DG METHOD FOR THE NUMERICAL PRICING OF TWO-ASSET EUROPEAN-STYLE ASIAN OPTIONS WITH FIXED STRIKE

Jiří Hozman, Liberec, Tomáš Tichý, Ostrava

Received June 30, 2017. First published December 11, 2017.

Abstract. The evaluation of option premium is a very delicate issue arising from the assumptions made under a financial market model, and pricing of a wide range of options is generally feasible only when numerical methods are involved. This paper is based on our recent research on numerical pricing of path-dependent multi-asset options and extends these results also to the case of Asian options with fixed strike. First, we recall the three-dimensional backward parabolic PDE describing the evolution of European-style Asian option contracts on two assets, whose payoff depends on the difference of the strike price and the average value of the basket of two underlying assets during the life of the option. Further, a suitable transformation of variables respecting this complex form of a payoff function reduces the problem to a two-dimensional equation belonging to the class of convection-diffusion problems and the discontinuous Galerkin (DG) method is applied to it in order to utilize its solving potentials. The whole procedure is accompanied with theoretical results and differences to the floating strike case are discussed. Finally, reference numerical experiments on real market data illustrate comprehensive empirical findings on Asian options.

Keywords: option pricing; discontinuous Galerkin method; Asian option; basket option; fixed strike

MSC 2010: 65M60, 35Q91, 91G60, 91G80

1. Introduction

From the financial perspective, one of the most important applications of advanced mathematics is related to pricing of financial derivatives and especially options due to their complex payoff functions (see also [19]). The reason is that proper usage

Both authors were supported through the Czech Science Foundation (GAČR) under project 16-09541S. The support is greatly acknowledged. Furthermore, the second author also acknowledges the support provided within SP2017/32, an SGS research project of VSB—TU Ostrava.

DOI: 10.21136/AM.2017.0176-17

of advanced mathematical methods, in line with no-arbitrage arguments, can help market participants to recover correct price of various products, prevent arbitrage opportunities, increase market balance and finally support economic stability. Without knowing efficient mathematical methods to recover no-arbitrage option prices, the market could not be balanced and price shocks would not be an exception.

Modern option pricing dates back to the seminal papers published in 70's, [3] and [23]. Commonly, the starting point for deriving pricing formulas is the construction of a system of partial differential equations (PDEs) accompanied by boundary and terminal (or initial) conditions, which arise from the no-arbitrage conditions. Unfortunately, if the system is too complex, such as for exotic options, it does not lead to analytical formulas and one should adopt some of the available numerical approximative techniques. These include, for example, Monte Carlo simulation (see e.g. [4]), lattices and trees (originally proposed for option pricing in [7]), finite difference method [30], finite element method [1], or discontinuous Galerkin (DG) method formulated for the first time in [24]. The latter approach, despite some our recent research [16], [17], and [18], remains rather unexplored as concerns option pricing problems and we believe that it might be relevant first of all in case of exotic options with very complex conditions.

In this paper we therefore extend our previous results on the topic ([19], [20]) and focus on the pricing of two-asset Asian options with fixed strike using DG method. There has been substantial research on the pricing of Asian options on one asset, assuming either fixed strike (see e.g. [2] and [27], where a relevant PDE system is solved numerically) or floating strike, which can satisfy a one-dimensional PDE system, similarly to fixed strike options, after numéraire change (see especially [28] and [29]). Moreover, some equivalence of floating and fixed strike Asian options has been proved in [15] and further extended in [10]. However, there are hardly any results on numerical solution of the PDE system for two-asset Asian options, not speaking of the case when DG method is involved.

We proceed as follows. In Section 2 we first define relevant PDE system, including its dimensionality reduction and the initial and boundary conditions, followed by its variational formulation. Next, in Section 3 the DG approximation is developed and finally, in Section 4, two illustrative examples assuming Asian put option with fixed strike are provided. The corresponding steps of the whole presented approach for Asian options with fixed strike are compared to the study of the floating strike case [19] and mutual differences are discussed.

2. PDE model for two-asset Asian options with fixed strike

An option is a special type of financial derivative giving its holder a right to trade an underlying asset S at (European options) or either at or prior to (American options) its maturity time T. The simplest forms of the right to trade the underlying asset are the right to buy it (call option) and the right to sell it (put option) for fixed exercise price $\mathcal{K}$. Due to the simplicity of the payoff function such options are also called plain-vanilla options. If there are some additional conditions concerning the option exercising or determining the payoff, we speak of exotic options. A key feature of options is that the payoff function is nonlinear—any option is striked at the maturity time only if it brings the holder positive cashflow (compare with other types of financial derivatives, such as forwards, futures or swaps, which must be exercised regardless the will of the holder).

In this paper we consider only European-style Asian option contracts on two assets. Such options exist either as put or call options and their value can be expressed as a function of the actual time t, the underlying asset prices $S_1(t)$, $S_2(t)$ and the path-dependent quantity $A(t)$, which represents in some sense the weighted average of the underlying asset prices and can be measured either continuously or discretely. For our purposes we use the continuous arithmetic average defined as

$$A(t) = \frac{1}{t}\int_0^t (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u, \quad \alpha_1 > 0, \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1. \tag{2.1}$$

Let $V = V(S_1(t), S_2(t), A(t), t)$ denote the value of an arithmetic Asian option with continuous sampling (2.1), i.e., S_1, S_2, and A are considered as independent variables. Using the general framework from the derivation of Black-Scholes model based on a construction of the risk-free portfolio containing one unit of an option V and $-\Delta_1$ units of the underlying asset S_1 and $-\Delta_2$ units of the underlying asset S_2, application of the multidimensional Itô's lemma and elimination of stochastic fluctuations by delta hedging (i.e., $\Delta_i = \partial V/\partial S_i$), we obtain the three-dimensional PDE model for pricing Asian option contracts on two assets

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \varrho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \\ &+ (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} + \frac{\alpha_1 S_1 + \alpha_2 S_2 - A}{t}\frac{\partial V}{\partial A} - rV = 0\end{aligned} \tag{2.2}$$

for $t \in (0, T)$, $S_1 > 0$, $S_2 > 0$, and $A > 0$. The detailed derivation of the pricing equation (2.2) is available in [19].

The value of an option is also influenced by several parameters introduced in (2.2), namely volatility $\sigma_i > 0$ and dividend yield $q_i \geqslant 0$ of the asset S_i, risk-free interest rate of return $r \geqslant 0$ and correlation $\varrho \in (-1,1)$ between the underlying assets. For a detailed explanation of these market parameters the reader is referred to the book [13]. Notice that in this study we assume that these model parameters are constant over the life of the option to simplify the problem analysis.

Let us note that (2.2) represents a backward linear parabolic PDE degenerated in variable A as $t \to 0+$, i.e., it tends to the two-factor Black-Scholes equation for pricing standard basket options with two underlying assets.

R e m a r k 2.1. More precisely, neglecting the term $(\alpha_1 S_1 + \alpha_2 S_2 - A)t^{-1}\partial V/\partial A$ in (2.2) comes from the following argument. Based on the definition of the average (2.1), the value of an Asian option V does not depend on the variable A at $t=0$, i.e., $\partial V/\partial A = 0$. Since $(\alpha_1 S_1 + \alpha_2 S_2 - A)/t = (\mathrm{d}A/\mathrm{d}t)(t)$, it is sufficient to prove that $(\mathrm{d}A/\mathrm{d}t)(0)$ is finite. Using a technique similar to that in [19], we put $I(t) = \int_0^t (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u$. Then

$$\begin{aligned}
\text{(2.3)}\qquad \frac{\mathrm{d}A}{\mathrm{d}t}(0) &= \lim_{t\to 0+} \frac{\mathrm{d}A}{\mathrm{d}t}(t) = \lim_{t\to 0+}\Big(\frac{1}{t}\frac{\mathrm{d}I}{\mathrm{d}t}(t) - \frac{1}{t^2}I(t)\Big) \\
&= \lim_{t\to 0+}\frac{1}{t}\Big(\frac{\mathrm{d}I}{\mathrm{d}t}(t) - \frac{I(t)-I(0)}{t}\Big) = \lim_{t\to 0+}\frac{(\mathrm{d}I/\mathrm{d}t)(t) - (\mathrm{d}I/\mathrm{d}t)(0)}{t} \\
&= \frac{\mathrm{d}^2 I}{\mathrm{d}t^2}(0) = \frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1 S_1(t) + \alpha_2 S_2(t))(0) < \infty.
\end{aligned}$$

The last inequality in (2.3) comes from the standard assumptions on a movement of the prices of the underlying assets under pure diffusion processes, i.e., no jumps are allowed, see [6].

Since the option price is exactly known at its maturity time T only, the equation (2.2) is closed by the terminal condition given by the so-called payoff function. Moreover, according to the way in which the average A is incorporated into the payoff function, we distinguish four basic types of Asian options, see Table 1. The classification identifies four subtypes of Asian options in total. An Asian put gives the holder the right to sell a basket of assets for its average price over some prescribed period and an Asian call allows him or her to buy it for this average. The floating strike options have payoff similar to a vanilla option but with the corresponding path-dependent variable A replacing the exercise price $\mathcal{K}$. On the other hand, the payoff of a fixed strike option corresponds to a vanilla option with swapping the basket of the asset prices $\alpha_1 S_1 + \alpha_2 S_2$ for its average A. Notice that the fixed strike options are also known as rate options and the floating strike options as average strike ones.

This slight difference in the structural form of the payoff between floating and fixed strike options has a big impact on their valuation. Whereas Asian options with floating strike admit several simple ways of similarity reductions, see [21], [30] and

payoff V_0	put	call
floating strike	$(A(T) - \alpha_1 S_1(T) - \alpha_2 S_2(T))^+$	$(\alpha_1 S_1(T) + \alpha_2 S_2(T) - A(T))^+$
fixed strike	$(\mathcal{K} - A(T))^+$	$(A(T) - \mathcal{K})^+$

Table 1. Payoff functions for four basic types of Asian options, $(\cdot)^+ = \max(\cdot, 0)$.

our recent paper [19], the pricing of Asian options with the fixed strike is a more complex issue. Therefore, the rest of the paper is oriented on Asian two-asset basket options with fixed strike.

