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Diagonal cubic equations

by

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1. Introduction. Let λ_j be positive integers. We shall be concerned with solutions of the diophantine equation

$$\lambda_1 x_1^3 + \lambda_2 x_2^3 + \ldots + \lambda_7 x_7^3 = 0 \tag{1.1}$$

which are small in the following sense. Let $\Lambda = \lambda_1\lambda_2\ldots\lambda_7$. Then we seek for solutions to (1.1) which satisfy

$$0 < \sum_{i=1}^{7} \lambda_i |x_i|^3 \ll \Lambda^g \tag{1.2}$$

with g as small as possible.

A first positive answer to the problem was found by R. C. Baker [1]. He established the solubility of (1.1) and (1.2) whenever $g = 61$. In this paper we prove the following theorem.

Theorem. *Let $g = 14$. Then there are solutions to* (1.1), (1.2).

The key for improvement is the mean value estimate for a class of exponential sums presented in [2]. We state the special case we require as a lemma. Let X be a large parameter, and write

$$\theta_1 = 15/113, \quad \theta_2 = 14/113, \quad \eta = 84/113. \tag{1.3}$$

We note that $\theta_1 + \theta_2 + \eta = 1$. Put

$$M_1 = X^{\theta_1}, \quad M_2 = X^{\theta_2}, \quad Y = X^{\eta}. \tag{1.4}$$

As usual we write $e(\alpha) = \exp(2\pi i\alpha)$. Now define the exponential sums

$$g(\alpha, P) = \sum_{P < X \le 2P} e(\alpha x^3), \tag{1.5}$$

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(1.6) $$G(\alpha, X) = \sum_{M_1 < p_1 \le 2M_1} \sum_{M_2 < p_2 \le 2M_2} g(\alpha(p_1p_2)^3, Y),$$

where, here and throughout, p, p_i denote prime numbers.

LEMMA 1. *In the notation introduced above, for any* $\varepsilon > 0$ *we have*

$$\int_0^1 |G(\alpha, X)|^6 \, d\alpha \ll X^{3+2\theta_1+\varepsilon}.$$

This is the case $l = 3$ of the Theorem in Brüdern [2].

2. Preliminaries. Now define

(2.1) $$\chi_3(X) = \{p_1p_2y : X^{\theta_1} < p_1 \le 2X^{\theta_1},\ X^{\theta_2} < p_2 \le 2X^{\theta_2},\ X^{\eta} < y \le 2X^{\eta}\}$$

and

(2.2) $$\chi_4(X) = \{py : X^{\theta} < p \le 2X^{\theta},\ y \in \chi_3(X^{1-\theta})\}$$

in which $\theta = 233/1815$.

Given positive integers μ_1, μ_2, we write $S(\mu_1, \mu_2, B, E)$ for the number of solutions of the equation

(2.3) $$\mu_1 x_1^3 + \mu_2 p^3(y_1^3 + y_2^3) = \mu_1 x_2^3 + \mu_2 p^3(y_3^3 + y_4^3)$$

in integers $x_1, x_2, y_1, y_2, y_3, y_4$ and primes p satisfying

(2.4) $$B < x_1, x_2 \le 2B, \quad p \nmid \mu_1 x_1 x_2, \quad y_i \in \chi_3(E^{1-\theta}) \quad (1 \le i \le 4),$$

(2.5) $$E^{\theta} < p \le 2E^{\theta}, \qquad p \equiv 2 \pmod 3).$$

LEMMA 2. *Let* $B \ge 1$, $E \ge 1$. *Let* μ_1, μ_2 *be positive integers, and*

(2.6) $$\mu_2 E^3 \ll \mu_1 B^3 \ll \mu_2 E^3, \quad \mu_1 \gg \mu_2, \quad E^{\theta} \le B^{1/7}.$$

Then

(2.7) $$S(\mu_1, \mu_2, B, E) \ll B^{1+\varepsilon} E^{2-\theta}.$$

Proof. By the argument of Vaughan [7], p. 125, we have $S(\mu_1, \mu_2, B, E) \ll E^{\varepsilon} S_0$, where S_0 is the number of solutions to (2.3) subject to (2.4), (2.5) and $x_1 \equiv x_2 \pmod{p^3}$. The solutions with $x_1 = x_2$ contribute to S_0 at most $BE^{\theta}E^{2(1-\theta)+\varepsilon} = BE^{2-\theta+\varepsilon}$. For the solutions with $x_1 \ne x_2$ we again follow through the argument in Vaughan [7] and find that there are at most $2S_1$ remaining solutions where S_1 equals the number of solutions to

$$\mu_1 h(3x^2 + h^2p^6) = 4\mu_2(y_1^3 + y_2^3 - y_3^3 - y_4^3),$$

where $x \le 4B$, $E^\theta < p \le 2E^\theta$, $0 < h \le H$, $y_i \in \chi_3(E^{1-\theta})$, and $H = CBE^{-3\theta}$ for some constant C. On writing

$$\Omega(\alpha) = \sum_{E^\theta < p \le 2E^\theta} \sum_{h \le H} \sum_{x \le 4B} e(\alpha h(3x^2 + h^2 p^6))$$

we have, by Hölder's inequality,

$$S_1 = \int_0^1 \Omega(\mu_1\alpha)|G(4\mu_2\alpha, E^{1-\theta})|^4\,d\alpha$$
$$\le \Big(\int_0^1 |\Omega(\mu_1\alpha)|^3\,d\alpha\Big)^{1/3}\Big(\int_0^1 |G(\alpha, E^{1-\theta})|^6\,d\alpha\Big)^{2/3}.$$

The argument of Vaughan [8], p. 39 shows that for $B^{1/8} < E^\theta \le B^{1/7}$ we have

$$\int_0^1 |\Omega(\alpha)|^3\,d\alpha \ll H^2(E^\theta B)^{3/2+\varepsilon}.$$

Since $\Omega(\alpha)$ has period 1, we have

$$\int_0^1 |\Omega(\mu_1\alpha)|^3\,d\alpha = \frac{1}{\mu_1}\int_0^{\mu_1} |\Omega(\alpha)|^3\,d\alpha = \int_0^1 |\Omega(\alpha)|^3\,d\alpha$$

and by Lemma 1 we have

$$\int_0^1 |G(\alpha, E^{1-\theta})|^6\,d\alpha \ll (E^{1-\theta})^{3+2\theta_1+\varepsilon}.$$

The lemma follows easily.

3. Outline of the method. For the proof of the Theorem, as in Baker [1] it suffices to prove the Theorem in the special case when $\lambda_1, \ldots, \lambda_7$ are cube-free, and no prime divides more than four of the λ_i. We may suppose that

$$\lambda_5 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4 \ge \lambda_6 \ge \lambda_7 \ge \lambda_1. \tag{3.1}$$

Let

$$N = C_0 \Lambda^g, \quad P = N^{1/3}, \quad P_j = P\lambda_j^{-1/3}, \tag{3.2}$$

where C_0 is sufficiently large. Let δ denote a sufficiently small positive absolute constant, and let $\varepsilon = \delta^2$.

Let $R(\lambda_1, \ldots, \lambda_7)$ denote the number of solutions of the equation

$$\lambda_1 x^3 - \lambda_2 p^3 y^3 + \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 z_3^3 + \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 z_4^3 + \lambda_5 u^3 + \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 v_6^3 + \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 v_7^3 = 0 \tag{3.3}$$

in integers $x, y, z_3, z_4, u, v_6, v_7$ and primes p, p_{ij} satisfying

$$P_1 < x \le 2P_1, \quad W < y \le 2W, \quad R_i < z_i \le 2R_i, \quad U < u \le 2U, \quad V_i < v_i \le 2V_i, \tag{3.4}$$

$$p \nmid x, \quad p_{ij} \nmid y \quad (3 \le i \le 4, 1 \le j \le 3), \quad p_{ij} \nmid u \quad (6 \le i \le 7, 1 \le j \le 3), \tag{3.5}$$

$$Y < p \le 2Y, \quad p \equiv 2 \pmod 3, \quad p \nmid \Lambda, \tag{3.6}$$

$$Z_{ij} < p_{ij} \le 2Z_{ij}, \quad p_{ij} \equiv 2 \pmod 3, \quad (i = 3, 4, 6, 7,\ 1 \le j \le 3), \quad p_{ij} \nmid \Lambda. \tag{3.7}$$

Here

$$Y = C_1 P_1^{1/7}, \quad W = P_2 Y^{-1}, \quad R_i = \frac{1}{200}(P_i Y^{-1})^{1176/1815} \quad (i = 3, 4), \tag{3.8}$$

$$U = \frac{1}{200} P_5, \quad V_i = \frac{1}{200} P_i^{1176/1815} \quad (i = 6, 7), \tag{3.9}$$

$$Z_{i1} = 2^{-i}(P_i Y^{-1})^{233/1815}, \quad Z_{i2} = 2^{-i}(P_i Y^{-1})^{210/1815}, \quad Z_{i3} = 2^{-i}(P_i Y^{-1})^{196/1815} \quad (i = 3, 4), \tag{3.10}$$

$$Z_{i1} = 2^{-i} P_i^{233/1815}, \quad Z_{i2} = 2^{-i} P_i^{210/1815}, \quad Z_{i3} = 2^{-i} P_i^{196/1815} \quad (i = 6, 7), \tag{3.11}$$

where C_1 is sufficiently large.

The inequalities

$$N \ll \lambda_1 P_1^3, \lambda_2 Y^3 W^3, \lambda_3 Y^3 Z_{31}^3 Z_{32}^3 Z_{33}^3 R_3^3, \lambda_4 Y^3 Z_{41}^3 Z_{42}^3 Z_{43}^3 R_4^3, \lambda_5 U^3, \lambda_6 Z_{61}^3 Z_{62}^3 Z_{63}^3 V_6^3, \lambda_7 Z_{71}^3 Z_{72}^3 Z_{73}^3 V_7^3 \ll N \tag{3.12}$$

are easily verified.

Let (a_1, a_2) or more generally $(a_1, \ldots, a_n)$ denote greatest common divisor, while $[a_1, \ldots, a_n]$ denotes least common multiple. We write

$$f_d(X, \alpha) = \sum_{\substack{X < x \le 2X \\ (x,d)=1}} e(\alpha x^3) \tag{3.13}$$

and use the notation f in place of f_1.

