Diagonal p-permutation functors, semisimplicity, and functorial equivalence of blocks

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Abstract

Let k be an algebraically closed field of characteristic $p > 0$, let R be a commutative ring, and let $\mathbb{F}$ be an algebraically closed field of characteristic 0. We consider the R-linear category $\mathcal{F}^\Delta_{Rpp_k}$ of diagonal p-permutation functors over R. We first show that the category $\mathcal{F}^\Delta_{\mathbb{F}pp_k}$ is *semisimple*, and we give a parametrization of its simple objects, together with a description of their evaluations.

Next, to any pair (G, b) of a finite group G and a block idempotent b of kG, we associate a diagonal p-permutation functor $RT^\Delta_{G,b}$ in $\mathcal{F}^\Delta_{Rpp_k}$. We find the *decomposition* of the functor $\mathbb{F}T^\Delta_{G,b}$ as a direct sum of simple functors in $\mathcal{F}^\Delta_{\mathbb{F}pp_k}$. This leads to a characterization of *nilpotent blocks* in terms of their associated functors in $\mathcal{F}^\Delta_{\mathbb{F}pp_k}$.

Finally, for such pairs (G, b) of a finite group and a block idempotent, we introduce the notion of *functorial equivalence* over R, which (in the case $R = \mathbb{Z}$) is slightly weaker than p-permutation equivalence, and we prove a corresponding *finiteness theorem*: for a given finite p-group D, there is only a finite number of pairs (G, b), where G is a finite group and b a block idempotent of kG with defect isomorphic to D, up to functorial equivalence over $\mathbb{F}$.

Keywords: diagonal p-permutation functor, semisimple, block, functorial equivalence.
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1 Introduction

In the past decades, various categories have been considered, where objects are finite groups and morphisms are obtained from various types of double group actions. The linear representations of these categories give rise to interesting functor categories: Examples include biset functors ([Bc10a]), p-permutation functors ([D15]), simple modules over Green biset functors ([Ro12]), modules over shifted representation functors ([BR21]), fibered biset functors ([BC18], and generalizations of these ([BO20], [B16]).

In the present paper, we consider another example of a similar context: For an algebraically closed field k of positive characteristic p, and a commutative ring R, we define the following category Rpp_k^Δ:

- The objects of Rpp_k^Δ are the finite groups.

- For finite groups G and H, the set of morphisms $\mathrm{Hom}_{Rpp_k^\Delta}(G, H)$ from G to H in Rpp_k^Δ is equal to $R \otimes_{\mathbb{Z}} T^\Delta(H, G)$, where $T^\Delta(H, G)$ is the Grothendieck group of the category of *diagonal p-permutation (kH, kG)-bimodules*. These are p-permutation bimodules which admit only indecomposable direct summands with twisted diagonal vertices (or equivalently, p-permutation bimodules which are projective when considered as left or right modules).

- The composition in Rpp_k^Δ is induced by R-linearity from the usual tensor product of bimodules: if G, H, and K are finite groups, if M is a diagonal p-permutation (kH, kG)-bimodule and N is a diagonal p-permutation (kK, kH)-bimodule, then $N \otimes_{kH} M$ is a diagonal p-permutation (kK, kG)-bimodule. The composition of (the isomorphism class of) N and (the isomorphism class of) M is by definition (the isomorphism class of) $N \otimes_{kH} M$.

- The identity morphism of the group G is the (isomorphism class of the) (kG, kG)-bimodule kG.

The category Rpp_k^Δ is an R-linear category. The R-linear functors from Rpp_k^Δ to the category $_R\mathsf{Mod}$ of R-modules are called *diagonal p-permutation functors* over R. These functors, together with natural transformations between them, form an R-linear abelian category $\mathcal{F}_{Rpp_k}^\Delta$.

These diagonal p-permutation functors have been introduced in [BY20], in the case R is a field $\mathbb{F}$ of characteristic 0. Even though this will also be our assumption in most of the present paper, we give the above more general definition, as we will also need to consider the case $R = \mathbb{Z}$ in various places.

The main motivation for considering diagonal p-permutation functors comes from block theory, and in particular the notion of p-permutation equivalence of blocks of finite groups, introduced in [BX08] and developed in [BP20]. This notion inserts in the following chain of equivalences of blocks of group algebras, namely

$$\text{Puig's equiv.} \implies \text{Rickard's splendid derived equiv.} \implies p\text{-permutation equiv.},$$

and these equivalences are related to important *structural conjectures*, such as Broué's abelian defect group conjecture (Conjecture 9.7.6 in [L18]), and *finiteness conjectures*, such as Puig's conjecture (Conjecture 6.4.2 in [L18]), or Donovan's conjecture (Conjecture 6.1.9 in [L18]).