Let $V_0 = V_0(S_1, S_2, A)$ denote one of the terminal data (put or call) from Table 1 (last row), i.e.,

$$V(S_1(T), S_2(T), A(T), T) = V_0(S_1(T), S_2(T), A(T)). \tag{2.4}$$

Taking the conventional assumptions of frictionless markets into account and the martingale theory (see [12]), the solution of PDE (2.2) at time t is equivalent to the expected value of the discounted payoff (2.4), i.e.,

$$V(S_1(t), S_2(t), A(t), t) = \mathrm{e}^{-r(T-t)}\mathbb{E}(V_0(S_1(T), S_2(T), A(T)). \tag{2.5}$$

Summing up, it is necessary to solve the Cauchy problem given by three-dimensional PDE (2.2) and terminal data (2.4). The core of this study lies in a more sophisticated and technically more demanding transformation of variables which reduces the problem to a two-dimensional equation belonging to the class of convection-diffusion problems and also overcomes A-degeneracy of the pricing equation.

2.1. Transformation of variables and reduced problem. The option value V is defined in a spatial (S_1, S_2, A)-domain and these three spatial dimensions of the governing PDE (2.2) increase the complexity of numerical methods incorporated into the process of pricing of such options. In fact, this undesirable feature can be eliminated by a suitable transformation of variables leading to the reduced problem in spatial dimensions decreased by one.

As shown in [19], one possible reduction for floating strike options might be achieved by introducing the new variable $x_i = S_i/A$, $i = 1, 2$. Unfortunately, we cannot use this approach in case of fixed strike options, because the assumption of homogeneity in variable A is not fulfilled for the payoff functions from Table 1 (last row).

In contrast to this and inspired by [27] for Asian option contract on one asset, we have proposed a more sophisticated change of variables accompanied with the forward time running, $\hat{t} = T - t$, i.e.,

$$x_i = x_i(\hat{t}) = \frac{\mathcal{K} - At/T}{S_i} = \frac{\mathcal{K} - A(T-\hat{t})/T}{S_i}, \quad i = 1, 2. \tag{2.6}$$

This transformation respects the form of a payoff function and converts the space-time (S_1, S_2, A, t)-domain, which is a subset of $\mathbb{R}^3_+ \times (0, T)$, to the $(x, \hat{t})$-domain with component $x = [x_1, x_2]$ lying in the first and the third quadrants only, i.e., $\operatorname{sgn}(x_1) = \operatorname{sgn}(x_2)$. Note that the x-domain can be decomposed into two subdomains, which have only the origin $[0, 0]$ as their common point.

Using simple calculation

$$\begin{aligned}\mathcal{K} - A &= \operatorname{sgn}(\mathcal{K} - A)\sqrt{\frac{|\mathcal{K} - A|}{S_1}}\sqrt{\frac{|\mathcal{K} - A|}{S_2}}\sqrt{S_1}\sqrt{S_2} \\ &= \operatorname{sgn}(x_1(0))\sqrt{x_1(0)x_2(0)}\sqrt{S_1S_2} = \operatorname{sgn}(x_2(0))\sqrt{x_1(0)x_2(0)}\sqrt{S_1S_2},\end{aligned} \tag{2.7}$$

we can rewrite the payoff functions from Table 1 (last row) as

$$V_0(S_1, S_2, A) = \sqrt{S_1S_2} \cdot u^0(x), \quad u^0(x) := \begin{cases} (-\operatorname{sgn}(x_1)\sqrt{x_1x_2})^+ & \text{for call,} \\ (\operatorname{sgn}(x_1)\sqrt{x_1x_2})^+ & \text{for put.} \end{cases} \tag{2.8}$$

Further, to get a simplified problem defined (in Subsection 2.3) on a simply connected region, we can use the geometric properties of the x-domain and restrict the studied option pricing problem only on one of the two possible quadrants. Moreover, in order to evaluate the option value at maturity, the point $[x_1(T), x_2(T)] = [\mathcal{K}/S_1, \mathcal{K}/S_2]$ has to be included in this quadrant. Therefore, we consider $x_1 \geqslant 0$ and $x_2 \geqslant 0$ for the rest of the paper. Since $u^0(x)$ has zero value for $x \in \mathbb{R}^2_+$ for call options and its support lies in $\mathbb{R}^2_+$ for puts, it is more convenient to value put options only. Thus, in the following we focus on puts. The treatment for the case of a call option is explained in Subsection 2.4.

Next the put option price transforms into

$$\begin{aligned}V(S_1, S_2, A, t) &= \mathrm{e}^{-r\hat{t}}\mathbb{E}((\mathcal{K} - A)^+) = \sqrt{S_1S_2} \cdot \mathrm{e}^{-r\hat{t}}\mathbb{E}(u^0(x)) \\ &= \sqrt{S_1S_2} \cdot u(x, \hat{t}).\end{aligned} \tag{2.9}$$

and its corresponding partial derivatives are calculated by the chain rule as

$$\frac{\partial V}{\partial t} = -\sqrt{S_1S_2}\Big(\frac{\partial u}{\partial \hat{t}} + \frac{A}{S_1T}\frac{\partial u}{\partial x_1} + \frac{A}{S_2T}\frac{\partial u}{\partial x_2}\Big), \tag{2.10}$$

$$\frac{\partial V}{\partial A} = -\sqrt{S_1S_2}\Big(\frac{T-\hat{t}}{S_1T}\frac{\partial u}{\partial x_1} + \frac{T-\hat{t}}{S_2T}\frac{\partial u}{\partial x_2}\Big), \tag{2.11}$$

$$\frac{\partial V}{\partial S_1} = \frac{1}{2}\sqrt{\frac{S_2}{S_1}}u - \sqrt{\frac{S_2}{S_1}}x_1\frac{\partial u}{\partial x_1}, \quad \frac{\partial V}{\partial S_2} = \frac{1}{2}\sqrt{\frac{S_1}{S_2}}u - \sqrt{\frac{S_1}{S_2}}x_2\frac{\partial u}{\partial x_2}, \tag{2.12}$$

further

$$\frac{\partial^2 V}{\partial S_1^2} = -\frac{1}{4S_1}\sqrt{\frac{S_2}{S_1}}u + \frac{1}{S_1}\sqrt{\frac{S_2}{S_1}}x_1\frac{\partial u}{\partial x_1} + \frac{1}{S_1}\sqrt{\frac{S_2}{S_1}}x_1^2\frac{\partial^2 u}{\partial x_1^2}, \tag{2.13}$$

$$\begin{aligned}\frac{\partial^2 V}{\partial S_1 \partial S_2} = {} & \frac{1}{4\sqrt{S_1S_2}}u - \frac{1}{2\sqrt{S_1S_2}}x_1\frac{\partial u}{\partial x_1} - \frac{1}{2\sqrt{S_1S_2}}x_2\frac{\partial u}{\partial x_2} \\ & + \frac{1}{\sqrt{S_1S_2}}x_1x_2\frac{\partial^2 u}{\partial x_1 \partial x_2},\end{aligned} \tag{2.14}$$

$$\frac{\partial^2 V}{\partial S_2^2} = -\frac{1}{4S_2}\sqrt{\frac{S_1}{S_2}}u + \frac{1}{S_2}\sqrt{\frac{S_1}{S_2}}x_2\frac{\partial u}{\partial x_2} + \frac{1}{S_2}\sqrt{\frac{S_1}{S_2}}x_2^2\frac{\partial^2 u}{\partial x_2^2}. \tag{2.15}$$

Now, substituting (2.9) and partial derivatives (2.10)–(2.15) into (2.2) and dividing by the factor $\sqrt{S_1S_2}$, we obtain the new pricing equation with dimensions reduced by one, i.e.,

$$\begin{aligned}\frac{\partial u}{\partial \hat{t}} & - \frac{1}{2}\sigma_1^2x_1^2\frac{\partial^2 u}{\partial x_1^2} - \varrho\sigma_1\sigma_2x_1x_2\frac{\partial^2 u}{\partial x_1\partial x_2} - \frac{1}{2}\sigma_2^2x_2^2\frac{\partial^2 u}{\partial x_2^2} \\ & + \Big(-\frac{\sigma_1^2}{2} + \frac{\varrho\sigma_1\sigma_2}{2} + r - q_1 + \frac{\alpha_1}{Tx_1} + \frac{\alpha_2}{Tx_2}\Big)x_1\frac{\partial u}{\partial x_1} \\ & + \Big(-\frac{\sigma_2^2}{2} + \frac{\varrho\sigma_1\sigma_2}{2} + r - q_2 + \frac{\alpha_1}{Tx_1} + \frac{\alpha_2}{Tx_2}\Big)x_2\frac{\partial u}{\partial x_2} \\ & + \Big(\frac{\sigma_1^2}{8} - \frac{\varrho\sigma_1\sigma_2}{4} + \frac{\sigma_2^2}{8} + \frac{q_1}{2} + \frac{q_2}{2}\Big)u = 0\end{aligned} \tag{2.16}$$

for $x_1 > 0$, $x_2 > 0$ and $\hat{t} \in (0, T)$. For further numerical treatment it is more suitable to rewrite (2.16) in the nondivergence form as

$$\frac{\partial u}{\partial \hat{t}} - \sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\mathbb{D}(x)\nabla u)_i + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + cu = 0, \tag{2.17}$$

where $(\mathbb{D}(x)\nabla u)_i$ denotes the ith component of vector $\mathbb{D}(x)\nabla u$ with the matrix

$$\mathbb{D}(x) = \begin{pmatrix} d_{11}(x) & d_{12}(x) \\ d_{21}(x) & d_{22}(x) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_1^2x_1^2 & \varrho\sigma_1\sigma_2x_1x_2 \\ \varrho\sigma_1\sigma_2x_1x_2 & \sigma_2^2x_2^2 \end{pmatrix}. \tag{2.18}$$

The vector $(b_1(x), b_2(x))^{\mathrm{T}}$ represents the physical flux with components

$$b_i(x) = \Big(\frac{\sigma_i^2}{2} + \varrho\sigma_1\sigma_2 + r - q_i + \frac{\alpha_1}{Tx_1} + \frac{\alpha_2}{Tx_2}\Big)x_i, \quad i = 1, 2, \tag{2.19}$$

and the nonnegative constant

$$c = \frac{\sigma_1^2}{8} - \frac{\varrho\sigma_1\sigma_2}{4} + \frac{\sigma_2^2}{8} + \frac{q_1}{2} + \frac{q_2}{2} \tag{2.20}$$

stands for the reaction coefficient.

