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DIAGONAL EQUIVALENCE TO MATRICES WITH PRESCRIBED ROW AND COLUMN SUMS. II

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ABSTRACT. Let A be a nonnegative $m \times n$ matrix and let $r = (r_1, \cdots, r_m)$ and $c = (c_1, \cdots, c_n)$ be positive vectors such that $\Sigma_{i=1}^m r_i = \Sigma_{j=1}^n c_j$. It is well known that if there exists a nonnegative $m \times n$ matrix B with the same zero pattern as A having the ith row sum r_i and jth column sum c_j, there exist diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 has ith row sum r_i and jth column sum c_j. However the known proofs are at best cumbersome. It is shown here that this result can be obtained by considering the minimum of a certain real-valued function of n positive variables.

It has been shown originally by Sinkhorn and Knopp [8] and Brualdi, Parter, and Schneider [3] that if A is a nonnegative fully indecomposable matrix, i.e. A contains no $s \times (n-s)$ zero submatrix, then there exists a doubly stochastic matrix of the form D_1AD_2 where D_1 and D_2 are diagonal matrices with positive main diagonals. Later Djoković [4], and independently, London [5], proved the same theorem by considering the minimum of

$$f(x) = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \Big/ \prod_{j=1}^{n} x_j \tag{1}$$

for vectors $x = (x_1, \cdots, x_n)$ with positive coordinates.

In the meantime Menon [6] had obtained the following modification of this result.

Theorem 1. *Let A be a nonnegative $m \times n$ matrix and let $r = (r_1, \cdots, r_m)$ and $c = (c_1, \cdots, c_n)$ be positive vectors such that $\Sigma_{i=1}^m r_i = \Sigma_{j=1}^n c_j$. If there exists a nonnegative $m \times n$ matrix B with the same zero pattern as A, i.e. $b_{ij} = 0 \Leftrightarrow a_{ij} = 0$, having ith row sum r_i and jth column sum c_j, then*

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there exist diagonal matrices D_1 *and* D_2 *with positive main diagonals such that* D_1AD_2 *has ith row sum* r_i *and jth column sum* c_j.

Brualdi [2] showed that the existence of B in Theorem 1 is equivalent to the conditions that

(1) $A[E|F) = 0,\ A(E|F] \neq 0 \Rightarrow \sum_{i\in E} r_i < \sum_{j\in F} c_j$, and

(2) $A[E|F) = 0,\ A(E|F] = 0 \Rightarrow \sum_{i\in E} r_i = \sum_{j\in F} c_j$,

if $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$ holds.

The notation used has the following meaning. If E is a proper nonvoid subset of $M = \{1, \cdots, m\}$ and F is a proper nonvoid subset of $N = \{1, \cdots, n\}$, then $A[E|F)$ is that submatrix of A obtained by deleting from A those rows whose indices do not belong to E and those columns whose indices belong to F. The rows and columns of this submatrix appear in the same order as in A: rows are counted from top to bottom; columns are counted from left to right. $A(E|F]$ is that submatrix of A obtained by deleting from A those rows whose indices belong to E and those columns whose indices do not belong to F, where, as before, the rows and columns of this submatrix appear in the same order as in A. Observe that the submatrix $A(E|F]$ is the same as the submatrix $A[M - E|N - F)$. In the course of the paper two other submatrix notations are used. $A[E|F]$ is used to denote the submatrix $A[E|N - F) = A(M - E|F]$ in A; $A(E|F)$ is used to denote the submatrix $A[M - E|F) = A(E|N - F]$ in A.

Menon and Schneider [7] have given another proof of the Menon-Brualdi results.

It is the intent of this paper to show how the Djoković-London formula can be modified to yield the Menon-Brualdi-Schneider results.

We shall require the following lemma which follows at once from the concavity of the logarithm function. See [1, p. 7].

Lemma. *Let* $x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_n$ *be nonnegative real numbers and put* $\lambda_1 + \cdots + \lambda_n = \lambda$. *Then if* 0^0 *is taken to be* 1,

$$\left(\sum_{k=1}^{n} \lambda_k x_k\right)^{\lambda} \geq \lambda^{\lambda}\left(\prod_{k=1}^{n} x_k^{\lambda_k}\right).$$

We now prove the intended result. We shall assume that whenever there is a submatrix $A[E|F) = 0$ in A, $A(E|F] \neq 0$, for otherwise we could establish the result for the submatrices $A[E|F]$ and $A(E|F)$. We assume that Brualdi's condition (1) holds and that $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$.