In this paper, we introduce yet another equivalence, weaker than p-permutation equivalence, between blocks of group algebras, which we call *functorial equivalence* over R: to each pair (G, b) of a finite group G and a block idempotent b of the group algebra kG, we associate a canonical diagonal p-permutation functor over R, denoted by $RT_{G,b}^\Delta$. This functor is a direct summand of the representable functor RT_G^Δ at G, obtained from the (kG, kG)-bimodule kGb, viewed as an idempotent endomorphism of G in the category Rpp_k^Δ. When (H, c) is a pair of a finite group H and a block idempotent c of kH, we say that (G, b) and (H, c) are *functorially equivalent over R* if the functors $RT_{G,b}^\Delta$ and $RT_{H,c}^\Delta$ are isomorphic in $\mathcal{F}_{Rpp_k}^\Delta$.

We obtain the following main results:

- The category $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$ of diagonal p-permutation functors over $\mathbb{F}$ is a *semisimple* $\mathbb{F}$-linear abelian category (Theorem 6.15).

- The *simple* diagonal p-permutation functors over $\mathbb{F}$ are parametrized by triples (L, u, V), where (L, u) is a D^Δ-pair (see Section 3), and V is a simple $\mathbb{F}\mathrm{Out}(L, u)$-module (see Notation 6.8).

- The *evaluation* $S_{L,u,V}(G)$ of the simple functor $S_{L,u,V}$ parametrized by the triple (L, u, V), at a finite group G, is explicitly computed in Corollary 7.4.

- The *multiplicity* of any simple functor $S_{L,u,V}$ in the functor $\mathbb{F}T^{\Delta}_{G,b}$ associated to a block idempotent b of a finite group G is explicitly given by three equivalent formulas (Theorem 8.22): One in terms of fixed points of some subgroups of $\mathrm{Out}(L, u)$ on V, the second one in terms of the "u-invariant" (G, b)-Brauer pairs (P, e), and the third one in terms of the "u-invariant" local pointed subgroups P_γ of G on kGb.

- We give two characterizations (Theorem 9.2) of *nilpotent blocks* in terms of their associated diagonal p-permutation functors: let b be a block idempotent of the group algebra kG. Then b is nilpotent if and only if one of the equivalent following conditions holds:

  - If $S_{L,u,V}$ is a simple summand of $\mathbb{F}T^{\Delta}_{G,b}$, then $u = 1$.
  - The functor $\mathbb{F}T^{\Delta}_{G,b}$ is isomorphic to the representable functor $\mathbb{F}T^{\Delta}_{D}$ for some p-group D.

- We show (Proposition 10.4) that if (G, b) and (H, c) are functorially equivalent blocks over $\mathbb{F}$, then kGb and kHc have the *same number of simples modules*, and b and c have *isomorphic defect groups*.

- We show (Proposition 10.4) that if b and c are *nilpotent* blocks of G and H, respectively, then (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$ *if and only* if b and c have isomorphic defect groups.

- We prove a *finiteness theorem* (Theorem 10.6) for functorial equivalence of blocks: for a given finite p-group D, there are only finitely many pairs (G, b), where G is a finite group, and b is a block idempotent of kG with defect isomorphic to D, up to functorial equivalence over $\mathbb{F}$.

- We give a sufficient condition (Theorem 11.1) for two pairs (G, b) and (H, c) to be functorially equivalent over $\mathbb{F}$ in the situation of *Broué's abelian defect group conjecture*.

The paper is organized as follows: Sections 2 to 5 are devoted to technical tools used in the proof of our semisimplicity Theorem 6.15. Section 2 deals with Brauer quotients of tensor products of diagonal p-permutation bimodules, Section 3 recalls the definitions of pairs, D^{Δ}-pairs, and idempotents of p-permutation rings. The main theorem of this section is Theorem 3.7. In Section 4 we state some results from Clifford theory, and in Section 5 a theorem on some equivalences of abelian categories. Section 6 is devoted to the proof of our semisimplicity theorem (Corollary 6.15). In Section 7, we compute the evaluations of the simple diagonal p-permutation functors (Corollary 7.4). In Section 8, we introduce the diagonal p-permutation functors associated to blocks of finite groups, and we describe their decomposition as a direct sum of simple functors (Theorem 8.22). In Section 9, we apply these results to the characterization of nilpotents blocks in terms of the associated functors (Theorem 9.2). In Section 10, we introduce functorial equivalence of blocks, and state some of its basic consequences (Proposition 10.4); next we prove our finiteness theorem for functorial equivalence over $\mathbb{F}$ (Theorem 10.6). Finally, Section 11 considers the case of blocks with abelian defect groups (Theorem 11.1).

2 Brauer character formula

Throughout G, H and K denote finite groups. Let $\mathbb{F}$ denote an algebraically closed field of characteristic 0 and let k denote an algebraically closed field of positive characteristic p. We assume that all the modules considered are finitely generated.