Finally, notice that this technically more demanding transformation of variables in comparison with the approach from [19] is balanced by the fact that the transformation (2.6) overcomes the undesirable degeneracy of (2.2) in variable A and a singularity at $\hat{t} = T$ is not present either, because the convection and reaction parts are independent of time compared with [19].

2.2. Comparison to the single-asset case. In this part we theoretically justify the choice of the change of variables (2.6) by a comparison of the derived pricing equation (2.17)–(2.20) with PDE models of Asian option contracts on one asset, well-known from literature, see [9], [27], [28], and [32].

To compare two similar problems but with different dimensions we have to restrict the more complex problem to the domain of the latter. We start with the reformulation of the equation (2.17)–(2.20) for a domain represented by a general straight line $x_2 = kx_1$ with a positive slope k. Putting

$$z = \frac{\sqrt{x_1 x_2}}{\sqrt{k}} \quad \text{and} \quad W(z,\hat{t}) = u(x_1,x_2,\hat{t}) \quad \text{on } x_2 = kx_1, \tag{2.21}$$

we obtain

$$\begin{aligned} &\frac{\partial u}{\partial \hat{t}} = \frac{\partial W}{\partial \hat{t}}, \quad \frac{\partial u}{\partial x_1} = \frac{z}{2x_1}\frac{\partial W}{\partial z}, \quad \frac{\partial u}{\partial x_2} = \frac{z}{2x_2}\frac{\partial W}{\partial z} \\ &\frac{\partial^2 u}{\partial x_1^2} = \frac{z^2}{4x_1^2}\frac{\partial^2 W}{\partial z^2} - \frac{z}{4x_1^2}\frac{\partial W}{\partial z}, \quad \frac{\partial^2 u}{\partial x_2^2} = \frac{z^2}{4x_2^2}\frac{\partial^2 W}{\partial z^2} - \frac{z}{4x_2^2}\frac{\partial W}{\partial z} \\ &\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{z^2}{4x_1x_2}\frac{\partial^2 W}{\partial z^2} + \frac{z}{4x_1x_2}\frac{\partial W}{\partial z}. \end{aligned} \tag{2.22}$$

Then substituting (2.21)–(2.22) into equation (2.16) and using the relation $x_2 = kx_1$, the equation (2.16) takes now the one-dimensional form (w.r.t. spatial variable z)

$$\begin{aligned} \frac{\partial W}{\partial \hat{t}} &- \Big(\frac{\sigma_1^2}{8} + \frac{\varrho\sigma_1\sigma_2}{4} + \frac{\sigma_2}{8}\Big) z^2 \frac{\partial^2 W}{\partial z^2} \\ &+ \Big(\Big(-\frac{\sigma_1^2}{8} + \frac{\varrho\sigma_1\sigma_2}{4} - \frac{\sigma_2^2}{8} + r - \frac{q_1}{2} - \frac{q_2}{2}\Big) z + \frac{\alpha_1}{T} + \frac{\alpha_2}{kT}\Big)\frac{\partial W}{\partial z} \\ &+ \Big(\frac{\sigma_1^2}{8} - \frac{\varrho\sigma_1\sigma_2}{4} + \frac{\sigma_2^2}{8} + \frac{q_1}{2} + \frac{q_2}{2}\Big) W = 0 \end{aligned} \tag{2.23}$$

for $z > 0$ and $\hat{t} \in (0,T)$. According to (2.8) the corresponding payoff function (for put options) is

$$W^0(z) = (\sqrt{k}z)^+. \tag{2.24}$$

Furthermore, to set the well-posed initial boundary-value problem the equation (2.23) is also completed by appropriate boundary conditions. At $z = 0$, recalling [27] we use the homogeneous Dirichlet type of the boundary condition $W(0, \hat{t}) = 0$. The boundary condition for $z \to \infty$ is commonly evaluated at a sufficiently large point $z^{\max} \gg 0$. Provided that the physical flux induced by the convection part in (2.23) is positively oriented along the z-axis one can set the artificial Neumann boundary condition at the far-field boundary based on the asymptotic behaviour of the Asian option with fixed strike and thus corresponding to the discounted initial condition (2.24), i.e.,

$$\frac{\partial W}{\partial z}(z^{\max}, \hat{t}) = \mathrm{e}^{-r\hat{t}}\sqrt{k}. \tag{2.25}$$

Note that for practical computation the positive orientation of the physical flux is always satisfied. Therefore, the prescribed Neumann boundary condition in the form (2.25) has sense.

Finally, setting $k = 1$ (a diagonal line of the first and the third quadrants), we obtain from (2.6) identical underlying assets $S_1 = S_2$ and weights $\alpha_1 = \alpha_2$. This implies a unit correlation factor $\varrho = 1$ and an equality of volatilities $\sigma_1 = \sigma_2 = \sigma$. Assuming zero dividend yields ($q_1 = q_2 = 0$), then (2.23) can be rewritten as

$$\frac{\partial W}{\partial \hat{t}} - \frac{\sigma^2}{2} z^2 \frac{\partial^2 W}{\partial z^2} + \Bigl(rz + \frac{1}{T}\Bigr)\frac{\partial W}{\partial z} = 0, \tag{2.26}$$

which represents a well-known PDE ruling the price of Asian options studied in several works, e.g. [9], [27] and [32].

2.3. Domain geometry and boundary conditions. Before its numerical treatment the problem (2.17)–(2.20) has to be restricted to a bounded domain $\Omega \times (0, T)$. For this purpose let $x_i^{\max}$ denote the maximal sufficient value of variable x_i. Since $x_1 \neq 0$ and $x_2 \neq 0$ due to (2.6), the set Ω is newly defined as a convex quadrilateral with sides Γ_i, lying on the lines

$$x_1 = x_1^{\max}, \quad x_2 = x_2^{\max}, \quad x_2 = k_1 x_1, \quad x_2 = k_2 x_1, \tag{2.27}$$

with $0 < k_1 < k_2 < \infty$, see Figure 1 with the detailed description of boundary $\overline{\partial\Omega} = \bigcup_{i=1}^{4} \overline{\Gamma_i}$.

Remark 2.2. The choice of artificial boundaries as parts of straight lines with positive slopes (i.e., $x_2 = k_1 x_1$ and $x_2 = k_2 x_1$, $k_2 > k_1$) is essential to guarantee the boundedness of terms x_1/x_2 and x_2/x_1 on Ω, appearing in the components of the

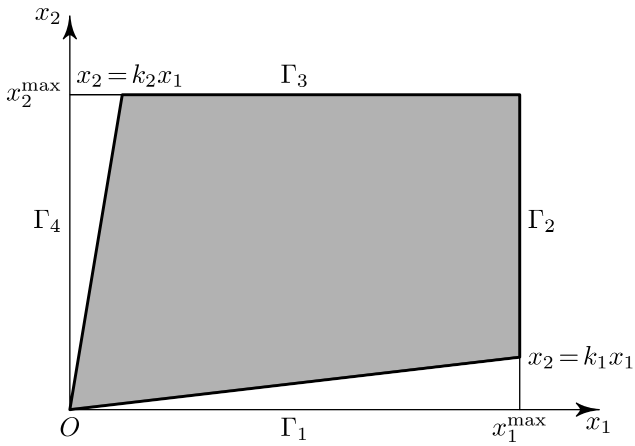

Figure 1. The computational domain Ω and appropriate parts of the boundary Γ_i, $i = 1, \ldots, 4$.

physical flux (2.19). This property is later used for the semidiscrete problem to be well-posed, specifically the integrals in (3.5). Therefore, if we use the curve boundary in the vicinity of the point $[0, 0]$, the above conclusions cannot be used generally.

The second advantage of the quadrilateral domain is its simple connection to the single-asset case from Subsection 2.2 for easy setting of boundary conditions in (2.28).

For the particular situation $x \in \Gamma_1 \cup \Gamma_4$ we simply prescribe Dirichlet boundary conditions

$$u(x, \hat{t}) = u_1(x, \hat{t}) \quad \text{on } \Gamma_1, \quad u(x, \hat{t}) = u_4(x, \hat{t}) \quad \text{on } \Gamma_4, \tag{2.28}$$

where u_1 and u_2 are the solutions of (2.23)–(2.25) with $k = k_1$ and $k = k_2$, respectively. On the far-field boundary, similarly to (2.25), Neumann boundary conditions are reformulated as

$$\frac{\partial u}{\partial x_1}(x, \hat{t}) = \mathrm{e}^{-r\hat{t}} \frac{1}{2} \sqrt{\frac{x_2}{x_1}}, \quad \frac{\partial u}{\partial x_2}(x, \hat{t}) = \mathrm{e}^{-r\hat{t}} \frac{1}{2} \sqrt{\frac{x_1}{x_2}}, \quad x \in \Gamma_2 \cup \Gamma_3. \tag{2.29}$$

Again, note that the couple (2.29) is correctly set if $b_1(x) \geqslant 0$ on Γ_2 and $b_2(x) \geqslant 0$ on Γ_3, which is commonly fulfilled in practice by virtue of (2.19) and constant market parameters considered.

Finally, let us denote by (OPP) the option pricing problem for Asian two-asset basket option with fixed strike formulated as the initial-boundary value problem for unknown function $u(x, \hat{t})\colon \Omega \times (0, T) \to \mathbb{R}$ governed by (2.17)–(2.20) with (2.8) and (2.28)–(2.29).

2.4. Put-call parity for Asian options with fixed strike. One of the most important symmetry results in financial option pricing is the so-called put-call parity. This parity establishes the relationship between the price of a call option and put option, both with the identical underlying assets S_i, strike price $\mathcal{K}$ and expiry T. Such parity relationship is very useful for transferring the prices of one type of option to another and also simplify the analysis and implementation of numerical schemes. This is especially advantageous for Asian options with fixed strike.