Let $\mathbf{p}$ denote the ordered set $p, p_{31}, p_{32}, p_{33}, p_{41}, p_{42}, p_{43}, p_{61}, p_{62}, p_{63}, p_{71}, p_{72}, p_{73}$ and let

$$A = \prod_{i=3}^{4} \prod_{j=1}^{3} p_{ij}, \quad B = \prod_{i=6}^{7} \prod_{j=1}^{3} p_{ij}, \tag{3.14}$$

$$\begin{aligned} F(\mathbf{p}, \alpha) = {} & f_p(P_1, \lambda_1 \alpha) \overline{f_A(W, \lambda_2 p^3 \alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \\ & \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ & \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha), \end{aligned} \tag{3.15}$$

$$F(\alpha) = \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} F(\mathbf{p};\alpha). \tag{3.16}$$

The summation here is over ordered sets $\mathbf{p}$ satisfying (3.6), (3.7); we often use this type of notation below.

Let $\Re$ be the unit interval $[LN^{-1}, 1+LN^{-1}]$. Then clearly

$$R(\lambda_1,\dots,\lambda_7) = \int_{\Re} F(\alpha)\,d\alpha. \tag{3.17}$$

Here

$$L = P_1/10. \tag{3.18}$$

When $1 \le a \le q \le L$ and $(a,q)=1$, we take $M(q,a)$ to be the interval $\{\alpha : |\alpha - a/q| \le q^{-1}LN^{-1}\}$, and let M denote the union of all such $M(q,a)$. Then $M(q,a)$ are disjoint subsets of M, as we easily verify using (3.2) and (3.18). Let

$$m = \Re \setminus M. \tag{3.19}$$

We shall prove that

$$\int_M F(\alpha)\,d\alpha \gg \lambda_1^{2/21} \Lambda^{-1/3-\varepsilon} N^{26/21} (\log N)^{-13}, \tag{3.20}$$

$$\int_m F(\alpha)\,d\alpha \ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}. \tag{3.21}$$

It follows from (3.19)–(3.21) and (3.17) that

$$R(\lambda_1,\dots,\lambda_7) \gg \lambda_1^{2/21} \Lambda^{-1/3-\varepsilon} N^{26/21} (\log N)^{-13}. \tag{3.22}$$

Thus the Theorem will follow once (3.20)–(3.21) have been proved.

We recall the standard notations

$$S(q,b) = \sum_{r=1}^{q} e(br^3/q), \qquad J(\beta, X) = \int_X^{2X} e(\beta x^3)\,dx. \tag{3.23}$$

The estimate

$$S(q,b) \ll q^{1+\varepsilon}\psi(q), \qquad (q,b)=1, \tag{3.24}$$

where $\psi(q)$ is the multiplicative function with

$$\psi(p^{3h+r}) = p^{-h-r/2} \qquad (h = 0,1,\dots; 0 \le r \le 2), \tag{3.25}$$

follows from Lemmas 4.3 and 4.4 of [5]. By partial integration,

$$J(\beta, X) \ll \frac{X}{1 + X^3|\beta|} \tag{3.26}$$

for positive X and real β.

LEMMA 3. *Let d be an integer with $\ll 1$ divisors. Let*

$$s_d(q,c) = q^{-1}\sum_{b\mid d}\mu(d)\frac{S(q,b^3c)}{b} \tag{3.27}$$

and let λ be a positive integer. Then

$$\begin{aligned} f_d(X,\lambda\alpha) &= s_d(q,\lambda a)J(\lambda(\alpha-a/q),X)\\ &\quad + O(q^{1/2+\varepsilon}(1+\lambda X^3|\alpha-a/q|)^{1/2}) \end{aligned} \tag{3.28}$$

for any $X > 0$, real α and rational number a/q.

This is Lemma 2 of [1].

For $\alpha \in M(q,a)$, we introduce the notations

$$s(p,\alpha) = s_p(q,\lambda_1 a)J(\lambda_1(\alpha-a/q),P_1), \tag{3.29}$$

$$g(p,\alpha) = s_A(q,\lambda_2 p^3 a)J(\lambda_2(\alpha-a/q)p^3,W). \tag{3.30}$$

It is convenient to write

$$\Delta = \Delta(\alpha,a,q) = 1 + N|\alpha-a/q|. \tag{3.31}$$

LEMMA 4. *Let $\mathbf{p}$ satisfy* (3.6) *and* (3.7). *Let $\alpha \in M(q,a)$ where $1 \le a \le q \le L, (a,q) = 1$. Then*

$$f_p(P_1,\lambda_1\alpha) - s(p,\alpha) \ll q^{1/2+\varepsilon}\Delta^{1/2}, \tag{3.32}$$

$$f_A(W,\lambda_2 p^3\alpha) - g(p,\alpha) \ll q^{1/2+\varepsilon}\Delta^{1/2}. \tag{3.33}$$

Proof. The proof is similar to that of Lemma 3 of [1]; see [1] and [6].

4. The minor arcs. In this section we use the notations

$$H = C_2P_1Y^{-3}, \quad Q = P_1Y^{-1}, \tag{4.1}$$

where C_2 is a sufficiently large absolute constant; also

$$M = L(2Y)^{-3}. \tag{4.2}$$

Further, let

$$\begin{aligned} S(\alpha) = \Big| &\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}}\ \sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} f_A(W,\lambda_2\alpha)\\ &\times f(R_3,\lambda_3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2 \end{aligned} \tag{4.3}$$

and

$$(4.4)\qquad \Phi_p(\alpha) = \sum_{\substack{P_1 < y \le 2P_1 \\ p \mid y}} 1 + 2\mathrm{Re} \sum_{h \le H} \sum_{\substack{2P_1 + hp^3 < y \le 4P_1 - hp^3 \\ p \nmid y,\ y \equiv h \,(\mathrm{mod}\, 2)}} e\Big(\frac{\lambda_1 \alpha}{4}(3hy^2 + h^3 p^6)\Big).$$

Let n denote the set of real numbers in $(0, 1]$ with the property that whenever $|\alpha - a/q| \le q^{-1} L N^{-1}$ and $(a, q) = 1$, we have $q > M$. Let

$$(4.5)\qquad T = \int_{n} \sum_{\substack{p \\ (3.6)}} \Phi_p(\alpha) S(\alpha)\, d\alpha.$$

LEMMA 5. *Let*

$$J_1 = \int_0^1 \Big| \sum_{\substack{p_{61}, p_{62}, p_{63} \\ (3.7)}} \sum_{\substack{p_{71}, p_{72}, p_{73} \\ (3.7)}} f_B(U, \lambda_5 \alpha) \times f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\Big|^2 \, d\alpha.$$

Then $J_1 \ll P_5 P_6^{2\theta} P_7^{2+\varepsilon}$.

P r o o f. By considering the underlying diophantine equation we have

$$J_1 \ll \int_0^1 \Big| \sum_{\substack{p_{61}, p_{62}, p_{63} \\ (3.7)}} \sum_{\substack{p_{71}, p_{72}, p_{73} \\ (3.7)}} f_{p_{61} p_{71}}(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) \times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\Big|^2 \, d\alpha.$$

From the inequality $|zw| \le \frac{1}{2}|z|^2 + \frac{1}{2}|w|^2$, we have

$$J_1 \ll \sum_{k=6}^{7} \int_0^1 \Big(\sum_{\substack{p_{61} \\ (3.7)}} \sum_{\substack{p_{71} \\ (3.7)}} |f_{p_{61} p_{71}}(U, \lambda_5 \alpha)| \Big| \sum_{\substack{p_{k2}, p_{k3} \\ (3.7)}} f(V_k, \lambda_k p_{k1}^3 p_{k2}^3 p_{k3}^3 \alpha)\Big|^2 \Big)^2 d\alpha.$$

Hence, by Cauchy's inequality,

$$\begin{aligned} J_1 &\ll Z_{61} Z_{71} \sum_{k=6}^{7} \int_0^1 \sum_{\substack{p_{61} \\ (3.7)}} \sum_{\substack{p_{71} \\ (3.7)}} |f_{p_{61} p_{71}}(U, \lambda_5 \alpha)|^2 \times \Big| \sum_{\substack{p_{k2}, p_{k3} \\ (3.7)}} f(V_k, \lambda_k p_{k1}^3 p_{k2}^3 p_{k3}^3 \alpha)\Big|^4 \, d\alpha \\ &\ll Z_{71}^2 Z_{61} S(\lambda_5, \lambda_6, P_5, P_6) + Z_{61}^2 Z_{71} S(\lambda_5, \lambda_7, P_5, P_7). \end{aligned}$$

By (3.1) and (3.2) we have

$$P_6 \le P_7, \quad P_6^{\theta} \le P_7^{\theta} \le P_5^{1/7}.$$

By (3.11), (3.12) and Lemma 2,

$$J_1 \ll P_5 P_6^{2\theta} P_7^{2+\varepsilon}.$$

This completes the proof of Lemma 5.

LEMMA 6. *Let*

$$J_2 = \int_{m} \Big| \sum_{\substack{p \\ (3.6)}} \sum_{\substack{p_{31}, p_{32}, p_{33} \\ (3.7)}} \sum_{\substack{p_{41}, p_{42}, p_{43} \\ (3.7)}} f_p(P_1, \lambda_1\alpha) f_A(W, \lambda_2 p^3\alpha) \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^2 \, d\alpha.$$

Then $J_2 \ll YT$.

Proof. The proof is similar to that of Lemma 15 of [1].