Let M be a (kG, kH)-bimodule. For $m \in M$ and $(g,h) \in G \times H$, the formula $(g,h)m = gmh^{-1}$ induces an isomorphism between categories ${}_{kG}\mathsf{mod}_{kH}$ and ${}_{k[G \times H]}\mathsf{mod}$. In what follows, we will often use this isomorphism to identify left $k[G \times H]$-modules via (kG, kH)-bimodules.

Let M be a kG-module and let P be a p-subgroup of G. The Brauer construction of M at P will be denoted by $M[P]$.

2.1 Lemma *Let $N \trianglelefteq G$ be a normal subgroup of G and let V be an $\mathbb{F}G$-module with character χ. Then the character of $\mathrm{Def}^G_{G/N} V$ is given by*

$$gN \mapsto \frac{1}{|N|} \sum_{n \in N} \chi(gn)$$

for $gN \in G/N$.

Proof By [Bc10a, Lemma 7.1.3] the character of $\mathrm{Def}^G_{G/N} V$ is given by the formula

$$gN \mapsto \frac{1}{|G|} \sum_{\substack{u \in G/N, h \in G \\ gN \cdot u = u \cdot h}} \chi(h) = \frac{1}{|G|} |G/N| \sum_{n \in N} \chi(gn) = \frac{1}{|N|} \sum_{n \in N} \chi(gn) \, ,$$

as desired. $\qquad\square$

Let X be a subgroup of $G \times H$ and let L be a finite dimensional $\mathbb{F}X$-module. Let also Y be a subgroup of $H \times K$, and M a finite dimensional $\mathbb{F}Y$-module. Since $k_1(X) \times k_2(X)$ is a subgroup of X, the module L can be viewed as an $\big(\mathbb{F}k_1(X), \mathbb{F}k_2(X)\big)$-bimodule. Similarly, M can be viewed as an $\big(\mathbb{F}k_1(Y), \mathbb{F}k_2(Y)\big)$-bimodule. Set $S = k_2(X) \cap k_1(Y)$. Then the tensor product $L \otimes_{\mathbb{F}S} M$ is an $\big(\mathbb{F}k_1(X), \mathbb{F}k_2(Y)\big)$-bimodule. For $(a,b) \in X * Y$, choose $h \in H$ such that $(a,h) \in X$ and $(h,b) \in Y$. Then

$$(a,b)(l \otimes m) = (a,h)l \otimes (h,b)m$$

is a well defined element of $L \otimes_{\mathbb{F}S} M$, and this defines a structure of $\mathbb{F}(X * Y)$-module on $L \otimes_{\mathbb{F}S} M$. This construction is first used in [Bc10b].

Let $X \times_H Y := \{((g,h),(\tilde{h},k)) \in X \times Y \mid h = \tilde{h}\}$. Consider the surjective group homomorphism

$$\nu : X \times_H Y \to X * Y \, , \quad ((g,h),(h,k)) \mapsto (g,k) \, .$$

The kernel of ν is $\{((1,h),(h,1)) \mid h \in S\}$ and we denote by $\bar{\nu} : (X \times_H Y)/\ker(\nu) \to X * Y$ the isomorphism induced by ν.

2.2 Lemma *Let $X \leqslant G \times H$ and $Y \leqslant H \times K$ be subgroups. Let also L be a finite dimensional $\mathbb{F}X$-module and M a finite dimensional $\mathbb{F}Y$-module. Then the character $\chi_{L \otimes_{\mathbb{F}S} M}$ of the $\mathbb{F}(X * Y)$-module $L \otimes_{\mathbb{F}S} M$ is given by*

$$\chi_{L \otimes_{\mathbb{F}S} M}(a,b) = \frac{1}{|S|} \sum_{\substack{h \in H \\ (a,h) \in X \\ (h,b) \in Y}} \chi_L(a,h) \, \chi_M(h,b),$$

for all $(a,b) \in X * Y$, where χ_L and χ_M are the characters of L and M, respectively.

Proof Let $(a, b) \in X * Y$ be an arbitrary element and $c \in H$ such that $(a, c) \in X$ and $(c, b) \in Y$. By [Bo20, Proposition 2.8], we have

$$\chi_{L \otimes_{\mathbb{F}S} M} = \left(\mathrm{Iso}(\bar{\nu}) \circ \mathrm{Def}_{(X \times_H Y)/\ker(\nu)}^{X \times_H Y} \circ \mathrm{Res}_{X \times_H Y}^{X \times Y} \right) (\chi_L \times \chi_M).$$

Hence by Lemma 2.1 we have

$$\begin{aligned}
\chi_{L \otimes_{\mathbb{F}S} M}(a, b) &= \frac{1}{|S|} \sum_{h \in S} \chi_L(a, hc) \chi_M(hc, b) \\
&= \frac{1}{|S|} \sum_{h \in Sc} \chi_L(a, h) \chi_M(h, b) \\
&= \frac{1}{|S|} \sum_{\substack{h \in H \\ (a,h) \in X \\ (h,b) \in Y}} \chi_L(a, h)\, \chi_M(h, b),
\end{aligned}$$

as desired. $\square$

2.3 Notation (a) Let $U \leqslant G$ and $W \leqslant K$ be subgroups and $\gamma : W \to U$ a group isomorphism. We set

$$\Delta(U, \gamma, W) = \left\{ (\gamma(w), w) : w \in W \right\}$$

for the corresponding twisted diagonal subgroup of $G \times K$.