Assuming the forward time running $\hat{t}$ and let $V_C(\hat{t})$ and $V_P(\hat{t})$ be the values of the corresponding call option and put option, respectively. At $\hat{t} = 0$ (at an expiration date), these values are given by payoff functions from Table 1 (last row) and their difference is

$$V_C(0) - V_P(0) = (A - \mathcal{K})^+ - (\mathcal{K} - A)^+ = (A - \mathcal{K})^+ - (A - \mathcal{K})^- = A - \mathcal{K}. \tag{2.30}$$

Taking into account the risk-neutral expectations, the difference (2.30) at $\hat{t} = T$ (today) has the form

$$V_C(T) - V_P(T) = \mathrm{e}^{-rT}\mathbb{E}(A - \mathcal{K}) = \mathrm{e}^{-rT}\mathbb{E}(A) - \mathrm{e}^{-rT}\mathcal{K}. \tag{2.31}$$

Using the definition of A from (2.1) and the martingale theory (see [12]), we can write

$$\begin{aligned} \mathbb{E}(A) &= \mathbb{E}\left(\frac{1}{T}\int_0^{\mathrm{T}} (\alpha_1 S_1(u) + \alpha_2 S_2(u))\,\mathrm{d}u\right) \\ &= \frac{\alpha_1}{T}\mathbb{E}\left(\int_0^{\mathrm{T}} S_1(u)\,\mathrm{d}u\right) + \frac{\alpha_2}{T}\mathbb{E}\left(\int_0^{\mathrm{T}} S_2(u)\,\mathrm{d}u\right) \\ &= \frac{\alpha_1}{(r-q_1)T}(\mathrm{e}^{(r-q_1)T} - 1)S_1^{\mathrm{ref}} + \frac{\alpha_2}{(r-q_2)T}(\mathrm{e}^{(r-q_2)T} - 1)S_2^{\mathrm{ref}}, \end{aligned} \tag{2.32}$$

where S_1^{ref} and S_2^{ref} are the actual values of the underlying assets. Finally, (2.31) and (2.32) yield the put-call parity relationship

$$\begin{aligned} &V_C(S_1^{\mathrm{ref}}, S_2^{\mathrm{ref}}, A^{\mathrm{ref}}, T; K, r, q_1, q_2) - V_P(S_1^{\mathrm{ref}}, S_2^{\mathrm{ref}}, A^{\mathrm{ref}}, T; K, r, q_1, q_2) \\ &\quad = \frac{\alpha_1}{(r-q_1)T}(\mathrm{e}^{-q_1T} - \mathrm{e}^{-rT})S_1^{\mathrm{ref}} + \frac{\alpha_2}{(r-q_2)T}(\mathrm{e}^{-q_2T} - \mathrm{e}^{-rT})S_2^{\mathrm{ref}} - \mathrm{e}^{-rT}\mathcal{K}. \end{aligned} \tag{2.33}$$

Note that the afore-mentioned procedure can be easily generalized for option contracts on more than two assets.

We use the established parity (2.33) in this study to evaluate the prices of call options provided the put option prices are known. Therefore, without loss of generality we can focus on put options only.

2.5. Variational problem. In order to define the concept of the weak solution to (OPP), it is necessary to introduce the well-known Lebesgue spaces $L^2(\Omega)$ with the induced norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$ by the standard scalar product $(\cdot,\cdot)$ and $L^\infty(\Omega)$ with the norm $\|\cdot\|_\infty = \operatorname*{ess\,sup}_\Omega |\cdot|$. Moreover, we need the (nonweighted) Sobolev space

$$V(\Omega) \equiv H^1(\Omega) := \{v \in L^2(\Omega)\colon \ \nabla v \in (L^2(\Omega))^2\} \tag{2.34}$$

with a scalar product $(u,v)_V = (u,v) + (\nabla u, \nabla v)$ and induced norm $\|\cdot\|_V = (\cdot,\cdot)_V^{1/2}$. Furthermore, in consistency with the prescribed Dirichlet boundary conditions (2.28) on $\Gamma_1 \cup \Gamma_4$ and to deal with them, we define the space $V_0 := \{v \in V(\Omega)\colon \ v|_{\Gamma_1\cup\Gamma_4} = 0\}$.

The variational formulation of (OPP) is standardly derived in three steps. First we multiply the equation (2.17) by an arbitrary test function $v \in V_0$ and afterwards perform the integration by parts in two dimensions on the second order term. Lastly we use (2.29) and then we obtain

$$\Big(\frac{\partial u}{\partial \hat t}, v\Big) + (\mathcal{L}u, v) = (f(\hat t), v)_{\Gamma_N} \quad \forall\, v \in V_0, \ \text{a.e. } \hat t \in (0,T), \tag{2.35}$$

with the mapping $(\mathcal{L}\cdot,\cdot)\colon V \times V \to \mathbb{R}$ defined via the bilinear form

$$(\mathcal{L}u, v) = \int_\Omega \mathbb{D}(x)\nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{i=1}^2 \int_\Omega b_i(x)\frac{\partial u}{\partial x_i} v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x \tag{2.36}$$

and the right-hand side $f(\hat t)\colon V \to \mathbb{R}$ as

$$\begin{aligned}(f(\hat t), v)_{\Gamma_N} &= \int_{\Gamma_2} g_2(x,\hat t) v \,\mathrm{d}S + \int_{\Gamma_3} g_3(x,\hat t) v \,\mathrm{d}S \\ &= \frac{1}{2}\int_{\Gamma_2} \Big(d_{11}(x)\mathrm{e}^{-r\hat t}\sqrt{\frac{x_2}{x_1}} + d_{12}(x)\mathrm{e}^{-r\hat t}\sqrt{\frac{x_1}{x_2}}\Big) v \,\mathrm{d}S \\ &\quad + \frac{1}{2}\int_{\Gamma_3} \Big(d_{21}(x)\mathrm{e}^{-r\hat t}\sqrt{\frac{x_2}{x_1}} + d_{22}(x)\mathrm{e}^{-r\hat t}\sqrt{\frac{x_1}{x_2}}\Big) v \,\mathrm{d}S.\end{aligned} \tag{2.37}$$

Obviously, $g_2(\hat t) \in L^2(\Gamma_2)$ and $g_3(\hat t) \in L^2(\Gamma_3)$ for fixed $\hat t \in [0,T]$.

Remark 2.3. The difference in the definition of the operator $\mathcal{L}$ compared to approach [19] lies in its time independence and domain in the nonweighted Sobolev space $H^1(\Omega)$. This domain arises from the new condition $\nabla u \in (L^2(\Omega))^2$ in order for all integrals in (2.36) to be well-posed. More precisely, the convection term in (2.36) is defined correctly provided that $b_i(x) \in L^\infty(\Omega)$. Since $k_1 \leqslant x_2/x_1 \leqslant k_2$ from (2.27), this holds true. On the other hand, to work with the weighted Sobolev space, for which $x_i \partial u/\partial x_i \in L^2(\Omega)$, is inapplicable because $b_i(x)/x_i \notin L^\infty(\Omega)$.

R e m a r k 2.4. Since $\mathbb{D}(x)$ is zero at point $[0,0]$, the diffusion part of operator $\mathcal{L}$ is not strictly elliptic in $V(\Omega)$ and the usual arguments for the existence and uniqueness of the weak solution do not apply.

The standard way in the Black-Scholes framework to get around the degeneracy from Remark 2.4 is to turn spatial variables into logarithmic scale using transformation $y_i = \ln x_i$, see [30]. The disadvantage of this approach is that we obtain the problem defined on the unbounded domain $(-\infty, \ln x_1^{\max}) \times (-\infty, \ln x_2^{\max})$ which has to be restricted to a bounded one again in order to solve an initial boundary value problem.

One simple possibility is to set a new domain $(\ln \varepsilon, \ln x_1^{\max}) \times (\ln \varepsilon, \ln x_2^{\max})$ for sufficiently small but fixed $\varepsilon > 0$. Actually, this restriction is equivalent to the treatment of (OPP) on $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(0)}$, where $B_\varepsilon(0) = \{[x_1, x_2] \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < \varepsilon^2\}$. Concerning the new domain Ω_ε we have also the new boundary

$$\overline{\partial\Omega_\varepsilon} = \overline{\Gamma_{0,\varepsilon}} \cup \overline{\Gamma_{1,\varepsilon}} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} \cup \overline{\Gamma_{4,\varepsilon}}, \tag{2.38}$$

where $\Gamma_{1,\varepsilon} = \Gamma_1 \setminus \overline{B_\varepsilon(0)}$, $\Gamma_{4,\varepsilon} = \Gamma_4 \setminus \overline{B_\varepsilon(0)}$ and $\Gamma_{0,\varepsilon} = \{[x_1, x_2] \in \Omega \colon x_1^2 + x_2^2 = \varepsilon^2\}$. Thus we reformulate the Dirichlet boundary conditions (2.28) as

$$\begin{gathered} u(x, \hat{t}) = \varepsilon \sqrt{\frac{x_2}{x_1}} \quad \text{on } \Gamma_{0,\varepsilon}, \\ u(x, \hat{t}) = u_{1,\varepsilon}(x, \hat{t}) \quad \text{on } \Gamma_{1,\varepsilon}, \quad u(x, \hat{t}) = u_{4,\varepsilon}(x, \hat{t}) \quad \text{on } \Gamma_{4,\varepsilon}, \end{gathered} \tag{2.39}$$

where $u_{1,\varepsilon}(x, \hat{t})$ and $u_{2,\varepsilon}(x, \hat{t})$ are solutions of (2.23)–(2.25) on $(\varepsilon, z^{\max})$ with Dirichlet boundary condition $W(\varepsilon, \hat{t}) = \sqrt{k}\varepsilon$ at the left endpoint for $k = k_1$ and $k = k_2$, respectively. Finally, we denote by $(\mathrm{OPP})_\varepsilon$ the original option pricing problem (OPP) restricted to Ω_ε with the swapped Dirichlet boundary conditions (2.28) for (2.39).