Let

$$F(\beta, \gamma; h) = \sum_{\substack{2P_1 < y \le 4P_1 \\ y \equiv h \ (\mathrm{mod}\ 2)}} e\big(\tfrac{3}{4}\beta y^2 - \gamma y\big), \tag{4.6}$$

$$G_h(\varrho, \sigma) = \sum_{\substack{Y < p \le 2Y,\ p \equiv 2 \ (\mathrm{mod}\ 3) \\ p \le (2P_1/h)^{1/3}}} e\big(\tfrac{1}{4}\varrho p^6 + \sigma p^3\big), \tag{4.7}$$

$$F_p(\alpha; h) = \sum_{\substack{2P_1 + hp^3 < y \le 4P_1 - hp^3 \\ y \equiv h \ (\mathrm{mod}\ 2)}} e\big(\tfrac{3}{4}\lambda_1 \alpha h y^2\big), \tag{4.8}$$

$$\Psi_p(\alpha) = 2\mathrm{Re} \sum_{h \le H} F_p(\alpha; h) e\big(\tfrac{1}{4}\lambda_1 \alpha h^3 p^6\big), \tag{4.9}$$

$$D_p(\alpha; h) = \sum_{\substack{2P_1/p + hp^2 < y \le 4P_1/p - hp^2 \\ y \equiv h \ (\mathrm{mod}\ 2)}} e\big(\tfrac{3}{4}\lambda_1 \alpha h p^2 y^2\big), \tag{4.10}$$

$$\Xi_p(\alpha) = 2\mathrm{Re} \sum_{h \le H} D_p(\alpha; h) e\big(\tfrac{1}{4}\lambda_1 \alpha h^3 p^6\big), \tag{4.11}$$

$$T_1(p) = \int_{n} \Psi_p(\alpha) S(\alpha) \, d\alpha, \tag{4.12}$$

$$T_2(p) = \int_{n} \Xi_p(\alpha) S(\alpha) \, d\alpha, \tag{4.13}$$

$$T_3 = \int_0^1 S(\alpha) \, d\alpha. \tag{4.14}$$

By (4.4),

$$\Phi_p(\alpha) = \Psi_p(\alpha) - \Xi_p(\alpha) + O(P_1). \tag{4.15}$$

Hence

$$T = \sum_{\substack{p \\ (3.6)}} (T_1(p) - T_2(p)) + O(P_1 Y T_3) \tag{4.16}$$

from (4.5).

We estimate T_3 via Lemma 2. By (3.1), (3.2), (3.8) and (3.10) we have

$$(P_3Y^{-1})^{\theta} \le (P_4Y^{-1})^{\theta} \le (P_2Y^{-1})^{1/7}.$$

Just as in Lemma 5, we have

$$T_3 \ll (P_2Y^{-1})(P_3Y^{-1})^{2\theta}(P_4Y^{-1})^{2+\varepsilon} \ll P_2P_3^{2\theta}P_4^{2+\varepsilon}Y^{-3-2\theta}. \tag{4.17}$$

Let

$$T_5(\gamma,\theta) = \int_{n} \sum_{h\le H} |F(\alpha\lambda_1 h, \gamma; h) G_h(\alpha\lambda_1 h^3, \theta\gamma h)| S(\alpha)\, d\alpha. \tag{4.18}$$

By a very minor adaptation of the proof of (5.26) of [7], one finds that

$$\sum_{\substack{p \\ (3.6)}} T_1(p) \ll (\log P_1) \sup_{\substack{0\le\gamma\le 1 \\ \theta=\pm1}} T_5(\gamma,\theta) + N^{(26+12\theta)/21}\Lambda^{-1}. \tag{4.19}$$

We omit the details.

Lemma 7. *Suppose that* $\alpha \in \mathbb{R}$ *and*

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{24qHP_1}, \qquad (a,q) = 1.$$

Then

$$\sum_{h\le H} |F(\alpha h, \gamma; h)|^2 \ll P_1^{\delta}\left[\frac{q^{-1}HP_1^2}{1+Q^3|\alpha - a/q|} + HP_1 + q\right].$$

This is Lemma 16 of [1].

Lemma 8. *Let* α, $\gamma \in \mathbb{R}$. *Suppose that*

$$\left|\alpha - \frac{a}{q}\right| \le q^{-1}Q^{-3}H^{3/4}, \qquad (a,q) = 1, \qquad q \le Q^3H^{-3/4}.$$

Then

$$\sum_{h\le H} |G_h(\alpha h^3, \gamma h)|^2 \ll P_1^{\varepsilon}\left[\frac{HY^2q^{-1/3}}{(1+Q^3|\alpha - a/q|)^{1/3}} + H^{3/4}Y^2\right].$$

This is essentially Lemma 8 of [7].

LEMMA 9. *Let* $\gamma \in \mathbb{R}$ *and* $\theta \in \{-1, 1\}$. *Then*

$$T_5(\gamma,\theta) \ll P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-2-2\theta} + \lambda_1^{1+\varepsilon} P_1^{2+\varepsilon} P_3^{1+\varepsilon} P_4^{1+\varepsilon} Y^{-1} + \lambda_1^{1/2} P_1^{1+\varepsilon} P_2^{1/2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{-1/2}.$$

Proof. Let n_1 denote the set of α in n with the property that whenever

$$|\lambda_1\alpha - r/b| \le r^{-1} H^{7/4} Q^{-3}, \quad (b,r) = 1, \tag{4.20}$$

one has $r > H^{7/4}$. Let $\alpha \in n_1$. We apply Lemma 7 with $\lambda_1\alpha$ in place of α. By Dirichlet's theorem there exist integers b, r satisfying (4.20) with $r \le Q^3 H^{-7/4}$; here we must have $r > H^{7/4}$. Now the condition of Lemma 7 is easily verified, thus

$$\sum_{h\le H} |F(\alpha h\lambda_1, \gamma; h)|^2 \ll P_1^{\delta}(P_1^2 H^{-3/4} + HP_1 + Q^3 H^{-7/4}) \ll P_1^{2+\delta} H^{-3/4}. \tag{4.21}$$

We may choose c, s so that $(c,s) = 1, s \le Q^3 H^{-3/4}$, and

$$|\lambda_1\alpha - c/s| \le s^{-1} H^{3/4} Q^{-3}.$$

Then $|\lambda_1\alpha - c/s| \le s^{-1} H^{7/4} Q^{-3}$, so that $s > H^{7/4} > H^{3/4}$. By Lemma 8, applied to $\lambda_1\alpha$ in place of α, we have

$$\sum_{h\le H} |G_h(\lambda_1\alpha h^3, \theta\gamma h)|^2 \ll P_1^{\varepsilon} H^{3/4} Y^2.$$

Hence by Cauchy's inequality, (4.14), (4.17) and (4.21),

$$\int_{n_1} \sum_{h\le H} |F(\alpha h\lambda_1, \gamma; h) G_h(\lambda_1\alpha h^3, \theta\gamma h)| S(\alpha)\, d\alpha \ll P_1^{1+\delta} Y T_3 \ll P_1^{1+\delta} P_2 P_3^{2\theta} P_4^{2+\varepsilon} Y^{-2-2\theta}. \tag{4.22}$$

It remains to consider $n \setminus n_1$. Let $\alpha \in n \setminus n_1$. There are integers b, r satisfying

$$|\lambda_1\alpha - b/r| \le r^{-1} H^{7/4} Q^{-3}, \quad (b,r) = 1, \quad r \le H^{7/4}. \tag{4.23}$$

We write $b/(\lambda_1 r) = a/q$ in lowest terms. Clearly $q = rd$ where $d = \lambda_1/(\lambda_1, b)$. Also, $(r, \lambda_1/d) = (r, (\lambda_1, b)) = 1$. Because $\alpha \in n$, we know that either $|\alpha - a/(rd)| > r^{-1} d^{-1} L N^{-1}$, or $rd > M$, or both.

Let d be a given divisor of λ_1. Let

$$N_1(d,r,a) = \left\{\alpha : \left|\alpha - \frac{a}{rd}\right| \le r^{-1} d^{-1} L N^{-1}\right\},$$

$$N_2(d,r,a) = \left\{\alpha : \left|\alpha - \frac{a}{rd}\right| \le r^{-1} H^{7/4} Q^{-3} \lambda_1^{-1}\right\}.$$

It is clear from (3.2), (3.8), (3.18), (4.1) that $N_2(d,r,a)$ contains $N_1(d,r,a)$.

Let $N_1(d)$ denote the union of $N_1(d,r,a)$ with

$$1 \le a \le rd, \quad (a, rd) = (r, \lambda_1/d) = 1, \tag{4.24}$$

$$r \le Md^{-1}. \tag{4.25}$$

Let $N_2(d)$ denote the union of $N_2(d,r,a)$ with (4.24) and $r \le H^{7/4}$. By the discussion following (4.22) we have, modulo one,

$$n \setminus n_1 \subset \bigcup_{d|\lambda_1} \{N_2(d) \setminus N_1(d)\}.$$

Let $L(d,r,a)$ denote $N_2(d,r,a)$ when $Md^{-1} < r \le H^{7/4}$, and (4.24) holds; and denote $N_2(d,r,a) \setminus N_1(d,r,a)$ when $r \le Md^{-1}$ and (4.24) holds. For fixed d, we have

$$N_2(d) \setminus N_1(d) = \bigcup_{r \le H^{7/4}; a} L(d,r,a).$$

Let $\alpha \in L(d,r,a)$. The fraction $\lambda_1 a/(rd)$ may be written in the form b/r with $(b,r) = 1$. Since

$$(\lambda_1 a, rd) = d(\lambda_1 a/d, r) = d,$$

by Lemma 7, with $\lambda_1 \alpha$ in place of α, and

$$\left|\lambda_1 \alpha - \frac{b}{r}\right| = \lambda_1 \left|\alpha - \frac{a}{rd}\right| \le r^{-1} H^{7/4} Q^{-3} \le \frac{1}{24 r H P_1},$$

we have

$$\sum_{h \le H} |F(\alpha h \lambda_1, \gamma; h)|^2 \ll P_1^{\delta} \left[\frac{r^{-1} H P_1^2}{1 + Q^3 |\lambda_1 \alpha - b/r|} + HP_1 + r\right] \tag{4.26}$$
$$\ll \frac{r^{-1} H P_1^{2+\delta}}{1 + NY^{-3} |\alpha - a/(rd)|}.$$

Here we require the observations that

$$Q^3 \lambda_1 = NY^{-3}, \quad HP_1 + r \ll HP_1 \ll r^{-1} H P_1^2,$$
$$(HP_1 + r) r Q^3 |\lambda_1 \alpha - b/r| \ll HP_1 H^{7/4} \ll HP_1^2.$$

Choose c, s so that $|\lambda_1 \alpha - c/s| \le s^{-1} H^{3/4} Q^{-3}$, $(c,s) = 1$ and $s \le Q^3 H^{-3/4}$. If $b/r = c/s$, then by the definition of $L(d,r,a)$, we know that either

$$|\lambda_1 \alpha - c/s| > s^{-1} d^{-1} L N^{-1} \lambda_1 \ge s^{-1} L N^{-1},$$

or $s > Md^{-1}$, or both. By (3.1), (3.2) we have either $s > H^{3/4}$ or $Q^3 s |\lambda_1 \alpha - c/s| \ge H^{3/4}$, or both. If $b/r \ne c/s$, then

$$\frac{1}{rs} \le \left|\frac{b}{r} - \frac{c}{s}\right| \le (H^{3/4} s^{-1} + H^{7/4} r^{-1}) Q^{-3} \le \frac{1}{2sr} + H^{7/4} r^{-1} Q^{-3},$$

whence $s > \frac{1}{2}Q^3H^{-7/4} > H^{3/4}$ once more. Now we may apply Lemma 8 with $\lambda_1\alpha$ in place of α, obtaining

$$\sum_{h\le H}|G_h(\lambda_1\alpha h^3,\theta\gamma h)|^2 \ll P_1^{\varepsilon}H^{3/4}Y^2. \tag{4.27}$$