(b) Let $\Delta(U, \gamma, W)$ be a twisted diagonal subgroup of $G \times K$. Let $\Gamma_H(U, \gamma, W)$ denote the set of triples (α, V, β) where V is a subgroup of H, and $\alpha : V \to U$ and $\beta : W \to V$ are group isomorphisms with the property that $\gamma = \alpha \circ \beta$. The group $N_{G \times K}(\Delta(U, \gamma, W)) \times H$ acts on $\Gamma_H(U, \gamma, W)$ in the following way: if $((x, z), h) \in N_{G \times K}(\Delta(U, \gamma, W)) \times H$ and $(\alpha, V, \beta) \in \Gamma_H(U, \gamma, W)$, then

$$((x, z), h) \cdot (\alpha, V, \beta) = (i_x \circ \alpha \circ i_h^{-1}, {}^hV, i_h \circ \beta \circ i_z^{-1}),$$

where i_u denotes the conjugation by an element u.

2.4 Proposition *[BP20, Corollary 7.4(b)]* Let L be a p-permutation (kG, kH)-bimodule and M a p-permutation (kH, kK)-bimodule. Suppose that all of the indecomposable summands of L and M have twisted diagonal vertices. Let $\tilde{\Gamma}_H(U, \gamma, W)$ denote a set of representatives of $N_{G \times K}(\Delta(U, \gamma, W)) \times H$-orbits of $\Gamma_H(U, \gamma, W)$. Then the $kN_{G \times K}(\Delta(U, \gamma, W))$-module $(L \otimes_{kH} M)[\Delta(U, \gamma, W)]$ is isomorphic to

$$\bigoplus_{(\alpha, V, \beta)} \mathrm{Ind}_{N_{G \times H}(\Delta(U, \alpha, V)) * N_{H \times K}(\Delta(V, \beta, W))}^{N_{G \times K}(\Delta(U, \gamma, W))} \left(L[\Delta(U, \alpha, V)] \otimes_{kC_H(V)} M[\Delta(V, \beta, W)] \right),$$

where $(\alpha, V, \beta) \in \tilde{\Gamma}_H(U, \gamma, W)$.

We say that (P, u) is a *pair* of a finite group S, if P is a p-subgroup of S and u is a p'-element of $N_S(P)$. Let (P, u) be a pair of S and let X be a p-permutation kS-module. Then $\tau_{P,u}^S(X) \in \mathbb{F}$ is defined as the value at u of the Brauer character of $X[P]$ (see [BT10] Notation 2.1 and Notation 2.15 for details).

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In the next proposition, we will consider three finite groups G, H and K, and we will need to lift p-permutation (kG, kH)-bimodules and (kH, kK)-bimodules from characteristic p to characteristic 0. To do this, we can choose a p-modular system $(\mathbb{K}, \mathcal{O}, k)$ containing k, where $\mathcal{O}$ is a complete discrete valuation ring with residue field k and field of fractions $\mathbb{K}$ of characteristic 0. For a finite group S and a p-permutation kS-module X, there is a unique p-permutation $\mathcal{O}S$-lattice $X^{\mathcal{O}}$, up to isomorphism, such that $X^{\mathcal{O}}/J(\mathcal{O})X^{\mathcal{O}} \cong X$. We denote by X^0 the $\mathbb{K}S$-module $\mathbb{K} \otimes_{\mathcal{O}} X^{\mathcal{O}}$. For a p-subgroup P of S and a (possibly p-singular) element u of $N_S(P)$, we define $\widehat{\tau}_{P,u}^S(X)$ as the value at u of the (ordinary) character of $X[P]^0$. As $\widehat{\tau}_{P,u}^S(X)$ is a sum of roots of unity, we may assume that $\widehat{\tau}_{P,u}^S(X)$ lies in our algebraically closed field $\mathbb{F}$ of characteristic 0. Moreover $\widehat{\tau}_{P,u}^S(X) = \tau_{P,u}^S(X)$ when u is p-regular. With this notation:

2.5 Proposition *Let $(s,t) \in N_{G \times K}\big(\Delta(U, \gamma, W)\big)$. Let L be a p-permutation (kG, kH)-bimodule and M a p-permutation (kH, kK)-bimodule. Suppose that all of the indecomposable summands of L and M have twisted diagonal vertices. Then*

$$\widehat{\tau}_{\Delta(U,\gamma,W),(s,t)}^{G \times K}(L \otimes_{kH} M) = \frac{1}{|H|} \sum_{\substack{(\alpha,V,\beta) \in \Gamma_H(U,\gamma,W),\, h \in H \\ (s,h) \in N_{G \times H}(\Delta(U,\alpha,V)) \\ (h,t) \in N_{H \times K}(\Delta(V,\beta,W))}} \widehat{\tau}_{\Delta(U,\alpha,V),(s,h)}^{G \times H}(L)\, \widehat{\tau}_{\Delta(V,\beta,W),(h,t)}^{H \times K}(M).$$

Proof We use Proposition 2.4, where we set for simplicity

$$N_{U,\gamma,W} = N_{G \times K}\big(\Delta(U, \gamma, W)\big)$$
$$N_{U,\alpha,V} = N_{G \times H}\big(\Delta(U, \alpha, V)\big)$$
$$N_{V,\beta,W} = N_{H \times K}\big(\Delta(V, \beta, W)\big)$$
$$N_{U,\alpha,V,\beta,W} = N_{G \times H}\big(\Delta(U, \alpha, V)\big) * N_{H \times K}\big(\Delta(V, \beta, W)\big).$$

We write $\tilde{\Gamma}_{U,\gamma,W} = \tilde{\Gamma}_H(U, \gamma, W)$ for short, for a set of representatives of the orbits of $N_{U,\gamma,W} \times H$-action on $\Gamma_{U,\gamma,W} = \Gamma_H(U, \gamma, W)$. For $(\alpha, V, \beta) \in \Gamma_{U,\gamma,W}$, we denote by $S_{U,\alpha,V,\beta,W}$ the stabilizer of (α, V, β) in $N_{U,\gamma,W} \times H$, i.e.,

$$S_{U,\alpha,V,\beta,W} = \big\{\big((x,z),h\big) \in N_{U,\gamma,W} \times H \mid i_x \circ \alpha = \alpha \circ i_h,\ i_h \circ \beta = \beta \circ i_z\big\}.$$

With this notation, we have an isomorphism of $kN_{U,\gamma,W}$-modules

$$(L \otimes_{kH} M)\big[\Delta(U, \gamma, W)\big] \cong \bigoplus_{(\alpha,V,\beta) \in \tilde{\Gamma}_{U,\gamma,W}} \mathrm{Ind}_{N_{U,\alpha,V\beta,W}}^{N_{U,\gamma,W}} \left(L\big[\Delta(U, \alpha, V)\big] \otimes_{kC_H(V)} M\big[\Delta(V, \beta, W)\big]\right).$$

This isomorphism can be lifted to $\mathcal{O}$, to give an isomorphism of $\mathbb{F}N_{U,\gamma,W}$-modules

$$(L \otimes_{kH} M)\big[\Delta(U, \gamma, W)\big]^0 \cong \bigoplus_{(\alpha,V,\beta) \in \tilde{\Gamma}_{U,\gamma,W}} \mathrm{Ind}_{N_{U,\alpha,V\beta,W}}^{N_{U,\gamma,W}} \left(L\big[\Delta(U, \alpha, V)\big]^0 \otimes_{\mathbb{F}C_H(V)} M\big[\Delta(V, \beta, W)\big]^0\right),$$

and $\widehat{\tau}_{\Delta(U,\gamma,W),(s,t)}^{G \times K}(L \otimes_{kH} M)$ is equal to the trace of (s,t) acting on the right hand side. This

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implies that

$$
\widehat{\tau}^{G\times K}_{\Delta(U,\gamma,W),(s,t)}(L\otimes_{kH}M) = \sum_{(\alpha,V,\beta)\in\tilde{\Gamma}_{U,\gamma,W}}\frac{1}{|N_{U,\alpha,V,\beta,W}|}\sum_{\substack{(a,b)\in N_{U,\gamma,W}\\(^a s,{}^b t)\in N_{U,\alpha,V,\beta,W}}}\theta_{U,\alpha,V,\beta,W}(^a s,{}^b t)
$$

$$
= \sum_{(\alpha,V,\beta)\in\Gamma_{U,\gamma,W}}\frac{|S_{U,\alpha,V,\beta,W}|}{|N_{U,\gamma,W}||H||N_{U,\alpha,V,\beta,W}|}\sum_{\substack{(a,b)\in N_{U,\gamma,W}\\(^a s,{}^b t)\in N_{U,\alpha,V,\beta,W}}}\theta_{U,\alpha,V,\beta,W}(^a s,{}^b t)
$$

where $\theta_{U,\alpha,V,\beta,W}$ is the character of $L\big[\Delta(U,\alpha,V)\big]^0\otimes_{\mathbb{F}C_H(V)}M\big[\Delta(V,\beta,W)\big]^0$.