Put

$$\phi(x) = \prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j=1}^{n} x_j^{c_j},$$

where $x = (x_1, \cdots, x_n)$ is positive, i.e. $x \in (R^n)^+$. We shall consider the problem of determining the minimum of ϕ on $(R^n)^+$. Since $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$, $\phi(\lambda x) = \phi(x)$ for all $\lambda > 0$ and thus we can restrict our attention to the set K of $x \in (R^n)^+$ for which $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2} = 1$.

Suppose on K, $x \to \Delta$, the boundary of $(R^n)^+$. Let $F = \{j | x_j \to 0\}$ and then set $E = \{i | a_{ij} = 0 \text{ for all } j \notin F\}$. Since $x \to \Delta$ on K, F is a nonvoid proper subset of $\{1, \cdots, n\}$. Since every $c_j > 0$, E is a proper subset of $\{1, \cdots, m\}$. If $E = \varnothing$, $\phi(x) \to \infty$ as $x \to \Delta$. If $E \neq \varnothing$, we write $\phi(x) = \phi_1(x)\phi_2(x)$ where

$$\phi_1(x) = \prod_{i\in E}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\in F} x_j^{c_j} = \prod_{i\in E}\left(\sum_{j\in F} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\in F} x_j^{c_j} \tag{3}$$

and

$$\phi_2(x) = \prod_{i\notin E}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^{r_i} \Big/ \prod_{j\notin F} x_j^{c_j}. \tag{4}$$

Since ϕ_2 has a positive limit as $x \to \Delta$, we concentrate on ϕ_1.

Let B be as in Theorem 1. Then $B[E|F) = 0$ and therefore $\sum_{j\in F} b_{ij} = r_i$ for each $i \in E$ and hence from the Lemma

$$\left(\sum_{j\in F} a_{ij}x_j\right)^{r_i} \geq r_i^{r_i} \frac{\prod_{j\in F} a_{ij}^{b_{ij}}}{\prod_{j\in F} b_{ij}^{b_{ij}}} \prod_{j\in F} x_j^{b_{ij}} = \theta_i \prod_{j\in F} x_j^{b_{ij}} \tag{5}$$

for all $i \in E$, where 0^0 is taken to be 1. Whence

$$\phi_1(x) \geq \prod_{i\in E} \theta_i \Big/ \prod_{j\in F} x_j^{c_j - \sum_{i\in E} b_{ij}}. \tag{6}$$

Since $\sum_{i=1}^m b_{ij} = c_j$, $j = 1, \cdots, n$, certainly $\sum_{i\in E} b_{ij} \leq c_j$ for every $j \in F$. However since $\sum_{j\in F} c_j > \sum_{i\in E} r_i = \sum_{i\in E}\sum_{j\in F} b_{ij}$, $\sum_{i\in E} b_{ij_0} < c_{j_0}$ for at least one $j_0 \in F$. Thus $\phi_1(x) \to \infty$ and so $\phi(x) \to \infty$ as $x \to \Delta$.

It follows that ϕ achieves a minimum on $(R^n)^+$. At such a point $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)$, $\partial \ln \phi(x)/\partial x_k = 0$ for $k = 1, \cdots, n$. Whence

$$\sum_{i=1}^{m} r_i\left(a_{ik} \Big/ \sum_{j=1}^{n} a_{ij}\bar{x}_j\right) - c_k/\bar{x}_k = 0, \tag{7}$$

$k = 1, \cdots, n$. Put $\bar{y}_i = r_i / \sum_{j=1}^n a_{ij} \bar{x}_j$, $i = 1, \cdots, m$, and then set $D_1 = \operatorname{diag}(\bar{y}_1, \cdots, \bar{y}_m)$, $D_2 = \operatorname{diag}(\bar{x}_1, \cdots, \bar{x}_n)$. Then $D_1 A D_2$ satisfies the conclusion of Theorem 1.

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