Therefore, by (4.26) and Cauchy's inequality

$$\sum_{h\le H}|F(\alpha h\lambda_1,\gamma;h)G_h(\lambda_1\alpha h^3,\theta\gamma h)| \ll \frac{YH^{7/8}P_1^{1+\delta}r^{-1/2}}{(1+NY^{-3}|\alpha-a/(rd)|)^{1/2}}. \tag{4.28}$$

It follows that

$$\begin{aligned}\int\limits_{N_2(d)\setminus N_1(d)}\sum_{h\le H}&|F(\alpha h\lambda_1,\gamma;h)G_h(\lambda_1\alpha h^3,\theta\gamma h)|S(\alpha)\,d\alpha \\ &\ll P_1^{1+\delta}H^{7/8}Y\int\limits_{N_2(d)\setminus N_1(d)} r^{-1/2}\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{-1/2}S(\alpha)\,d\alpha.\end{aligned} \tag{4.29}$$

We write

$$\Gamma=\Gamma(\alpha,a,q)=q^{1/2+\varepsilon}(1+NY^{-3}|\alpha-a/q|)^{1/2}.$$

By Lemma 3, we have

$$f_A(W,\lambda_2\alpha)=s_A(q,\lambda_2a)J(\lambda_2(\alpha-a/q),W)+O(\Gamma)=\Gamma_1+O(\Gamma).$$

Hence by (3.26) we have

$$\begin{aligned}|f_A(W,\lambda_2\alpha)|^2 &\ll \Gamma_1^2+\Gamma^2 \ll |s_A(q,\lambda_2a)J(\lambda_2(\alpha-a/q),W)|^2+\Gamma^2\\ &\ll \frac{s_A^2(q,\lambda_2a)W^2}{(1+NY^{-3}|\alpha-a/q|)^2}+\Gamma^2.\end{aligned}$$

Hence

$$\begin{aligned}\int\limits_{N_2(d)\setminus N_1(d)} &r^{-1/2}\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{-1/2}S(\alpha)\,d\alpha\\ \ll \int\limits_{N_2(d)\setminus N_1(d)} &r^{-1/2}\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{-1/2}(\Gamma_1^2+\Gamma^2)\\ &\times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} f(R_3,\lambda_3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha.\end{aligned} \tag{4.30}$$

By Hua's inequality we have

$$\int_0^1\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} f(R_3,\lambda_3p_{31}^3p_{32}^3p_{33}^3\alpha)\Big|^8\,d\alpha \ll (P_3Y^{-1})^{5+\varepsilon},$$

$$\int_0^1 \Big| \sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^8 \, d\alpha \ll (P_4 Y^{-1})^{5+\varepsilon}$$

and

$$\int_0^1 \Big| \sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}} \sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^2 \, d\alpha$$
$$\ll \Big(\int_0^1 \Big| \sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha)\Big|^4 \, d\alpha\Big)^{1/2}$$
$$\times \Big(\int_0^1 \Big| \sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^4 \, d\alpha\Big)^{1/2}$$
$$\ll ((P_3 Y^{-1})^{2+\varepsilon} (P_4 Y^{-1})^{2+\varepsilon})^{1/2}.$$

By the definition of Γ,

$$\Gamma^2 = q^{1+\varepsilon}\Big(1 + NY^{-3}\Big|\alpha - \frac{a}{q}\Big|\Big) = (rd)^{1+\varepsilon}\Big(1 + NY^{-3}\Big|\alpha - \frac{a}{rd}\Big|\Big).$$

For $\alpha \in N_2(d) \setminus N_1(d)$,

$$r^{1/2+\varepsilon} d^{1+\varepsilon}\Big(1 + NY^{-3}\Big|\alpha - \frac{a}{rd}\Big|\Big)^{1/2} \ll d^{1+\varepsilon} H^{7/8+\varepsilon} \ll \lambda_1^{1+\varepsilon} H^{7/8+\varepsilon}.$$

Hence

$$\text{(4.31)} \quad \int_{N_2(d)\setminus N_1(d)} r^{-1/2}\Big(1 + NY^{-3}\Big|\alpha - \frac{a}{rd}\Big|\Big)^{-1/2} \Gamma^2$$
$$\times \Big| \sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}} \sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^2 \, d\alpha$$
$$\ll \int_{N_2(d)\setminus N_1(d)} r^{1/2+\varepsilon} d^{1+\varepsilon}\Big(1 + NY^{-3}\Big|\alpha - \frac{a}{rd}\Big|\Big)^{1/2}$$
$$\times \Big| \sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}} \sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_3, \lambda_3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^2 \, d\alpha$$
$$\ll \lambda_1^{1+\varepsilon} H^{7/8+\varepsilon} (P_3 P_4 Y^{-2})^{1+\varepsilon}.$$

Now we consider

$$\int\limits_{N_2(d)\setminus N_1(d)} \Gamma_1^4\,d\alpha = \int\limits_{N_2(d)\setminus N_1(d)} \frac{s_A^4(q,\lambda_2 a)W^4}{(1+NY^{-3}|\alpha-a/(rd)|)^4}\,d\alpha \tag{4.32}$$

$$\ll W^4 \sum_{r\le H^{7/4}} \sum_{a\le rd} \int\limits_{L(d,r,a)} \frac{s_A^4(q,\lambda_2 a)}{(1+NY^{-3}|\alpha-a/(rd)|)^4}\,d\alpha.$$

Let $\kappa(q)$ and $\kappa^*(q)$ be the multiplicative functions defined by

$$\kappa(p^{3k}) = p^{-k},\qquad \kappa(p^{3k+1}) = 2p^{-k-1/2},\qquad \kappa(p^{3k+2}) = p^{-k-1},$$
$$\kappa^*(p) = 2p^{-1/2},\qquad \kappa^*(p^2) = p^{-3/4},\qquad \kappa^*(p^l) = p^{-l/3}\qquad (l\ge 3).$$

Then $\kappa(q)\le\kappa^*(q)$, and by Lemmas 4.3, 4.4 and Theorem 4.2 of Vaughan [5],

$$q^{-1}S(q,a)\ll\kappa(q)\qquad\text{when } (a,q)=1.$$

Now we consider

$$\sum_{q\le\Xi} q\kappa^*\biggl(\frac{q}{(q,\lambda_2)}\biggr)^4$$
$$\ll \prod_{p\le\Xi}\biggl(1+\sum_{l=1}^{\infty} p^l\kappa^*\biggl(\frac{p^l}{(p^l,\lambda_2)}\biggr)^4\biggr)$$
$$\ll \prod_{\substack{p\nmid\lambda_2\\ p\le\Xi}}\Bigl(1+16p^{-1}+p^{-1}+\sum_{l=3}^{\infty}p^{-l/3}\Bigr)$$
$$\times \prod_{\substack{p\|\lambda_2\\ p\le\Xi}}\Bigl(1+p+16+1+1+\sum_{l=1}^{\infty}p^{-l/3}\Bigr)$$
$$\times \prod_{\substack{p^2\|\lambda_2\\ p\le\Xi}}\Bigl(1+p+p^2+16p+p+p+p^{2/3}+p^{1/3}+\sum_{l=0}^{\infty}p^{-l/3}\Bigr).$$

We have

$$\prod_{\substack{p\|\lambda_2\\ p\le\Xi}}(20+p)\ll \prod_{\substack{p\|\lambda_2\\ p\le C(\varepsilon)}}(20+p)\prod_{\substack{p\|\lambda_2\\ p\ge C(\varepsilon)}}(20+p)\ll\prod_{p\|\lambda_2}p^{1+\varepsilon}.$$

Here we use the fact that for $p\ge C(\varepsilon)$ we have $20+p\le p^{1+\varepsilon}$. Similarly

$$\prod_{\substack{p^2\|\lambda_2\\ p\le\Xi}}(p^2+22p+1)\ll\prod_{p^2\|\lambda_2}p^{2+\varepsilon}.$$

The constant implied by $\ll$ depends only on ε. So we have

$$\sum_{q\le \Xi} q\kappa^*\left(\frac{q}{(q,\lambda_2)}\right)^4 \ll \lambda_2^{1+\varepsilon}(\log \Xi)^{20}. \tag{4.33}$$

By (3.9) of [1],

$$s_A(q,\lambda_2 a) \ll q^{\varepsilon}\kappa\left(\frac{q}{(q,\lambda_2 a)}\right) \ll q^{\varepsilon}\kappa^*\left(\frac{q}{(q,\lambda_2)}\right),$$

so we have

$$\int_{N_2(d)\setminus N_1(d)} \Gamma_1^4\, d\alpha \ll W^4Y^3N^{-1+\varepsilon}\lambda_2. \tag{4.34}$$

For $\alpha \in N_2(d)\setminus N_1(d)$, we have

$$r^{-1/2}\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{-1/2} \ll M^{-1/2}d^{1/2}.$$

Hence by Schwarz's inequality,

$$\begin{aligned}\int_{N_2(d)\setminus N_1(d)} & r^{-1/2}\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)^{-1/2}\Gamma_1^2 \\ &\times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} f(R_3,\lambda_3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha \\ &\ll (W^4Y^3N^{-1+\varepsilon}\lambda_2)^{1/2}M^{-1/2}d^{1/2}(P_3Y^{-1})^{(5+\varepsilon)/4}(P_4Y^{-1})^{(5+\varepsilon)/4}.\end{aligned} \tag{4.35}$$

By (4.29)–(4.35) we have

$$\begin{aligned}\int_{n\setminus n_1}\sum_{h\le H} & |F(\alpha h\lambda_1,\gamma;h)G_h(\lambda_1\alpha h^3,\theta\gamma h)|S(\alpha)\,d\alpha \\ &\ll P_1^{1+\delta}H^{7/8}Y[\lambda_1^{1+\varepsilon}H^{7/8+\varepsilon}(P_3P_4Y^{-2})^{1+\varepsilon} \\ &\quad + (W^4Y^3N^{-1+\varepsilon}\lambda_2)^{1/2}M^{-1/2}d^{1/2}(P_3P_4Y^{-2})^{(5+\varepsilon)/4}] \\ &\ll \lambda_1^{1+\varepsilon}P_1^{2+\varepsilon}P_3^{1+\varepsilon}P_4^{1+\varepsilon}Y^{-1}+\lambda_1^{1/2}P_1^{1+\varepsilon}P_2^{1/2+\varepsilon}P_3^{5/4+\varepsilon}P_4^{5/4+\varepsilon}Y^{-1/2}.\end{aligned} \tag{4.36}$$

By (4.22), (4.36) the lemma follows.