Now we observe that $p_1(S_{U,\alpha,V,\beta,W})=N_{U,\alpha,V,\beta,W}$ and $k_2(S_{U,\alpha,V,\beta,W})=C_H(V)$. It follows that

$$
|S_{U,\alpha,V,\beta,W}|=|N_{U,\alpha,V,\beta,W}||C_H(V)|,
$$

so

$$
\widehat{\tau}^{G\times K}_{\Delta(U,\gamma,W),(s,t)}(L\otimes_{kH}M) = \sum_{(\alpha,V,\beta)\in\Gamma_{U,\gamma,W}}\frac{|C_H(V)|}{|N_{U,\gamma,W}||H|}\sum_{\substack{(a,b)\in N_{U,\gamma,W}\\(^a s,{}^b t)\in N_{U,\alpha,V,\beta,W}}}\theta_{U,\alpha,V,\beta,W}(^a s,{}^b t).
$$

But for $(a,b)\in N_{U,\gamma,W}$, saying that $(^a s,{}^b t)\in N_{U,\alpha,V,\beta,W}$ amounts to saying that $(s,t)\in N^{(a,b)}_{U,\alpha,V,\beta,W}=N_{U,i_a^{-1}\circ\alpha,V,\beta\circ i_b,W}$. Moreover $\theta_{U,\alpha,V,\beta,W}(^a s,{}^b t)=\theta_{U,i_a^{-1}\circ\alpha,V,\beta\circ i_b,W}(s,t)$. So

$$
\widehat{\tau}^{G\times K}_{\Delta(U,\gamma,W),(s,t)}(L\otimes_{kH}M) = \frac{|C_H(V)|}{|N_{U,\gamma,W}||H|}\sum_{\substack{(a,b)\in N_{U,\gamma,W}\\(i_a^{-1}\circ\alpha,V,\beta\circ i_b)\in\Gamma_{U,\gamma,W}\\(s,t)\in N_{U,i_a^{-1}\circ\alpha,V,\beta\circ i_b,W}}}\theta_{U,i_a^{-1}\circ\alpha,V,\beta\circ i_b,W}(s,t)
$$

$$
= \frac{|C_H(V)|}{|H|}\sum_{\substack{(\alpha,V,\beta)\in\Gamma_{U,\gamma,W}\\(s,t)\in N_{U,\alpha,V,\beta,W}}}\theta_{U,\alpha,V,\beta,W}(s,t).
$$

Now by Lemma 2.2

$$
\theta_{U,\alpha,V,\beta,W}(s,t) = \frac{1}{|C_H(V)|}\sum_{\substack{h\in H\\(s,h)\in N_{U,\alpha,V}\\(h,t)\in N_{V,\beta,W}}}\widehat{\tau}^{G\times H}_{\Delta(U,\alpha,V),(s,h)}(L)\,\widehat{\tau}^{H\times K}_{\Delta(V,\beta,W),(h,t)}(M).
$$

It follows that

$$
\widehat{\tau}^{G\times K}_{\Delta(U,\gamma,W),(s,t)}(L\otimes_{kH}M) = \frac{1}{|H|}\sum_{\substack{(\alpha,V,\beta)\in\Gamma_{U,\gamma,W}\\h\in H\\(s,h)\in N_{U,\alpha,V}\\(h,t)\in N_{V,\beta,W}}}\widehat{\tau}^{G\times H}_{\Delta(U,\alpha,V),(s,h)}(L)\,\widehat{\tau}^{H\times K}_{\Delta(V,\beta,W),(h,t)}(M).
$$

$\square$

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2.6 Corollary *Let L be a p-permutation (kG, kH)-bimodule and let M be a p-permutation (kH, kK)-bimodule. Suppose that all of the indecomposable summands of L and M have twisted diagonal vertices. Then, for any diagonal pair $\big(\Delta(U, \gamma, W), (s,t)\big)$ of $G \times K$*

$$\tau^{G \times K}_{\Delta(U,\gamma,W),(s,t)}(L \otimes_{kH} M) = \frac{1}{|H|} \sum_{\substack{(\alpha,V,\beta) \in \Gamma_H(U,\gamma,W),\, h \in H_{p'} \\ (s,h) \in N_{G \times H}(\Delta(U,\alpha,V)) \\ (h,t) \in N_{H \times K}(\Delta(V,\beta,W))}} \tau^{G \times H}_{\Delta(U,\alpha,V),(s,h)}(L)\, \tau^{H \times K}_{\Delta(V,\beta,W),(h,t)}(M),$$

where $H_{p'}$ is the set of p'-elements of H.