In a similar way to V_0, we also introduce a new space

$$V_{0,\varepsilon} = \{v \in V(\Omega_\varepsilon) \colon v|_{\Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup \Gamma_{4,\varepsilon}} = 0\} \tag{2.40}$$

with seminorm $|v|_V = \|\nabla v\|$ as a norm on $V_{0,\varepsilon}$, see the Friedrichs inequality [25]. Using the same technique as in (2.35), we conclude that the operator $\mathcal{L}$ defined on $V_{0,\varepsilon}$ changes only the domains in the volume integrals for Ω_ε and the right-hand side $f(\hat{t})$ remains unchanged.

R e m a r k 2.5. Now, since $b_i(x)/x_i \in L^\infty(\Omega_\varepsilon)$, one can argue that it is also possible to use the framework of a weighted Sobolev space, cf. Remark 2.3, but this treatment will not allow us to later define the semidiscrete and discrete solutions on the whole domain Ω, see Subsections 3.2–3.3.

Finally, taking Remarks 2.3–2.4 into account we can proceed to the variational formulation of $(\mathrm{OPP})_\varepsilon$ and present the abstract theory of parabolic equations giving the existence and uniqueness of such a weak solution.

Definition 2.1. Assume that there exists a function $u_D \in C^0([0,T];L^2(\Omega_\varepsilon)) \cap L^2(0,T;V(\Omega_\varepsilon)$ with traces from (2.39) and $\partial u_D/\partial\hat{t} \in L^2(0,T;V'_{0,\varepsilon})$. The variational formulation of $(\mathrm{OPP})_\varepsilon$ is as follows: Find $u \in C^0([0,T];L^2(\Omega_\varepsilon)) \cap L^2(0,T;V(\Omega_\varepsilon))$ such that $u - u_D \in C^0([0,T];L^2(\Omega_\varepsilon)) \cap L^2(0,T;V_{0,\varepsilon})$ and $\partial u/\partial\hat{t} \in L^2(0,T;V'_{0,\varepsilon})$ satisfies

$$(u(0),v) = (u^0,v) \quad \forall\, v \in V_0^\varepsilon, \tag{2.41}$$

$$\Big(\frac{\partial u}{\partial\hat{t}}(\hat{t}),v\Big) + (\mathcal{L}u(\hat{t}),v) = (f(\hat{t}),v)_{\Gamma_N} \quad \forall\, v \in V_0^\varepsilon,\ \text{a.e. } \hat{t} \in (0,T). \tag{2.42}$$

where $u(\hat{t})$ denotes such function on Ω_ε that $u(\hat{t})(x),\ x \in \Omega_\varepsilon$.

Theorem 2.1. *Problem* (2.41)–(2.42) *has a unique weak solution.*

P r o o f. Since the proof procedure is similar to the approach from [19], Theorem 2.1, with respect to the nonweighted space $V(\Omega_\varepsilon) = H^1(\Omega_\varepsilon)$ it is enough to list several results (inequalities) guaranteeing that there exist three positive constants $c_L \leqslant C_L$, C_f, and a constant $\lambda \geqslant 0$ such that for all $u,v \in V_{0,\varepsilon}$, $\hat{t} \in [0,T]$,

$$|(\mathcal{L}u,v)| \leqslant C_L|u|_V|v|_V, \tag{2.43}$$

$$(\mathcal{L}u,u) \geqslant c_L|u|_V^2 - \lambda\|u\|^2, \tag{2.44}$$

$$(f(\hat{t}),v)_{\Gamma_N} \leqslant C_f|v|_V. \tag{2.45}$$

First, assuming invariable model parameters there exist two positive constants $0 < D_{\min} \leqslant D_{\max}$ such that the matrix $\mathbb{D}(x)$ satisfies for all $\zeta \in \mathbb{R}^2$, $\zeta \neq 0$,

$$D_{\min}|\zeta|^2 \leqslant \zeta^{\mathrm{T}}\mathbb{D}(x)\zeta \leqslant D_{\max}|\zeta|^2 \quad \forall\, x \in \overline{\Omega_\varepsilon}. \tag{2.46}$$

Next, one can easily estimate analogously to [19], Lemma 2.3,

$$\Big|\int_{\Omega_\varepsilon} \mathbb{D}(x)\nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\Omega_\varepsilon} cuv\,\mathrm{d}x\Big| \leqslant D_{\max}|u|_V|v|_V + c\|u\|\|v\|, \tag{2.47}$$

$$\Big|\sum_{i=1}^{2}\int_{\Omega_\varepsilon} b_i(x)\frac{\partial u}{\partial x_i}v\,\mathrm{d}x\Big| \leqslant C_b|u|_V\|v\|, \quad C_b > 0. \tag{2.48}$$

Then (2.47)–(2.48) together with the relation $\|\cdot\|_V^2 = \|\cdot\|^2 + |\cdot|_V^2$ and the norm equivalence give (2.43), i.e., the boundedness of the mapping $(\mathcal{L}\cdot,\cdot)$.

Secondly, we proceed as in [19], Lemma 2.4, and prove estimates

$$\left|\int_{\Omega_\varepsilon} \mathbb{D}(x)\nabla u\cdot\nabla u\,\mathrm{d}x\right| \geqslant D_{\min}|u|_V^2, \tag{2.49}$$

$$\left|\sum_{i=1}^{2}\int_{\Omega_\varepsilon} b_i(x,\hat t)\frac{\partial u}{\partial x_i}u\,\mathrm{d}x\right| \leqslant \frac{D_{\min}}{2}|u|_V^2 + C_Y\|u\|^2, \quad C_Y>0, \tag{2.50}$$

which imply the Gårding inequality (2.44), i.e., a sufficient condition for the strong ellipticity of the operator $\mathcal{L}$.

Further, from (2.37) one can easily deduce that $f(\hat t)$, $\hat t\in[0,T]$, is a continuous linear functional on $V_{0,\varepsilon}$, see the following sequence of inequalities

$$(f(\hat t),v)_{\Gamma_N} \leqslant \|f(\hat t)\|_{L^2(\Gamma_2\cup\Gamma_3)}\|v\|_{L^2(\Gamma_2\cup\Gamma_3)} \leqslant \|f(\hat t)\|_{L^2(\Gamma_2\cup\Gamma_3)}C_\gamma\|v\|_V, \tag{2.51}$$

where C_γ is the continuity constant of the trace mapping [22]. Using norm equivalence one concludes that estimate (2.45) holds.

The rest of the proof is based on the main theorem on first-order linear evolution equations, see [31]. Since $V(\Omega_\varepsilon)$ is densely embedded in $L^2(\Omega_\varepsilon)$ (see [22]), these spaces form a Gelfand triple $(V(\Omega_\varepsilon), L^2(\Omega), V(\Omega_\varepsilon)')$. This fact together with estimates (2.43)–(2.45) guarantees the fulfillment of the assumptions, which allow us to apply the abstract theory of variational parabolic problems and prove the existence of a unique weak solution of problem $(\mathrm{OPP})_\varepsilon$. □

3. Discontinuous Galerkin method

In a wide class of pricing problems resulting in a solution of partial differential equations, numerical methods are rather popular, especially if there exist no analytical pricing formulae in general. As was shown in Section 2, the pricing equation has a convection-diffusion character, which is very often the reason of many numerical difficulties, e.g., occurrence of spurious oscillations in discrete solutions. Therefore, the DG method is applied to problem (OPP) in order to utilize the potential of this method for solving such problems and enable better resolving of discrete option prices with respect to computational meshes as well as polynomial approximation degrees. This technique is based on piecewise polynomial, but generally discontinuous, approximation of the pth order describing a global solution on the whole domain, for a survey see [8] and [26].

In what follows we mention the routine steps, from a triangulation of Ω and a construction of the finite dimensional space S_h^p, over the space semidiscretization and time discretization to the final form of the numerical scheme for pricing Asian options with fixed strike.

3.1. Finite-dimensional space of discontinuous functions. Let $\mathcal{T}_h = \{K\}_{K\in\mathcal{T}_h}$, $h = \max\limits_{K\in\mathcal{T}_h} \operatorname{diam}(K)$, represent a triangulation of $\overline{\Omega}$ into a finite number of elements K. We do not put any conforming properties on the triangulation $\mathcal{T}_h$ as in the finite element method, e.g. [5]. Therefore, by $\mathcal{F}_h$ we denote the set of all open edges and parts of edges (resulting from hanging nodes) of all elements $K \in \mathcal{T}_h$. According to their position, we distinguish sets of inner edges $\mathcal{F}_h^I$, boundary edges of Dirichlet type $\mathcal{F}_h^D$, and boundary edges of Neumann type $\mathcal{F}_h^N$. Obviously, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N$. For shorter notation we put $\mathcal{F}_h^{ID} \equiv \mathcal{F}_h^I \cup \mathcal{F}_h^D$ and $\mathcal{F}_h^{DN} \equiv \mathcal{F}_h^D \cup \mathcal{F}_h^N$.

Secondly, for each $\Gamma \in \mathcal{F}_h$, we define a unit normal vector $\vec{n}_\Gamma$ with the following orientation. For $\Gamma \subset \partial\Omega$, $\vec{n}_\Gamma$ is oriented as the outward normal to $\partial\Omega$. If $\Gamma \in \mathcal{F}_h^I$, then $\Gamma \subset K_+ \cap K_-$ and we set $\vec{n}_\Gamma$ to be the outward normal to K_+ lying in K_-.

The approximate solution of the problem (OPP) is sought in the finite dimensional space of discontinuous piecewise polynomial functions, defined over the triangulation $\mathcal{T}_h$ as

$$S_h^p \equiv S_h^p(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega)\colon\ v|_K \in P_p(K)\ \forall\, K \in \mathcal{T}_h\}, \tag{3.1}$$

where $P_p(K)$ denotes the space of all polynomials of order less than or equal to p defined on K. For each function $v \in S_h^p$ restricted to $\Gamma \in \mathcal{F}_h^I$, we distinguish two traces $v|_\Gamma^+ = (v|_{K_+})|_\Gamma$ and $v|_\Gamma^- = (v|_{K_-})|_\Gamma$. Moreover, $[v]_\Gamma = v|_\Gamma^+ - v|_\Gamma^-$ and $\langle v\rangle_\Gamma = \frac{1}{2}(v|_\Gamma^+ + v|_\Gamma^-)$ denote the jump and the mean value of the function v over the edge Γ, respectively. For $\Gamma \subset \partial\Omega$, we simply put $\langle v\rangle_\Gamma = [v]_\Gamma = v|_\Gamma^+$.