As in Section 8 of [1], for the estimation of $\sum_{p,(3.6)} T_2(p)$ we have the same upper bound for $T_5(\gamma,\theta)$.

5. First steps on the major arcs. In this section we show that $R(\lambda_1,\ldots,\lambda_7)$ is well approximated by integrals similar to (3.17), but with $f_p(P_1,\lambda_1\alpha)$ and $f_A(W,\lambda_2p^3\alpha)$ replaced by suitable approximations. Let

$$D(\alpha)=q^{1/2}(1+N|\alpha-a/q|)^{1/2}.$$

By Lemma 4,

$$f_p(P_1,\lambda_1\alpha)-s(p,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}\ll D(\alpha)q^{\varepsilon},$$
$$f_A(W,\lambda_2\alpha)-g(p,\alpha)\ll q^{1/2+\varepsilon}\Delta^{1/2}\ll D(\alpha)q^{\varepsilon}.$$

Here

$$\Delta=1+N|\alpha-a/q|.$$

Now we introduce the numbers

$$v_1=\int\limits_M\Big|\sum_{\substack{p\\(3.6)}}\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}}(f_p(P_1,\lambda_1\alpha)-s(p,\alpha))f_A(W,\lambda_2p^3\alpha)$$
$$\times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha,$$
$$v_2=\int\limits_M\Big|\sum_{\substack{p\\(3.6)}}\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}}s(p,\alpha)(f_A(W,\lambda_2\alpha)-g(p,\alpha))$$
$$\times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha.$$

We use Cauchy's inequality and note that $D(\alpha)\ll L^{1/2}$ for $\alpha\in M$. Hence by (4.17),

$$v_1\ll YL^{1+\varepsilon}\sum_{Y<p\le 2Y}\int\limits_0^1\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}}f_A(W,\lambda_2p^3\alpha)\tag{5.1}$$
$$\times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha$$
$$\ll Y^2L^{1+\varepsilon}\int\limits_0^1\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}}f_A(W,\lambda_2\alpha)$$
$$\times f(R_3,\lambda_3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha$$
$$\ll Y^2L^{1+\varepsilon}T_3\ll L^{1+\varepsilon}P_2P_3^{2\theta}P_4^{2+\varepsilon}Y^{-1-2\theta}.$$

Now we consider v_2. By Cauchy's inequality and Schwarz's inequality we have

$$v_2\ll YP^{\varepsilon}\sum_{Y<p\le 2Y}\int\limits_M\Big|s(p,\alpha)D(\alpha)$$
$$\times\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}}f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2d\alpha$$
$$\ll YP^{\varepsilon}(I_1I_2)^{1/2},$$

where

$$I_1 = \sum_{Y<p\le 2Y} \int_{M} |s(p,\alpha)D(\alpha)|^4\, d\alpha,$$

$$I_2 = \sum_{Y<p\le 2Y} \int_0^1 \Big| \sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} \sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) \times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^4\, d\alpha.$$

By Hua's inequality and Cauchy's inequality,

$$I_2 \ll Y(P_3Y^{-1})^{(5+\varepsilon)/2}(P_4Y^{-1})^{(5+\varepsilon)/2} = (P_3P_4)^{(5+\varepsilon)/2}Y^{-4}.$$

As in (4.34), by (3.29), (3.26) and the definition of $D(\alpha)$ we have

$$I_1 \ll Y\lambda_1 N^{-1} P_1^{4+\varepsilon} L^2.$$

Hence

$$\begin{aligned} v_2 &\ll YP_1^{\varepsilon}(P_3^{(5+\varepsilon)/2}P_4^{(5+\varepsilon)/2}Y^{-4})^{1/2}(Y\lambda_1N^{-1}P_1^{4+\varepsilon}L^2)^{1/2} \\ &\ll Y^{-1/2}P_1^{2+\varepsilon}P_3^{5/4+\varepsilon}P_4^{5/4+\varepsilon}\lambda_1^{1/2}LN^{-1/2}. \end{aligned} \tag{5.2}$$

6. The second step on the major arcs. Let M_1 denote the set of all major arcs $M(q,a)$ subject to $1 \le a \le q \le X, (a,q) = 1$, where

$$X = P^{4/7}. \tag{6.1}$$

Let

$$v^* = \int_{M\setminus M_1} \Big| \sum_{\substack{p\\ (3.6)}} \sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} \sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} s(p,\alpha)g(p,\alpha) \times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)\Big|^2\, d\alpha, \tag{6.2}$$

where $s(p,\alpha)$ and $g(p,\alpha)$ are given by (3.29) and (3.30).

As in [3], by Lemmas 4.3 and 4.4 of Vaughan [5] when $p\nmid a$ one has

$$S(p^l, a) = \begin{cases} p^{[2l/3]}, & l \not\equiv 1 \pmod 3, \\ 0, & l \equiv 1 \pmod 3,\ p \equiv 2 \pmod 3, \\ p^{2(l-1)/3}S(p,a), & l \equiv 1 \pmod 3,\ p \equiv 1 \pmod 3. \end{cases} \tag{6.3}$$

Also, standard methods show that

$$S(q, ab^3) = S(q,a) \quad \text{when } (b,q) = 1; \tag{6.4}$$

$$S(qr, a) = S(q, ar^2)S(r, aq^2) \quad \text{when } (r,q) = (qr,a) = 1. \tag{6.5}$$

When $p\nmid q$, by (3.9) of [1],

$$s_A(q, \lambda_2 p^3 a) \ll q^{\varepsilon}\kappa\Big(\frac{q}{(\lambda_2 a p^3, q)}\Big) \ll q^{\varepsilon}\kappa\Big(\frac{q}{(\lambda_2, q)}\Big),$$

$$s_p(q,\lambda_1 a)=q^{-1}\{S(q,\lambda_1 a)-p^{-1}S(q,\lambda_1 p^3 a)\}\ll \kappa\bigg(\frac{q}{(\lambda_1,q)}\bigg).$$

When $p\,\|\,q$, write $q=pr$ with $p\nmid r$. By (3.9) of [1],

$$s_A(q,\lambda_2 p^3 a)\ll q^{\varepsilon}\kappa\bigg(\frac{q}{(\lambda_2 a p^3,q)}\bigg)\ll q^{\varepsilon}\kappa\bigg(\frac{r}{(\lambda_2,r)}\bigg),$$

$$S(q,\lambda_1 a)=(q,\lambda_1)S\bigg(\frac{q}{(q,\lambda_1)},\frac{\lambda_1}{(q,\lambda_1)}a\bigg)=(q,\lambda_1)S\bigg(\frac{pr}{(r,\lambda_1)},\frac{\lambda_1}{(r,\lambda_1)}a\bigg).$$

By (6.3) and (6.5) this vanishes. Similarly,

$$S(q,\lambda_1 p^3 a)\ll pr\kappa\bigg(\frac{r}{(r,\lambda_1)}\bigg),$$

so

$$s_p(q,\lambda_1 a)\ll p^{-1}\kappa\bigg(\frac{r}{(r,\lambda_1)}\bigg).$$

When $p^2\,|\,q$, let $q=p^l r, p\nmid r$ and $l\ge 2$. Then

$$\begin{aligned}S(q,\lambda_1 a)&=(r,\lambda_1)S\bigg(p^l\frac{r}{(r,\lambda_1)},\frac{\lambda_1}{(r,\lambda_1)}a\bigg)\\&=(r,\lambda_1)S\bigg(p^l,\frac{\lambda_1}{(r,\lambda_1)}a\frac{r^2}{(r,\lambda_1)^2}\bigg)S\bigg(\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a p^{2l}}{(r,\lambda_1)}\bigg).\end{aligned}$$

When $l=2$,

$$p^{-1}S(q,\lambda_1 a p^3)=p(\lambda_1,r)S\bigg(\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a p}{(r,\lambda_1)}\bigg).$$

By (6.3),

$$S\bigg(p^2,\frac{\lambda_1}{(r,\lambda_1)}a\frac{r^2}{(r,\lambda_1)^2}\bigg)=p,$$

$$S\bigg(\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a p^{2l}}{(r,\lambda_1)}\bigg)=S\bigg(\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a p}{(r,\lambda_1)}\bigg),$$

so $s_p(q,\lambda_1 a)=0$.