Proof For $(\alpha, V, \beta) \in \Gamma_H(U, \gamma, W)$, the indecomposable direct summands of the $\mathbb{F}N_{U,\alpha,V}$-module $L[\Delta(U, \alpha, V)]^0$ have twisted diagonal vertices. Similarly, the indecomposable direct summands of the $\mathbb{F}N_{V,\beta,W}$-module $M[\Delta(V, \beta, W)]^0$ have twisted diagonal vertices. Now assume that $(s, h) \in N_{U,\alpha,V}$ is such that $\widehat{\tau}^{G \times H}_{\Delta(U,\alpha,V),(s,h)}(L) \neq 0$. Then by [NT89, Theorem 4.7.4], the p-part of the element (s, h) is contained in a twisted diagonal p-subgroup of $G \times H$. Since s is a p'-element, this means that h is a p'-element as well. Therefore the summation over $h \in H$ in Proposition 2.5 reduces to a summation over $h \in H_{p'}$, and then both (s, h) and (h, t) are p'-elements, so we can replace $\widehat{\tau}$ by τ throughout. $\qquad\square$

3 Pairs

Recall that k denotes an algebraically closed field of characteristic p and $\mathbb{F}$ denotes an algebraically closed field of characteristic zero. Also, G, H and K denote finite groups.

3.1 (a) We denote by $T(G)$ the Grothendieck group of p-permutation kG-modules. Let $\mathcal{Q}_{G,p}$ denote the set of pairs (P, u) where P is a p-subgroup of G and u is a p'-element of $N_G(P)$. The group G acts on the set $\mathcal{Q}_{G,p}$ via conjugation and we write $[\mathcal{Q}_{G,p}]$ for a set of representatives of G-orbits on $\mathcal{Q}_{G,p}$.

Let M be a p-permutation kG-module. Recall from Section 2 that $\tau^G_{P,u}(M)$ is defined as the value at u of the Brauer character of $M[P]$. We extend this $\mathbb{F}$-linearly to obtain a map $\tau^G_{P,u}$ from $\mathbb{F}T(G) := \mathbb{F} \otimes_{\mathbb{Z}} T(G)$ to $\mathbb{F}$. It is well known that the set of maps $\tau_{P,u}$, for $(P, u) \in [\mathcal{Q}_{G,p}]$, is the set of all distinct $\mathbb{F}$-algebra homomorphisms from $\mathbb{F}T(G)$ to $\mathbb{F}$ (see e.g. Proposition 2.18 of [BT10] for details), and hence the primitive idempotents $F^G_{P,u}$ of $\mathbb{F}T(G)$ are indexed by $[\mathcal{Q}_{G,p}]$. The idempotent $F^G_{P,u}$ is defined by the property that for any $(Q, t) \in [\mathcal{Q}_{G,p}]$,

$$\tau^G_{Q,t}(F^G_{P,u}) = \begin{cases} 1 & \text{if } (Q, t) =_G (P, u), \\ 0 & \text{otherwise}. \end{cases}$$

(b) More generally, we often consider pairs (P, s) where P is a p-group and s is a generator of a p'-group acting on P. We write $P\langle s \rangle := P \rtimes \langle s \rangle$ for the corresponding semi-direct product. We say that two pairs (P, s) and (Q, t) are *isomorphic* and write $(P, s) \cong (Q, t)$, if there is a group isomorphism $f : P\langle s \rangle \to Q\langle t \rangle$ that sends s to a conjugate of t. The following type of pairs will play a crucial role in this paper.

3.2 Definition [BY20] A pair (P, s) is called a D^Δ-*pair*, if $C_{\langle s \rangle}(P) = 1$.

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See [BY20, Proposition 5.6] for more properties of D^Δ-pairs. Note that for an arbitrary pair (P, s), the pair $(\tilde{P}, \tilde{s}) := (PC_{\langle s \rangle}(P)/C_{\langle s \rangle}(P), sC_{\langle s \rangle}(P))$ is a D^Δ-pair.

3.3 Lemma (i) Let $\big(\Delta(U, \gamma, W), (s, t)\big)$ be a pair of $G \times K$. Then we have $(\tilde{U}, \tilde{s}) \cong (\tilde{W}, \tilde{t})$.

(ii) Let (U, s) be a pair of G and let (W, t) be a pair of K. Suppose that $(U, s) \cong (W, t)$. Then $\big(\Delta(U, \gamma, W), (s, t)\big)$ is a pair of $G \times K$ for some group isomorphism $\gamma : W \to U$.

(iii) Let (U, s) be a pair of G and let (W, t) be a pair of K. Assume that we have $(\tilde{U}, \tilde{s}) \cong (\tilde{W}, \tilde{t})$. Then $(\Delta(U, \gamma, W), (s, t))$ is a pair of $G \times K$ for some group isomorphism $\gamma : W \to U$.