3.2. Space semidiscretization. We recall the DG framework for the space semidiscrete formulation of (OPP). The complete derivation for Asian option contract with the floating strike on two underlying assets can be found in [19]. Here we apply this approach to the case of options with the fixed strike and introduce the semi-discrete solution $u_h(\hat{t}) \in S_h^p$ represented by the system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}}{\mathrm{d}\hat{t}}(u_h(\hat{t}), v_h) + \mathcal{C}_h(u_h(\hat{t}), v_h) = l_h(v_h)(\hat{t}) \quad \forall\, v_h \in S_h^p,\ \forall\, \hat{t} \in (0,T), \tag{3.2}$$

where the form $\mathcal{C}_h(\cdot,\cdot)$ stands for the semi-discrete variant of the operator $\mathcal{L}$ from (2.36) accompanied with penalties and stabilizations, and the right-hand side form $l_h(\cdot)$ enforces the fulfilment of boundary conditions. More precisely, one can easily decompose $\mathcal{C}_h$ into

$$\mathcal{C}_h(u,v) = a_h(u,v) + b_h(u,v) + (\gamma u, v) + J_h(u,v), \tag{3.3}$$

where

$$\begin{aligned}
(3.4)\qquad a_h(u,v) = {} & \sum_{K\in\mathcal{T}_h} \int_K \mathbb{D}(x)\nabla u\cdot\nabla v \,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \langle \mathbb{D}(x)\nabla u\cdot\vec{n}_\Gamma\rangle\,[v]\,\mathrm{d}S \\
& + \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \langle \mathbb{D}(x)\,\nabla v\cdot\vec{n}_\Gamma\rangle[u]\,\mathrm{d}S, \\
(3.5)\qquad b_h(u,v) = {} & -\sum_{K\in\mathcal{T}_h} \int_K (b_1(x),b_2(x))u\cdot\nabla v\,\mathrm{d}x \\
& + \sum_{\Gamma\in\mathcal{F}_h^{I}} \int_\Gamma H(u|_\Gamma^+, u|_\Gamma^-, \vec{n}_\Gamma)[v]_\Gamma\,\mathrm{d}S \\
& + \sum_{\Gamma\in\mathcal{F}_h^{DN}} \int_\Gamma H(u|_\Gamma^+, u^*|_\Gamma, \vec{n}_\Gamma)[v]_\Gamma\,\mathrm{d}S, \\
(3.6)\qquad \gamma(x) = {} & -2r + \frac{3}{2}q_1 + \frac{3}{2}q_2 - \frac{\alpha_1}{Tx_1} - \frac{\alpha_2}{Tx_2} - \frac{3}{8}\sigma_1^2 - \frac{9}{4}\varrho\sigma_1\sigma_2 - \frac{3}{8}\sigma_2^2, \\
(3.7)\qquad J_h(u,v) = {} & \frac{\min^2(\sigma_1,\sigma_2)}{2} \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \frac{1}{|\Gamma|}[u][v]\,\mathrm{d}S, \quad |\Gamma| = \mathrm{meas}_1(\Gamma).
\end{aligned}$$

Notice that term (3.4) contains the so-called nonsymmetric variant of interior penalty Galerkin stabilization (cf. [16]) and the concept of the upwinding is used in the treatment of the numerical flux H in (3.5), i.e.,

$$H(u|_\Gamma^+, u|_\Gamma^-, \vec{n}_\Gamma) = \begin{cases} \sum_{i=1}^2 b_i(x)n_i\cdot u|_\Gamma^+, & \text{if } c_s > 0, \\ \sum_{i=1}^2 b_i(x)n_i\cdot u|_\Gamma^-, & \text{if } c_s \leqslant 0, \end{cases} \qquad \vec{n}_\Gamma = (n_1,n_2)^{\mathrm{T}}, \tag{3.8}$$

where the characteristic speed $c_s = \sum_{i=1}^2 b_i(x)n_i$ and the function u^* on boundary edges $\Gamma\subset\partial\Omega$ is given by (2.28), if $\Gamma\in\mathcal{F}_h^D$, and extrapolated from interior of Ω, i.e., $u^* = u|_\Gamma^+$, if $\Gamma\in\mathcal{F}_h^N$. This extrapolation is meaningful provided that $\sum_{i=1}^2 b_i(x)n_i \geqslant 0$ on each $\Gamma\in\mathcal{F}_h^N$, which is commonly valid in practice due to (2.19) and the market parameters considered.

Finally, the right-hand side form l_h contains terms arising from the boundary conditions, i.e.,

$$
\begin{aligned}
(3.9) \qquad l_h(v)(\hat{t}) = &\sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \mathbb{D}(x)\nabla v\cdot\vec{n}_\Gamma u^*(\hat{t})\,\mathrm{d}S \\
&+ \frac{\min^2(\sigma_1,\sigma_2)}{2}\sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \frac{1}{|\Gamma|}u^*(\hat{t})v\,\mathrm{d}S \\
&+ \sum_{\substack{\Gamma\in\mathcal{F}_h^N\\ \Gamma\subset\Gamma_2}}\int_\Gamma g_2(\hat{t})v\,\mathrm{d}S + \sum_{\substack{\Gamma\in\mathcal{F}_h^N\\ \Gamma\subset\Gamma_3}}\int_\Gamma g_3(\hat{t})v\,\mathrm{d}S,
\end{aligned}
$$

where u^* is given by (2.28) and g_2, g_3 by (2.37).

Now, we are ready to define the semidiscrete solution of problem (OPP) and prove its existence and uniqueness.

Definition 3.1. We say that u_h is a semidiscrete solution of problem (OPP), if $u_h \in C^1(0,T;S_h^p)$ and the following conditions are satisfied:

$$
(3.10) \qquad (u_h(0),v_h) = (u^0,v_h) \quad \forall\, v_h\in S_h^p,
$$

$$
(3.11) \qquad \Big(\frac{\partial u_h(\hat{t})}{\partial \hat{t}}, v_h\Big) + \mathcal{C}_h(u_h(\hat{t}),v) = l_h(v_h)(\hat{t}) \quad \forall\, v_h\in S_h^p,\ \forall\,\hat{t}\in(0,T).
$$

Theorem 3.1. *Problem* (3.10)–(3.11) *has a unique semidiscrete solution.*

P r o o f. Since problem (3.11) represents a system of ODEs, the proof follows the standard knowledge from the theory of differential equations, see e.g. [11]. Assuming $u_h \in C^1(0,T;S_h^p)$, the existence of the solution is ensured by the boundedness of the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ on $S_h^p\times S_h^p$ and the regularity of functions u^*, g_2, and g_3 occurring in the form l_h. Moreover, the uniqueness of the solution arises from Lipschitz continuity of the right-hand side of the given system of ODEs, especially the assumption of Lipschitz continuity of the numerical flux H has to be taken into account. The analogous proof with $l_h\equiv 0$ can be found in [19], Theorem 3.1. □

3.3. Time discretization. Our aim is to present the high-order scheme also with respect to the time coordinate $\hat{t}$. Therefore, we introduce the numerical scheme for the time discretization based on the trapezoidal rule and giving the second order convergence in time. Actually, it is the average of forward and backward Euler scheme in time, well-known as the Crank-Nicolson method, which is practically unconditionally stable without any restrictive condition on the length of the time step. However, to avoid nonphysical oscillations in the discrete solution the choice of the time step has to depend on the mesh size h, see Section 4 for practical setting.

Consequently, using the bilinearity of $\mathcal{C}_h$ we define the discrete solution of problem (OPP) in the following way. The existence of a unique discrete solution is also proven.

Definition 3.2. Let $0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_s = T$ be a partition of the interval $[0, T]$ with constant time step $\tau = T/s$. We define the discrete solution of problem (OPP) as functions $u_h^k \approx u_h(\hat{t}_k)$, $\hat{t}_k \in [0, T]$, $k = 0, \ldots, s-1$, satisfying the conditions

$$u_h^0 \text{ is the } S_h^p\text{-approximation of } u^0, \tag{3.12}$$

$$\begin{aligned} &(u_h^{k+1}, v_h) + \frac{\tau}{2}\mathcal{C}_h(u_h^{k+1}, v_h) \\ &\quad = (u_h^k, v_h) - \frac{\tau}{2}\mathcal{C}_h(u_h^k, v_h) + \frac{\tau}{2}(l_h(v_h)(\hat{t}_{k+1}) + l_h(v_h)(\hat{t}_k)) \quad \forall\, v_h \in S_h^p. \end{aligned} \tag{3.13}$$

Theorem 3.2. *Problem* (3.12)–(3.13) *has a unique discrete solution.*

P r o o f. The proof follows the steps from [19], Theorem 3.2, where an implicit Euler scheme is considered. The essential ingredient of the proof is the ellipticity of the form $(\cdot,\cdot) + \frac{\tau}{2}\mathcal{C}_h(\cdot,\cdot)$ of the left-hand side of (3.13). This property comes from upper estimates for the form b_h and taking sufficiently small $\tau > 0$. Hence, the discrete problem has a unique solution. □

3.4. Numerical scheme. The discrete problem (3.13) is equivalent to a system of linear algebraic equations at each time level $\hat{t}_k$ and can always be expressed in a matrix form, cf. [17] and [18]. Indeed, setting the vector of real coefficients $U_k = \{\xi_j^k\}_{j=1}^{\mathrm{DOF}} \in \mathbb{R}^{\mathrm{DOF}}$ such that

$$u_h^k(x) = \sum_{j=1}^{\mathrm{DOF}} \xi_j^k v_j(x), \quad x \in \Omega, \quad S_h^p = \operatorname{span}(v_1, \ldots, v_{\mathrm{DOF}}), \tag{3.14}$$

where DOF denotes the number of degrees of freedom (corresponding to the dimension of S_h^p), one can rewrite (3.13) as

$$\Big(\mathbf{M} + \frac{\tau}{2}\mathbf{C}\Big)U_{k+1} = \Big(\mathbf{M} - \frac{\tau}{2}\mathbf{C}\Big)U_k + \frac{\tau}{2}(F_{k+1} + F_k), \tag{3.15}$$

where the matrix $\mathbf{M}$ is related to the mass matrix, the matrix $\mathbf{C}$ to the form $\mathcal{C}_h$ and the vector F_k represents the right-hand side form $l_h(\hat{t}_k)$. Finally, let us mention that the DG solution u_h^{k+1} at each time level is uniquely determined by the solution vector U_{k+1}, which is usually computed by a suitable sparse solver. Since the system matrix in (3.15) is nonsymmetric due to (3.4) and (3.5), GMRES solver is used in the forthcoming numerical experiments. Because of the relatively small number of degrees of freedom in practical computations (i.e., DOF $< 10^5$), no preconditioner is employed.