When $l\ge 3$, by (6.3)–(6.5) we have

$$\begin{aligned}&p^{-1}S(p^l r,\lambda_1 a p^3)\\&\quad=p^2(\lambda_1,r)S\bigg(p^{l-3}\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a}{(r,\lambda_1)}\bigg)\\&\quad=p^2(\lambda_1,r)S\bigg(\frac{r}{(r,\lambda_1)},\frac{\lambda_1 a p^{2l-6}}{(r,\lambda_1)}\bigg)S\bigg(p^{l-3},\frac{\lambda_1 a}{(r,\lambda_1)}\cdot\frac{r^2}{(r,\lambda_1)^2}\bigg),\end{aligned}$$

hence $s_p(q,\lambda_1 a)=0$. Combining this result with (3.29) and (3.30) we have

$$s(p,\alpha)g(p,\alpha) \ll \begin{cases} 0 & \text{if } p^2\mid q,\\ q^\varepsilon p^{-1}\kappa\Big(\dfrac{r}{(r,\lambda_1)}\Big)\kappa\Big(\dfrac{r}{(r,\lambda_2)}\Big)P_1W\Big(1+N\Big|\alpha-\dfrac{a}{q}\Big|\Big)^{-2} & \text{if } q=pr,\ p\nmid r,\\ q^\varepsilon \kappa\Big(\dfrac{q}{(q,\lambda_1)}\Big)\kappa\Big(\dfrac{q}{(q,\lambda_2)}\Big)P_1W\Big(1+N\Big|\alpha-\dfrac{a}{q}\Big|\Big)^{-2} & \text{if } p\nmid q.\end{cases}$$

Note that

$$4p^{-1}\kappa(r)^2=\kappa(pr)^2 \quad \text{for } p\nmid r.$$

For $p^2\nmid q$ we have

$$s(p,\alpha)g(p,\alpha)\ll q^\varepsilon\kappa\Big(\frac{q}{(q,\lambda_1)}\Big)\kappa\Big(\frac{q}{(q,\lambda_2)}\Big)P_1W\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-2}. \tag{6.6}$$

Hence

$$\begin{aligned}
&\sum_{\substack{Y<p\le 2Y\\ (3.6)}}\int_{M\setminus M_1}\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} s(p,\alpha)g(p,\alpha)\\
&\qquad\times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha\\
&\ll P_1^2W^2L^\varepsilon\sum_{\substack{Y<p\le 2Y\\ (3.6)}}\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^2\Big(\frac{q}{(q,\lambda_1)}\Big)\kappa^2\Big(\frac{q}{(q,\lambda_2)}\Big)\\
&\quad\times\sum_{\substack{a=1\\ (a,q)=1}}^{q}\int_{M(q,a)}\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-4}\\
&\quad\times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha\\
&\ll P_1^2W^2L^\varepsilon\sum_{\substack{Y<p\le 2Y\\ (3.6)}}\Big(\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)\sum_{\substack{a=1\\ (a,q)=1}}^{q}\int_{M(q,a)}\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-4}\\
&\qquad\times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)\Big|^4\,d\alpha\Big)^{1/2}
\end{aligned} \tag{6.7}$$

$$\times\Bigg(\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^4\Big(\frac{q}{(q,\lambda_2)}\Big)\sum_{\substack{a=1\\ (a,q)=1}}^{q}\int_{M(q,a)}\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-4}$$

$$\times\Big|\sum_{\substack{p_{41},p_{42},p_{43}\\ (3.7)}} f(R_4,\lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3\alpha)\Big|^4 d\alpha\Bigg)^{1/2}$$

Now we consider

$$\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)\sum_{\substack{a=1\\ (a,q)=1}}^{q}\int_{M(q,a)}\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-4}$$

$$\times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} f(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\alpha)\Big|^4 d\alpha$$

$$\ll\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)$$

$$\times\int_{-1/2}^{1/2}(1+N|\beta|)^{-4}\sum_{a=1}^{q}\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} f\Big(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\Big(\frac{a}{q}+\beta\Big)\Big)\Big|^4 d\beta.$$

Now,

$$\sum_{a=1}^{q}\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\ (3.7)}} f\Big(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\Big(\frac{a}{q}+\beta\Big)\Big)\Big|^4$$

$$=\sum_{x_1,\dots,x_4\in\chi_4(P_3Y^{-1})}\sum_{a=1}^{q} e\Big(\lambda_3 p^3\Big(\frac{a}{q}+\beta\Big)(x_1^3+x_2^3-x_3^3-x_4^3)\Big)$$

$$=q\sum_{\substack{x_i\in\chi_4(P_3Y^{-1})\\ \lambda_3p^3(x_1^3+x_2^3-x_3^3-x_4^3)\equiv 0\ (\mathrm{mod}\ q)}} e(\lambda_3 p^3\beta(x_1^3+x_2^3-x_3^3-x_4^3))\ll q\Psi(q,p),$$

where $\Psi(q,p)$ denotes the number of solutions of the congruence

$$\lambda_3 p^3(x_1^3+x_2^3-x_3^3-x_4^3)\equiv 0 \pmod q$$

subject to $x_i\in\chi_4(P_3Y^{-1})$. Now we have

$$\sum_{\substack{X<q\le L\\ p^2\nmid q}}\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)\sum_{\substack{a=1\\ (a,q)=1}}^{q}\int_{M(q,a)}\Big(1+N\Big|\alpha-\frac{a}{q}\Big|\Big)^{-4}$$

$$\times \Big| \sum_{\substack{p_{31},p_{32},p_{33} \\ (3.7)}} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha)\Big|^4 \, d\alpha$$

$$\ll N^{-1} \sum_{\substack{X<q\le L \\ p^2\nmid q}} q\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)\Psi(q,p) = N^{-1}\Big(\sum\nolimits_0 + \sum\nolimits_1\Big),$$

where $\sum_j$ is the part of the remaining sum with $p^j \,\|\, q$.

When $p\nmid q$ we have $\Psi(q,p) = \Psi(q,1) = \Psi(q)$, say. Thus by a simple splitting up argument and Cauchy's inequality,

$$\sum\nolimits_0 \ll (\log L) \sum_{\Xi<q\le 2\Xi} q\kappa^4\Big(\frac{q}{(q,\lambda_1)}\Big)\Psi(q) \tag{6.8}$$

$$\ll (\log L)\Big(\sum_{\Xi<q\le 2\Xi} q\kappa^8\Big(\frac{q}{(q,\lambda_1)}\Big)\Big)^{1/2}\Big(\sum_{q\le 2\Xi} q\Psi^2(q)\Big)^{1/2}$$

for some Ξ in the range $X < \Xi \le L$.

Let $\varrho(h)$ denote the number of solutions to

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = h$$

subject to $x_i \in \chi_4(P_3Y^{-1})$. Then

$$\Psi(q) = \sum_{|l|\le (P_3Y^{-1})^3(q/(q,\lambda_3))^{-1}} \varrho\Big(\frac{lq}{(q,\lambda_3)}\Big),$$

and by Cauchy's inequality,

$$\Psi(q)^2 \ll \Big(2(P_3Y^{-1})^3\Big(\frac{q}{(q,\lambda_3)}\Big)^{-1} + 1\Big) \sum_{|l|\le (P_3Y^{-1})^3(q/(q,\lambda_3))^{-1}} \varrho^2\Big(\frac{lq}{(q,\lambda_3)}\Big).$$

Of course, $\varrho(0) \ll (P_3Y^{-1})^{2+\varepsilon}$ so that

$$\sum_{q\le 2\Xi} q\Psi(q)^2 \tag{6.9}$$

$$\ll \Xi(P_3Y^{-1})^{7+\varepsilon} + (P_3Y^{-1})^3 \sum_{0<lq/(q,\lambda_3)\le (P_3Y^{-1})^3} (q,\lambda_3)\varrho^2\Big(\frac{lq}{(q,\lambda_3)}\Big)$$

$$\ll \Xi(P_3Y^{-1})^{7+\varepsilon} + (P_3Y^{-1})^{3+\varepsilon}\lambda_3 \sum_{h\le (P_3Y^{-1})^3} \varrho^2(h)$$

and the final sum is bounded by the number of solutions of

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_5^3 + x_6^3 + x_7^3 + x_8^3$$

subject to $x_i \in \chi_4(P_3Y^{-1})$. By Hua's inequality (Lemma 2.5 of [5]) this number is bounded by $O((P_3Y^{-1})^{5+\varepsilon})$. We now have

$$\sum_{q\le 2\Xi} q\Psi(q)^2 \ll \Xi(P_3Y^{-1})^{7+\varepsilon} + (P_3Y^{-1})^{8+\varepsilon}\lambda_3.$$

Note that $\kappa(q) \ll q^{-1/3+\varepsilon}$. Hence from (4.33),

$$\sum_{\Xi<q\le 2\Xi} q\kappa^8\left(\frac{q}{(q,\lambda_1)}\right) \ll \lambda_1^{7/3+\varepsilon}\Xi^{-4/3+\varepsilon}. \tag{6.10}$$

From (6.8) we have

$$\begin{aligned}\sum\nolimits_0 &\ll (\log L)(\lambda_1^{7/3+\varepsilon}\Xi^{-4/3+\varepsilon})^{1/2}(\Xi(P_3Y^{-1})^{7+\varepsilon} + (P_3Y^{-1})^{8+\varepsilon}\lambda_3)^{1/2}\\ &\ll (\log L)\lambda_1^{7/6+\varepsilon}X^{-2/3+\varepsilon}(P_3Y^{-1})^{4+\varepsilon}\lambda_3^{1/2}.\end{aligned}$$

An estimate for $\sum_1$ is obtained along similar lines. We write $q = pr$ with $p\nmid r$. Then $\Psi(pr,p) = \Psi(r,1) = \Psi(r)$. Therefore since κ is multiplicative,

$$\sum\nolimits_1 \ll p^{-1}\sum_{Xq^{-1}<r\le Lq^{-1}} r\kappa^4\left(\frac{r}{(r,\lambda_1)}\right)\Psi(r).$$

By the same argument as before, the sum is

$$\ll (\log L)\lambda_1^{7/6+\varepsilon}(P_3Y^{-1})^{4+\varepsilon}\lambda_3^{1/2}X^{-2/3+\varepsilon}p^{-1/3}.$$

So the bound for $\sum_1$ is even stronger than the bound for $\sum_0$. Hence we have

$$\begin{aligned}\sum_{\substack{X<q\le L\\ p^2\nmid q}} \kappa^4\left(\frac{q}{(q,\lambda_1)}\right) \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{M(q,a)} \left(1+N\left|\alpha-\frac{a}{q}\right|\right)^{-4}&\\ \times\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}} f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)\Big|^4\,d\alpha&\\ \ll N^{-1+\varepsilon}\lambda_1^{7/6+\varepsilon}X^{-2/3}(P_3Y^{-1})^{4+\varepsilon}\lambda_3^{1/2}&.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{Y<p\le 2Y}\int_{M\setminus M_1}\Big|\sum_{\substack{p_{31},p_{32},p_{33}\\(3.7)}}\sum_{\substack{p_{41},p_{42},p_{43}\\(3.7)}} s(p,\alpha)g(p,\alpha)&\\ \times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\Big|^2\,d\alpha&\\ \ll P_1^2W^2YL^{\varepsilon}(N^{-1+\varepsilon}\lambda_1^{7/6+\varepsilon}X^{-2/3}(P_3Y^{-1})^{4+\varepsilon}\lambda_3^{1/2})^{1/2}&\\ \times(N^{-1+\varepsilon}\lambda_2^{7/6+\varepsilon}X^{-2/3}(P_4Y^{-1})^{4+\varepsilon}\lambda_4^{1/2})^{1/2}&\\ \ll P_1^2W^2N^{-1+\varepsilon}\lambda_1^{7/12+\varepsilon}\lambda_2^{7/12+\varepsilon}YX^{-2/3}(P_3P_4Y^{-2})^{2+\varepsilon}\lambda_3^{1/4}\lambda_4^{1/4}&.\end{aligned} \tag{6.11}$$

