	6.16010 6.11722	7.03085 6.99444
0.15	5.43499 5.39115	6.06573 6.02129	6.77591 6.73512	7.54478 7.50910	8.35751 8.32639
0.20	7.03286 7.00067	7.62733 7.59162	8.29107 8.25682	9.01075 8.97995	9.77543 9.74803
0.25	8.64311 8.61628	9.21087 9.18118	9.84104 9.81180	10.5233 10.4964	11.2495 11.2250

Table 1. Comparison of P_1 (upper values in cells) and P_2 approximations (bottom values in cells) w.r.t. volatilities ($\varrho = 0.45$).

Another important factor, specific to basket options, is the mutual dependence of the two risk sources (foreign exchange rates), which is expressed here by the coefficient of linear correlation. According to Figure 3, it is evident that the relation between the option price and the correlation is positive and almost linear except for strongly negatively correlated exchanges rates when the option price falls down sharply. The reason is that the appreciation of EUR is connected with the depreciation of USD (and vice versa) so that there is hardly any profit from holding the option. Note finally that the difference between the linear and quadratic approximation is rather stable.

Finally, we focus on the structure of the basket, which is defined by weights α_i. We consider seven different scenarios with weights varying from 20% to 80%, and fixed correlation and volatilities. The reference option prices (in thousands at T^*) for all scenarios are recorded in Table 2. It is apparent that the results are not symmetric, i.e., the structure does matter—it is again natural, since both risk sources (foreign exchange rates) have different starting value as well as different volatilities.

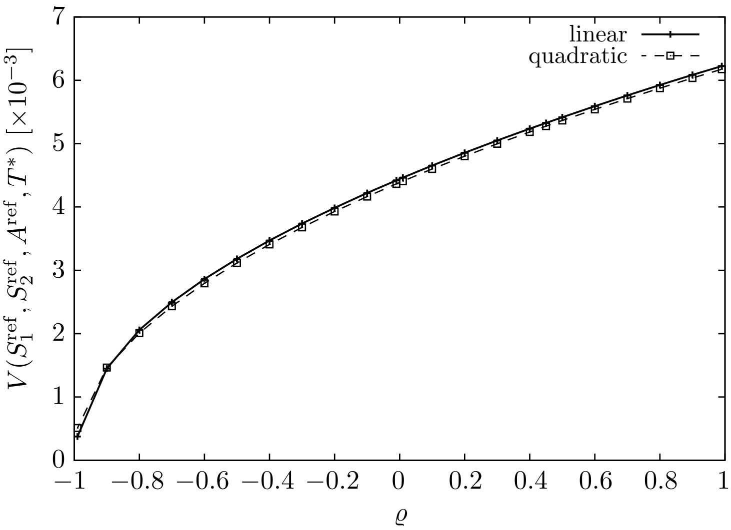

Figure 3. Dependence of option values between particular risk sources given by correlation ϱ for different orders of approximation with fixed $\sigma_1 = 0.1$ and $\sigma_2 = 0.15$.

weights		approximation	
α_1	α_2	P_1	P_2
0.20	0.80	6.53574	6.44199
0.30	0.70	6.11812	6.05392
0.40	0.60	5.76402	5.72306
0.50	0.50	5.51185	5.46052
0.60	0.40	5.32706	5.27593
0.70	0.30	5.24099	5.17977
0.80	0.20	5.19455	5.17476

Table 2. Option prices at reference node $[S_1^{\mathrm{ref}}, S_2^{\mathrm{ref}}, A^{\mathrm{ref}}]$ after 30 days for piecewise linear and quadratic treatment w.r.t. weights.

5. Conclusion

Various options have so specific payoff functions that analytical solution of the PDE system is impossible and one must apply some numerical approximative technique. In this paper we have focused on deriving a numerical approach for the pricing of Asian basket options with floating strike, which is based on the dimensionality reduction and the discontinuous Galerkin framework. We have also presented an illustrative example of two-asset option on two foreign exchange rates, including the sensitivity analysis. From this point of view the proposed numerical scheme provides an option pricing tool which seems to be robust with respect to various types of options as well as different market conditions.

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