The treatment in nonweighted function spaces allows us to use standard techniques for the theoretical analysis of the presented numerical scheme. We recall the derived error estimates for the scalar linear convection-diffusion equations, which include the studied problem (OPP), cf. [8], [26] and references cited therein.

In the case of sufficiently regular data and using the nonsymmetric discretization of diffusion terms (3.4) and with the interior and boundary penalty (3.7) together with the Crank-Nicolson method, a priori error estimates guarantee an $\mathcal{O}(h^p + \tau^2)$ convergence rate in the $L^\infty(L^2)$-norm and the energy norm for pth degree polynomial approximation over a polygonal space-time domain of mesh size h and equidistant time stepping τ.

These theoretical results are confirmed by the observed experimental order of convergence in the energy norm. And therewithal, a better behaviour of the experimental L^2-order of convergence is signalized, which is expected to be asymptotically $\mathcal{O}(h^{p+1})$ for odd polynomial order p, see book [8]. The second-order accuracy in time with respect to both types of norms is preserved.

4. Numerical examples

The numerical experiments presented in this section demonstrate the potency of the DG method in problems arising from the PDE approach applied to the option valuation. Throughout this section, we numerically price Asian basket put options with fixed strike. In the first experiment we numerically justify the derived numerical scheme on reference option prices while the second example provides experimental insight into the mutual comparison of Asian options with floating as well as fixed strikes on real market data. The whole implementation of the proposed numerical scheme is done in the solver Freefem++, for more details see [14].

4.1. Comparison to one-dimensional case. As the first numerical experiment we consider the reference model problem for fixed strike Asian call with only one underlying asset, frequently presented in vast literature, such as [9], [28], and [32]. In these works an experiment with model parameters $r = 0.15$, $S^{\rm ref} = 100.0$ and $T = 1.0$ is reported as the most difficult case. Moreover, the impact of different volatilities σ and strikes $\mathcal{K}$ is investigated. To be consistent with the aforementioned one-dimensional case we assume that

$$
\begin{gathered}
q_1 = q_2 = 0, \quad \alpha_1 = \alpha_2 = 0.5, \\
S_1^{\rm ref} = S_2^{\rm ref} = A^{\rm ref} = 100.0, \quad \varrho = 1.0,
\end{gathered}
\tag{4.1}
$$

where the unit correlation parameter ϱ corresponds to a perfect positive linear relationship between the two underlying assets S_1 and S_2 with volatilities of the same value σ. This is necessary to be able to compare the two-factor case with the one-factor one.

In order to obtain reliable comparison we employ piecewise quadratic approximations and similar mesh size $h \approx 0.02$ with $x_1^{\max} = x_2^{\max} = 2.0$ and the same time step $\tau = 1/400$ are used, as in [28]. The boundaries Γ_1 and Γ_4 are specified by $k_1 = 1/10$ and $k_2 = 10$ in (2.27). Moreover, we add an artificial edge lying on $x_2 = x_1$ to the triangulation to get a 2D mesh consistent with the diagonal cut to which the solution is restricted for easy comparison with 1D results.

In fact, we compute the discrete price of a put option, which is evaluated at the reference node for maturity T according to the relation (2.9) as

$$V_P(S_1^{\text{ref}}, S_2^{\text{ref}}, A^{\text{ref}}, T) = \sqrt{S_1^{\text{ref}} S_2^{\text{ref}}} \cdot u_h(\mathcal{K}/S_1^{\text{ref}}, \mathcal{K}/S_2^{\text{ref}}, T). \tag{4.2}$$

Then the value of the corresponding call option is given by put-call parity (2.33). The discrete solution u_h at maturity is depicted in Figure 2 (left) and the transformed call option values V_C along the diagonal cut in Figure 2 (right).

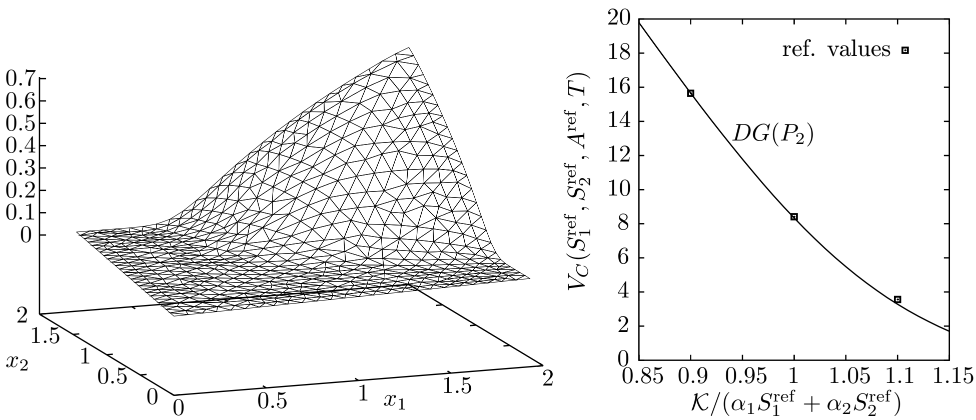

Figure 2. The case $\sigma_1 = \sigma_2 = 0.20$: The piecewise quadratic DG discrete solution u_h at maturity on mesh $\#\mathcal{T}_h \approx 1\,500$ (left) and the corresponding diagonal cut of the transformed call option (right).

The comparative results evaluated at the reference nodes with different volatilities and strikes are presented in Table 2. One can easily observe that the DG approach for piecewise quadratic approximations gives fairly the same results as the finite difference method in [28] and [32]. In particular, our results are mostly only slightly above those presented in [28], though we can observe some impact of various input parameters. This fact is also illustrated in Figure 2 (right), since the difference is more apparent in the right corner of the cut. However, since each resulting price obtained via the DG approach falls within the range specified in [27], it should also fulfil the no-arbitrage conditions. Note finally that changing the domain Ω by putting $k_1 \ll 1/10$ and $k_2 \gg 10$ does not improve the results significantly.

		put	call	ref. call values		bounds [27]	
σ	$\mathcal{K}$	DG(P_2)		[28]	[32]	lower	upper
	95	0.0115	11.1056	11.094	11.094	11.094	11.114
0.05	100	0.0044	6.7949	6.795	6.793	6.794	6.810
	105	0.2604	2.7474	2.744	2.744	2.744	2.761
	90	0.0093	15.4069	15.399	15.399	15.399	15.445
0.10	100	0.2493	7.0400	7.029	7.030	7.028	7.066
	110	3.2581	1.4416	1.415	1.410	1.413	1.451
	90	0.2541	15.6517	15.643	15.643	15.641	15.748
0.20	100	1.6223	8.4129	8.412	8.409	8.408	8.515
	110	5.3771	3.5610	3.560	3.554	3.554	3.661
	90	1.1174	16.5150	16.516	16.514	16.512	16.732
0.30	100	3.4232	10.2138	10.215	10.210	10.208	10.429
	110	7.5458	5.7293	5.736	5.729	5.728	5.948

Table 2. Comparison of DG discrete option prices w.r.t. different methods for fixed strike Asian call.

4.2. Comparison with floating strike case. As the second numerical experiment we consider reference benchmark on real market data from [19]. Our aim is to investigate the behaviour of Asian basket put option with fixed strike written on two underlying assets and compare the results with the case of the option with floating strike, namely the sensitivity to volatilities and correlation factors.

The financial background of this experiment was specified in more detail in [19]. Here we only recall that the underlying assets are represented by the exchange rates of EUR and USD, both with respect to GBP, and the structure of the basket is fixed (60% EUR and 40% USD) in the respective currencies for the prescribed period of one month. The fixed model parameters are set up as

$$T = 30/365, \quad r = 0.0, \quad q_1 = q_2 = 0.0, \quad \alpha_1 = 0.6, \quad \alpha_2 = 0.4, \tag{4.3}$$

and the reference node $[S_1^{\text{ref}}, S_2^{\text{ref}}, A^{\text{ref}}]$ is given by the closing values of both the underlying assets and their weighted average

$$S_1^{\text{ref}} = 0.83, \quad S_2^{\text{ref}} = 0.75, \quad A^{\text{ref}} = \alpha_1 S_1^{\text{ref}} + \alpha_2 S_2^{\text{ref}} = 0.798. \tag{4.4}$$

In order to realize the comparison between fixed and floating strike options the strike has to be chosen as $\mathcal{K} = A^{\text{ref}}$. Note that the choice of $r = q_1 = q_2 = 0$ is not quite realistic from a financial viewpoint, if the underlying assets are linked to exchange rates, but for easier capture of the impact of the volatilities and the correlation factor on the result, these zero values are reasonable.

The numerical experiments are carried out for the piecewise quadratic DG approximations on adaptively refined and sufficiently large computational domain Ω in order to well resolve the solution in the whole domain and to suppress the asymptotic treatment of boundary conditions on the far-field boundary, respectively. Here it is sufficient to put $x_i^{\max} \approx 2\mathcal{K}/S_i^{\text{ref}}$, $k_1 = 1/100$, and $k_2 = 100$. Finally, the time step is chosen proportional to one calendar day, i.e., $\tau = 1/365$.

We start with the experimental findings on the sensitivity of the option prices to the volatilities of both the underlying assets for a fixed correlation $\varrho = 0.45$. To capture various market conditions we consider 5 values of $\sigma_i \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ for each underlying asset and compute discrete DG solutions using piecewise quadratic approximations. The obtained results (in thousandths) evaluated at the reference node after 30 days are presented in Table 3 along with the floating strike option values from [19].