By (6.2) and Cauchy's inequality we have

$$v^* \ll P_1^2 W^2 N^{-1+\varepsilon} \lambda_1^{7/12+\varepsilon} \lambda_2^{7/12+\varepsilon} Y^{-2} X^{-2/3} (P_3 P_4)^{2+\varepsilon} \lambda_3^{1/4} \lambda_4^{1/4}. \tag{6.12}$$

7. The major arcs. For $\alpha \in M(q,a)$, let

$$h_i(\alpha) = s(q, \lambda_i p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3 a) J\left(\lambda_i\left(\alpha - \frac{a}{q}\right) p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3, R_i\right) \qquad (i = 3,4),$$

$$k(\alpha) = s_B(q, \lambda_5 a) J\left(\lambda_5\left(\alpha - \frac{a}{q}\right), U\right),$$

$$l_i(\alpha) = s(q, \lambda_i p_{i1}^3 p_{i2}^3 p_{i3}^3 a) J\left(\lambda_i\left(\alpha - \frac{a}{q}\right) p_{i1}^3 p_{i2}^3 p_{i3}^3, V_i\right) \qquad (i = 6,7).$$

Then as in Lemma 3 of [1], for $\alpha \in M(q,a)$, where $1 \le a \le q \le X, (a,q) = 1$, we have

$$f(R_i, \lambda_i p^3 p_{i1}^3 p_{i2}^3 p_{i3}^3 \alpha) - h_i(\alpha) \ll q^{1/2+\varepsilon} \Delta^{1/2} \qquad (i = 3,4),$$

$$f_B(U, \lambda_5 \alpha) - k(\alpha) \ll q^{1/2+\varepsilon} \Delta^{1/2},$$

$$f(V_i, \lambda_i p_{i1}^3 p_{i2}^3 p_{i3}^3 \alpha) - l_i(\alpha) \ll q^{1/2+\varepsilon} \Delta^{1/2} \qquad (i = 6,7).$$

Let

$$F^*(\mathbf{p}, \alpha) = s(p,\alpha)\overline{g(p,\alpha)} h_3(\alpha) h_4(\alpha) k(\alpha) l_6(\alpha) l_7(\alpha),$$

$$F^*(\alpha) = \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} F^*(\mathbf{p}, \alpha),$$

$$A_1 = q^{-1/3} P_1 \Delta^{-1}, \qquad A_2 = (q,p^3)^{1/3} q^{-1/3} W \Delta^{-1}, \qquad A_5 = q^{-1/3} U \Delta^{-1},$$

$$A_i = q^{-1/3} (q,p^3)^{1/3} (q,p_{i1}^3)^{1/3} (q,p_{i2}^3)^{1/3} (q,p_{i3}^3)^{1/3} R_i \Delta^{-1} \qquad (i = 3,4),$$

$$A_i = q^{-1/3} (q,p_{i1}^3)^{1/3} (q,p_{i2}^3)^{1/3} (q,p_{i3}^3)^{1/3} V_i \Delta^{-1} \qquad (i = 6,7).$$

Then as in Lemma 5 of [1] we have

$$h_i(\alpha) \ll \lambda_i^{1/3} q^\varepsilon A_i \ (i = 3,4), \ k(\alpha) \ll \lambda_5^{1/3} q^\varepsilon A_5, \ l_i(\alpha) \ll \lambda_i^{1/3} q^\varepsilon A_i \ (i = 6,7)$$

and

$$\begin{aligned} &s(p,\alpha)\overline{g(p,\alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) \\ &\qquad \times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_3 p_{71}^3 p_{72}^3 p_{73}^3 \alpha) - F^*(\mathbf{p}, \alpha) \\ &\ll \lambda_1^{1/3} \lambda_2^{1/3} q^{2\varepsilon} A_1 A_2 \sum_{i=3}^{7} q^{1/2+\varepsilon} \Delta^{1/2} \prod_{\substack{j=3 \\ j \ne i}}^{7} (\lambda_j^{1/3} q^\varepsilon A_j + q^{1/2+\varepsilon} \Delta^{1/2}). \end{aligned}$$

By Lemma 4 of [1] we see that

$$\sum_{\substack{p_{ij}\\(3.7)}} |s(p,\alpha)\overline{g(p,\alpha)}f(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\alpha)f(R_4,\lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3\alpha)$$

$$\times f_B(U,\lambda_5\alpha)f(V_6,\lambda_6 p_{61}^3 p_{62}^3 p_{63}^3\alpha)f(V_7,\lambda_3 p_{71}^3 p_{72}^3 p_{73}^3\alpha) - F^*(\mathbf{p},\alpha)|$$

$$\ll \Big\{\prod_{(3.7)} Z_{ij}\Big\}\lambda_1^{1/3}\lambda_2^{1/3}q^{-2/3+2\varepsilon}P_1 W \varDelta^{-2}(q,p^3)$$

$$\times [(q^{1/2+\varepsilon}\varDelta^{1/2})^5 + (q^{1/2+\varepsilon}\varDelta^{1/2})^4 q^{-1/3}U\varDelta^{-1}$$

$$+ (q^{1/2+\varepsilon}\varDelta^{1/2})^3 q^{-2/3}UV_7\varDelta^{-2} + (q^{1/2+\varepsilon}\varDelta^{1/2})^2 q^{-1}UV_6V_7\varDelta^{-3}$$

$$+ (q^{1/2+\varepsilon}\varDelta^{1/2})q^{-4/3}UV_6V_7R_4\varDelta^{-4}]$$

and

$$\int_{M_1}\sum_{\substack{\mathbf{p}\\(3.6),(3.7)}} |s(p,\alpha)\overline{g(p,\alpha)}f(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\alpha)f(R_4,\lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3\alpha)$$

$$\times f_B(U,\lambda_5\alpha)f(V_6,\lambda_6 p_{61}^3 p_{62}^3 p_{63}^3\alpha)f(V_7,\lambda_3 p_{71}^3 p_{72}^3 p_{73}^3\alpha) - F^*(\mathbf{p},\alpha)|\,d\alpha$$

$$= \sum_{\substack{\mathbf{p}\\(3.6),(3.7)}}\sum_{q\le X}\sum_{\substack{a\le q\\(a,q)=1}}\int_{M(q,a)} |s(p,\alpha)\overline{g(p,\alpha)}f(R_3,\lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3\alpha)$$

$$\times f(R_4,\lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3\alpha)f_B(U,\lambda_5\alpha)$$

$$\times f(V_6,\lambda_6 p_{61}^3 p_{62}^3 p_{63}^3\alpha)f(V_7,\lambda_3 p_{71}^3 p_{72}^3 p_{73}^3\alpha) - F^*(\mathbf{p},\alpha)|\,d\alpha$$

$$\ll \Big\{\prod_{(3.7)} Z_{ij}\Big\}\lambda_1^{1/3}\lambda_2^{1/3}P_1 W N^{-1}\sum_{\substack{p\\(3.6)}}\sum_{q\le X}(q,p^3)[q^{4/3+\varepsilon}L^{3/2} + U\log N q^{2+\varepsilon}$$

$$+ q^{7/6+\varepsilon}UV_7 + q^{1/3+\varepsilon}UV_6V_7 + q^{-1/2+\varepsilon}UV_6V_7R_4]$$

$$\ll \Big\{\prod_{(3.7)} Z_{ij}\Big\}\lambda_1^{1/3}\lambda_2^{1/3}P_1 W N^{-1}[X^{7/3+\varepsilon}L^{3/2} + U\log N X^{3+\varepsilon} + UV_7X^{13/6+\varepsilon}$$

$$+ UV_6V_7X^{4/3+\varepsilon} + UV_6V_7R_4X^{1/2+\varepsilon}].$$

By (3.1), (3.2), (3.8)–(3.11) and (6.1), the above is

$$\ll Y\Big\{\prod_{(3.7)} Z_{ij}\Big\}P_1 W R_3 R_4 U V_6 V_7 N^{-1-\varepsilon} \ll \lambda_1^{2/21}\varLambda^{-1/3}N^{26/21-\varepsilon}.$$

Now, we write

$$S(q,a,h)=\sum_{\substack{x=1\\(x,q,h)=1}}^{q} e(ax^3/q),$$

where h,q are natural numbers and a is an integer.

Let

$$\chi(q,p)=\begin{cases}0 & \text{if } p\,|\,q,\\ 1 & \text{otherwise.}\end{cases}$$

Then as in Lemma 6 of [1], by induction we have

$$s_p(q,c)=\left(1-\frac{1}{p}\right)^{\chi(q,p)} q^{-1}S(q,c,p),$$

$$s_A(q,c)=\prod_{i=3}^{4}\prod_{j=1}^{3}\left(1-\frac{1}{p_{ij}}\right)^{\chi(q,p_{ij})} q^{-1}S(q,c,A),$$

$$s_B(q,c)=\prod_{i=6}^{7}\prod_{j=1}^{3}\left(1-\frac{1}{p_{ij}}\right)^{\chi(q,p_{ij})} q^{-1}S(q,c,B).$$

Now we consider $\int_{M_1} F^*(\alpha)\,d\alpha$. We write

$$\begin{aligned}S(\mathbf{p},q)=\frac{1}{q^7}\sum_{\substack{a=1\\(a,q)=1}}^{q} & S(q,\lambda_1 a,p)S(q,\lambda_2 ap^3,A)S(q,\lambda_3 ap^3p_{31}^3p_{32}^3p_{33}^3)\\ &\times S(q,\lambda_4 ap^3p_{41}^3p_{42}^3p_{43}^3)S(q,\lambda_5 a,B)\\ &\times S(q,\lambda_6 ap_{61}^3p_{62}^3p_{63}^3)S(q,\lambda_7 ap_{71}^3p_{72}^3p_{73}^3)\end{aligned}$$

and for $q\le X, |\beta|\le q^{-1}LN^{-1}$,

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} F^*(\mathbf{p},\beta+a/q)=\theta(q,\mathbf{p})S(\mathbf{p},q)\varphi(\mathbf{p},\beta).$$

Here

$$\theta(q,\mathbf{p})=\prod_{\substack{\omega\in\{p,p_{ij}\}\\ \omega\nmid q}}\left(1-\frac{1}{\omega}\right)$$

and

$$\begin{aligned}\varphi(\mathbf{p},\beta)=J(\lambda_1\beta,P_1)&J(-\lambda_2p^3\beta,W)J(\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\beta,R_3)\\ &\times J(\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\beta,R_4)J(\lambda_5\beta,U)\\ &\times J(\lambda_6p_{61}^3p_{62}^3p_{63}^3\beta,V_6)J(\lambda_7p_{71}^3p_{72}^3p_{73}^3\beta,V_7).\end{aligned}$$

As in the proof of Lemma 10 of [1],

$$\int\limits_{M_1} F^*(\alpha)\,d\alpha = \sum_{\substack{\mathbf{p}\\(3.6),(3.7)}} \sum_{q\le X} \theta(q,\mathbf{p})S(\mathbf{p},q) \int\limits_{-LN^{-1}}^{LN^{-1}} \varphi(\mathbf{p},\beta)\,d\beta.$$

In view of Lemmas 8 and 9 of [1], we may insert the summation condition

$$(q, pAB) = 1$$

in the last summation over q. For these values of q we have

$$\theta(q,\mathbf{p}) = \prod_{\omega\in\{p,p_{ij}\}} \left(1-\frac{1}{\omega}\right).$$

As in Lemma 11 of [1], by (3.2) we have

$$\sum_{\substack{q>X\\(q,pAB)=1}} |S(\mathbf{p},q)| \ll \Lambda^{1/3}X^{\varepsilon-1/3} \ll \Lambda^{-2\varepsilon}$$

and

$$S_X(\mathbf{p}) = \sum_{\substack{q\le X\\(q,pAB)=1}} S(\mathbf{p},q) \gg \Lambda^{-\varepsilon}.$$

As in the proof of (2.7) of [1] we have

$$\int\limits_{M_1} F^*(p,\alpha)\,d\alpha \gg \lambda_1^{2/21}\Lambda^{-1/3-\varepsilon}N^{26/21}(\log N)^{-13}.$$

8. The proof of Theorem

LEMMA 10. *Let $\lambda_1 \ge \ldots \ge \lambda_s \ge 1$ be real numbers, let $\kappa_1,\ldots,\kappa_s$ be non-negative real numbers, and suppose that the real number c satisfies the inequalities*

$$\kappa_1+\ldots+\kappa_t \le tc \quad \textit{for } 1\le t\le s.$$

Then

$$\lambda_1^{\kappa_1}\ldots\lambda_s^{\kappa_s} \le (\lambda_1\ldots\lambda_s)^c.$$

This is Lemma 5 of [4].

Now, by Lemmas 6 and 9, (4.16) and (4.19), $J_2 \ll YT$. By (3.1), (3.2) and Lemma 10, it is easy to check that

$$J_2 \ll P_1^{1+\delta}P_2P_3^{2\theta}P_4^{2+\varepsilon}Y^{-1-2\theta} + \lambda_1^{1/2}P_1^{1+\varepsilon}P_2^{1/2+\varepsilon}P_3^{5/4+\varepsilon}P_4^{5/4+\varepsilon}Y^{1/2}.$$

By Cauchy's inequality and Lemma 5,

$$\begin{aligned}\int\limits_M F(\alpha)\,d\alpha &\ll J_2^{1/2}J_1^{1/2}\\ &\ll (P_1^{1+\delta}P_2P_3^{2\theta}P_4^{2+\varepsilon}Y^{-1-2\theta}\\ &\quad + \lambda_1^{1/2}P_1^{1+\varepsilon}P_2^{1/2+\varepsilon}P_3^{5/4+\varepsilon}P_4^{5/4+\varepsilon}Y^{1/2})^{1/2}(P_5P_6^{2\theta}P_7^{2+\varepsilon})^{1/2}\\ &\ll \lambda_1^{2/21}\Lambda^{-1/3}N^{26/21+\varepsilon}\\ &\quad\times(\lambda_1^{(2+\theta)/21}\lambda_2^{1/6}\lambda_3^{(1-\theta)/3}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3}N^{-(2-13\theta)/21}\\ &\quad + \lambda_1^{13/42}\lambda_2^{1/4}\lambda_3^{1/8}\lambda_4^{1/8}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3}N^{-(5-28\theta)/84}).\end{aligned}$$

By Lemma 10,

$$\begin{aligned}\lambda_1^{(2+\theta)/21}\lambda_2^{1/6}\lambda_3^{(1-\theta)/3}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3} &\ll \Lambda^{(2-\theta)/9},\\ \lambda_1^{13/42}\lambda_2^{1/4}\lambda_3^{1/8}\lambda_4^{1/8}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3} &\ll \Lambda^{5/24}\end{aligned}$$

and by (3.2) we have

$$\begin{aligned}&\lambda_1^{(2+\theta)/21}\lambda_2^{1/6}\lambda_3^{(1-\theta)/3}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3}N^{-(2-13\theta)/21}\\ &\qquad + \lambda_1^{13/42}\lambda_2^{1/4}\lambda_3^{1/8}\lambda_4^{1/8}\lambda_5^{1/6}\lambda_6^{(1-\theta)/3}N^{-(5-28\theta)/84} \ll N^{-3\varepsilon}\end{aligned}$$

so (3.21) follows.

By Lemma 5 and Cauchy's inequality,

$$\begin{aligned}&\int\limits_M \sum_{\substack{\mathbf{p}\\ (3.6),(3.7)}} (f_p(P_1,\lambda_1\alpha) - s(p,\alpha))\overline{f_A(W,\lambda_2p^3\alpha)}\\ &\qquad\times f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\\ &\qquad\times f_B(U,\lambda_5\alpha)f(V_6,\lambda_6p_{61}^3p_{62}^3p_{63}^3\alpha)f(V_7,\lambda_7p_{71}^3p_{72}^3p_{73}^3\alpha)\,d\alpha \ll v_1^{1/2}J_1^{1/2}.\end{aligned}$$

As above, by Lemma 10 and (3.2), (5.1) the above is

$$\ll (P_1^{1+\varepsilon}P_2P_3^{2\theta}P_4^{2+\varepsilon}Y^{-1-2\theta})^{1/2}(P_5P_6^{2\theta}P_7^{2+\varepsilon})^{1/2} \ll \lambda_1^{2/21}\Lambda^{-1/3}N^{26/21-\varepsilon}$$

hence

$$\begin{aligned}\int\limits_M F(\alpha)\,d\alpha = \int\limits_M &\sum_{\substack{\mathbf{p}\\ (3.6),(3.7)}} s(p,\alpha)\overline{f_A(W,\lambda_2p^3\alpha)}f(R_3,\lambda_3p^3p_{31}^3p_{32}^3p_{33}^3\alpha)\\ &\times f(R_4,\lambda_4p^3p_{41}^3p_{42}^3p_{43}^3\alpha)\\ &\times f_B(U,\lambda_5\alpha)f(V_6,\lambda_6p_{61}^3p_{62}^3p_{63}^3\alpha)f(V_7,\lambda_7p_{71}^3p_{72}^3p_{73}^3\alpha)\,d\alpha\\ &+ O(\lambda_1^{2/21}\Lambda^{-1/3}N^{26/21-\varepsilon}).\end{aligned}$$

By Lemma 5 and Cauchy's inequality,

$$\int_{M} \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} s(p,\alpha)(\overline{f_A(W,\lambda_2 p^3\alpha)} - \overline{g(p,\alpha)})$$
$$\times f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)$$
$$\times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\, d\alpha$$
$$\ll v_2^{1/2} J_1^{1/2}.$$

By Lemmas 5 and 10 and (5.2), the above is

$$\ll (\lambda_1^{1/2} P_1^{2+\varepsilon} P_3^{5/4+\varepsilon} P_4^{5/4+\varepsilon} Y^{-1/2} L N^{-1/2})^{1/2} (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2}$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\lambda_2^{1/3} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(7-28\theta)/84})$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\Lambda^{1/4} N^{-(7-28\theta)/84})$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}.$$

Hence

$$\int_M F(\alpha)\, d\alpha = \int_M \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} s(p,\alpha)\overline{g(p,\alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha)$$
$$\times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha)$$
$$\times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\, d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}).$$

By Lemma 5 and Cauchy's inequality,

$$\int_{M\setminus M_1} \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} s(p,\alpha)\overline{g(p,\alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha) f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha)$$
$$\times f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha) f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\, d\alpha$$
$$\ll (v^*)^{1/2} J_1^{1/2}.$$

By (6.12) and Lemma 10 the above is

$$\ll (P_1^2 W^2 N^{-1+\varepsilon} \lambda_1^{7/12+\varepsilon} \lambda_2^{7/12+\varepsilon} Y^{-2} X^{-2/3} (P_3 P_4)^{2+\varepsilon} \lambda_3^{1/4} \lambda_4^{1/4})^{1/2}$$
$$\times (P_5 P_6^{2\theta} P_7^{2+\varepsilon})^{1/2}$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\lambda_1^{179/504} \lambda_2^{7/24} \lambda_3^{1/8} \lambda_4^{1/8} \lambda_5^{1/6} \lambda_6^{(1-\theta)/3} N^{-(4-21\theta)/63})$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21+\varepsilon} (\Lambda^{11/48} N^{-(4-21\theta)/63})$$
$$\ll \lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}.$$

So

$$\int_{M} F(\alpha)\,d\alpha = \int_{M_1} \sum_{\substack{\mathbf{p} \\ (3.6),(3.7)}} s(p,\alpha)\overline{g(p,\alpha)} f(R_3, \lambda_3 p^3 p_{31}^3 p_{32}^3 p_{33}^3 \alpha)$$

$$\times f(R_4, \lambda_4 p^3 p_{41}^3 p_{42}^3 p_{43}^3 \alpha) f_B(U, \lambda_5 \alpha) f(V_6, \lambda_6 p_{61}^3 p_{62}^3 p_{63}^3 \alpha)$$

$$\times f(V_7, \lambda_7 p_{71}^3 p_{72}^3 p_{73}^3 \alpha)\,d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon})$$

$$= \int_{M_1} F^*(\alpha)\,d\alpha + O(\lambda_1^{2/21} \Lambda^{-1/3} N^{26/21-\varepsilon}).$$

Hence the Theorem follows.

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