Since the weight of the first underlying asset is higher ($\alpha_1 = 0.6$), the impact of its volatility on the option price is higher as well. These empirical findings are common to both types of options with fixed and floating strike. However, one can observe that the option values for fixed strike are several times higher than the values for floating strike, which reflects a realistic expectation on options with fixed strike. On the other hand, the growth of these values with increasing volatilities σ_1 and σ_2 is slower than in the case of floating strikes. This different behaviour of the two types of Asian options are given by the different character of the convection parts in the governing equations in the vicinity of a reference point, cf. [19].

σ_1 \ σ_2	0.05	0.10	0.15	0.20	0.25
0.05	20.7915 (2.25641)	20.9361 (3.06311)	21.0410 (3.95030)	21.1797 (4.87346)	21.3557 (5.81509)
0.10	21.0058 (3.79658)	21.1463 (4.49091)	21.2779 (5.27593)	21.4435 (6.11722)	21.6440 (6.99444)
0.15	21.3020 (5.39115)	21.4464 (6.02129)	21.6032 (6.73512)	21.7945 (7.50910)	22.0198 (8.32639)
0.20	21.6881 (7.00067)	21.8397 (7.59162)	22.0209 (8.25682)	22.2367 (8.97995)	22.4861 (9.74803)
0.25	22.1628 (8.61628)	22.3263 (9.18118)	22.5315 (9.81180)	22.7710 (10.4964)	23.0434 (11.2250)

Table 3. Discrete option values for P_2 approximations: Comparison between fixed strike options and floating strike options from [19] (values in brackets).

The second important factor is the dependence between the two foreign exchange rates, which is represented by the correlation coefficient ϱ. Figure 3 records the relation between the option price and the correlation, which is positive and almost linear for both types of options and corresponds to the well-known basket option behaviour, though fixed strike option is again less sensitive to the parameter changes.

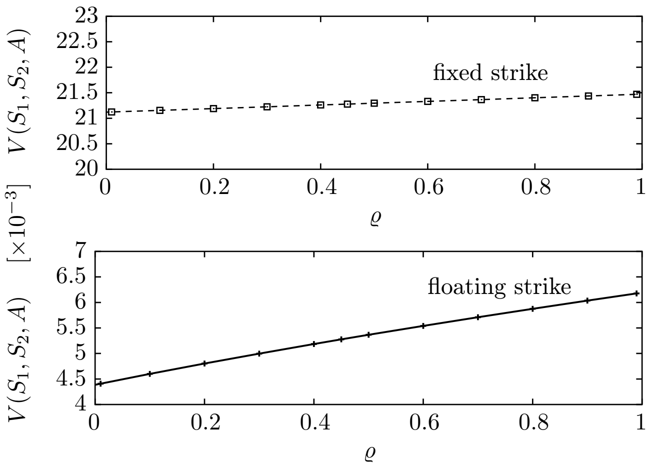

Figure 3. Comparison of the dependence of option values for particular risk sources given by correlation ϱ for floating (bottom) and fixed (top) strike Asian basket puts with fixed $\sigma_1 = 0.1$ and $\sigma_2 = 0.15$.

5. Conclusion

Obviously, the most straightforward way to option pricing is the usage of a closed-form formula. However, in many cases the option payoff function is too complex so that the relevant PDE system cannot be solved analytically or by some suitable transformation and it is inevitable to apply some of the numerical approximative techniques. In this contribution we have extended our previous results on the DG approach to Asian basket options [19] by considering the case of fixed strike. We have proposed a numerical scheme that leads to the option price and compared it to the case of floating strike, including experimental study assuming Asian put on two foreign exchange rates, including the sensitivity analysis. It follows that the proposed procedure seems to be sufficiently robust with respect to various options as well as market conditions and is comparable to other numerical approaches, at least in terms of the pricing error. Notwithstanding, it would be interesting to consider some advanced processes (see e.g. [6]) or at least dependence structures.

References

[1] *Y. Achdou, O. Pironneau*: Computational Methods for Option Pricing. Frontiers in Applied Mathematics 30, Society for Industrial and Applied Mathematics, Philadelphia, 2005. zbl MR doi

[2] *B. Alziary, J.-P. Décamps, P.-F. Koehl*: A P.D.E. approach to Asian options: analytical and numerical evidence. J. Bank. Financ. *21* (1997), 613–640.

[3] *F. Black, M. Scholes*: The pricing of options and corporate liabilities. J. Polit. Econ. *81* (1973), 637–654. zbl MR doi

[4] *P. Boyle, M. Broadie, P. Glasserman*: Monte Carlo methods for security pricing. J. Econ. Dyn. Control *21* (1997), 1267–1321. zbl MR doi

[5] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications 4, North-Holland Publishing Company, Amsterdam, 1978. zbl MR

[6] *R. Cont, P. Tankov*: Financial Modelling with Jump Processes. Chapman & Hall/CRC Financial Mathematics Series, Chapman and Hall/CRC, Boca Raton, 2004. zbl MR doi

[7] *J. C. Cox, S. A. Ross, M. Rubinstein*: Option pricing: a simplified approach. J. Financ. Econ. *7* (1979), 229–263. zbl doi

[8] *V. Dolejší, M. Feistauer*: Discontinuous Galerkin Method. Analysis and Applications to Compressible Flow. Springer Series in Computational Mathematics 48, Springer, Cham, 2015. zbl MR doi

[9] *F. Dubois, T. Lelièvre*: Efficient pricing of Asian options by the PDE approach. J. Comput. Finance *8* (2005), 55–63. doi

[10] *E. Eberlein, A. Papapantoleon*: Equivalence of floating and fixed strike Asian and lookback options. Stochastic Processes Appl. *115* (2005), 31–40. zbl MR doi

[11] *J. K. Hale*: Ordinary Differential Equations. Pure and Applied Mathematics 21, Wiley-Interscience a division of John Wiley & Sons, New York, 1969. zbl MR

[12] *J. M. Harrison, D. M. Kreps*: Martingales and arbitrage in multiperiod securities markets. J. Econ. Theory *20* (1979), 381–408. zbl MR doi

[13] *E. G. Haug*: The Complete Guide to Option Pricing Formulas. McGraw-Hill, New York, 2006.

[14] *F. Hecht*: New development in freefem++. J. Numer. Math. *20* (2012), 251–265. zbl MR doi

[15] *V. Henderson, R. Wojakowski*: On the equivalence of floating- and fixed-strike Asian options. J. Appl. Probab. *39* (2002), 391–394. zbl MR doi

[16] *J. Hozman*: Analysis of the discontinuous Galerkin method applied to the European option pricing problem. AIP Conference Proceedings 1570 (2013), 227–234. doi

[17] *J. Hozman, T. Tichý*: Black-Scholes option pricing model: Comparison of h-convergence of the DG method with respect to boundary condition treatment. ECON - Journal of Economics, Management and Business *24* (2014), 141–152. doi

[18] *J. Hozman, T. Tichý*: On the impact of various formulations of the boundary condition within numerical option valuation by DG method. Filomat *30* (2016), 4253–4263. zbl MR doi

[19] *J. Hozman, T. Tichý*: DG method for numerical pricing of multi-asset Asian options—The case of options with floating strike. Appl. Math., Praha *62* (2017), 171–195. zbl MR doi

[20] *J. Hozman, T. Tichý, D. Cvejnová*: A discontinuous Galerkin method for two-dimensional PDE models of Asian options. AIP Conference Proceedings *1738* (2016), Article no. 080011. doi

[21] *J. E. Ingersoll,* Jr.: Theory of Financial Decision Making. Rowman & Littlefield Publishers, New Jersey, 1987.

[22] *J. L. Lions, E. Magenes*: Non-Homogeneous Boundary Value Problems and Applications. Vol. I. Die Grundlehren der mathematischen Wissenschaften 181, Springer, Berlin, 1972. zbl MR doi

[23] *R. C. Merton*: Theory of rational option pricing. Bell J. Econ. Manage. Sci. *4* (1973), 141–183. zbl MR doi

[24] *W. H. Reed, T. R. Hill*: Triangular mesh methods for the neutron transport equation. Conf. Report, National Topical Meeting on Mathematical Models and Computational Techniques for Analysis of Nuclear Systems, Ann Arbor 1973. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, New Mexico, 1973.
[25] *K. Rektorys*: Variational Methods in Engineering Problems and in Problems of Mathematical Physics. Nakladatelsví Technické Literatury, Praha, 1974. (In Czech.) zbl MR
[26] *B. Rivière*: Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations. Theory and Implementation. Frontiers in Applied Mathematics 35, Society for Industrial and Applied Mathematics, Philadelphia, 2008. zbl MR doi
[27] *L. C. G. Rogers, Z. Shi*: The value of an Asian option. J. Appl. Probab. *32* (1995), 1077–1088. zbl MR doi
[28] *J. Večeř*: A new PDE approach for pricing arithmetic average Asian options. J. Comput. Finance *4* (2001), 105–113. doi
[29] *J. Večeř*: Unified pricing of Asian options. Risk *15* (2002), 113–116.
[30] *P. Wilmott, J. Dewynne, J. Howison*: Option Pricing: Mathematical Models and Computation. Financial Press, Oxford, 1993. zbl
[31] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/A: Linear Monotone Operators. Springer, New York, 1990. zbl MR doi
[32] *R. Zvan, P. A. Forsyth, K. Vetzal*: Robust numerical methods for PDE models of Asian options. J. Comput. Finance *1* (1998), 39–78. doi

Authors' addresses: *Jiří Hozman*, Technical University of Liberec, Faculty of Science, Humanities and Education, Department of Mathematics and Didactics of Mathematics, Studentská 1402/2, 461 17 Liberec, Czech Republic, e-mail: `jiri.hozman@tul.cz`; *Tomáš Tichý*, VŠB—Technical University of Ostrava, Faculty of Economics, Department of Finance, Sokolská třída 33, 702 00 Ostrava, Czech Republic, e-mail: `tomas.tichy@vsb.cz`.