Proof (i) The element (s, t) normalizes the group $\Delta(U, \gamma, W)$ means that s normalizes U, t normalizes W, and $\gamma(^t u) = {}^s \gamma(u)$ for all $u \in U$. Set $X := N_{\langle s \rangle \times \langle t \rangle}(\Delta(U, \gamma, W)) \leqslant \langle s \rangle \times \langle t \rangle$. Then we have

$$p_1(X) = \langle s \rangle, \quad p_2(X) = \langle t \rangle, \quad k_1(X) = C_{\langle s \rangle}(U), \quad k_2(X) = C_{\langle t \rangle}(W).$$

Hence we have a group isomorphism

$$\eta : \langle t \rangle / C_{\langle t \rangle}(W) \to \langle s \rangle / C_{\langle s \rangle}(U)$$

that sends $tC_{\langle t \rangle}(W)$ to $sC_{\langle s \rangle}(U)$. The map

$$\theta : W \langle t \rangle / C_{\langle t \rangle}(W) \to U \langle s \rangle / C_{\langle s \rangle}(U)$$

defined as $\theta(wt^i C_{\langle t \rangle}(W)) := \gamma(t)\eta(t^i C_{\langle t \rangle}(W))$ is an isomorphism that sends $tC_{\langle t \rangle}(W)$ to $sC_{\langle s \rangle}(U)$. Hence the pairs $\big(UC_{\langle s \rangle}(U)/C_{\langle s \rangle}(U), sC_{\langle s \rangle}(U)\big)$ and $\big(WC_{\langle t \rangle}(W)/C_{\langle t \rangle}(W), tC_{\langle t \rangle}(W)\big)$ are isomorphic. This proves the claim.

(ii) Let $f : W \langle t \rangle \to U \langle s \rangle$ be a group isomorphism that sends t to a conjugate of s. Let $u \in U$ be an element with the property that $f(t) = usu^{-1}$. Let $i_{u^{-1}}$ denote the automorphism of U induced by conjugation with u^{-1} and define $\gamma = i_{u^{-1}} \circ f : W \to U$. We have

$$\gamma(^t w) = u^{-1} f(^t w)u = u^{-1} f(t)f(w)f(t^{-1})u = su^{-1}f(w)us^{-1} = {}^s \gamma(w)$$

for all $w \in W$. This shows that $(\Delta(U, \gamma, W), (s, t))$ is a pair of $G \times K$.

(iii) By part (ii) there exists a group isomorphism

$$\tilde{\gamma} : WC_{\langle t \rangle}(W)/C_{\langle t \rangle}(W) \to UC_{\langle s \rangle}(U)/C_{\langle s \rangle}(U)$$

with the property that $\tilde{\gamma}(^{\bar t} \bar w) = {}^{\bar s} \tilde{\gamma}(\bar w)$ for all $\bar w \in WC_{\langle t \rangle}(W)/C_{\langle t \rangle}(W)$. The map $\tilde{\gamma}$ induces an isomorphism $\gamma : W \to U$ with the property that $\gamma(^t w) = {}^s \gamma(w)$ for all $w \in W$. This means that $(\Delta(U, \gamma, W), (s, t))$ is a pair of $G \times K$. $\qquad\square$

3.4 Notation For any p-permutation kG-module W, we set $\widetilde{W} := \mathrm{Ind}_{\Delta G}^{G \times G} W$. This defines an algebra homomorphism from $\mathbb{F}T(G)$ to $\mathbb{F}T^\Delta(G, G)$.

3.5 Remark Let (P, r) be a pair of G and let $\big(\Delta(U, \alpha, V), (s, z)\big)$ be a diagonal pair of $G \times G$. Then we have

$$\tau_{\Delta(U,\alpha,V),(s,z)}^{G \times G}(\widetilde{F_{P,r}^G}) = \begin{cases} |C_G(P\langle r \rangle)|, & \text{if } \big(\Delta(U, \alpha, V), (s, z)\big) =_{G \times G} \big(\Delta(P), (r, r)\big) \\ 0, & \text{otherwise}. \end{cases}$$

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Indeed, after identifying the group G with ΔG, we can also identify $F_{P,r}^G \in \mathbb{F}T(G)$ via $F_{\Delta P,(r,r)}^{\Delta G} \in \mathbb{F}T(\Delta G)$. Therefore by [BT10, Proposition 3.2] we have

$$\mathrm{Ind}_{\Delta G}^{G \times G} F_{\Delta P,(r,r)}^{\Delta G} = |C_G(P\langle r\rangle)| \cdot F_{\Delta P,(r,r)}^{G \times G}$$

which implies the equality above.

3.6 Lemma *Let $\big(\Delta(P,\gamma,Q),(r,u)\big)$ be a pair of $G \times K$. Then*

$$\widetilde{F_{P,r}^G} \otimes_{kG} F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K} = F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \quad in \quad \mathbb{F}T^\Delta(G,K).$$

Proof Let $\big(\Delta(P',\gamma',Q'),(r',u')\big)$ be a pair of $G \times K$. Then by Corollary 2.6,

$$\tau_{\Delta(P',\gamma',Q'),(r',u')}^{G \times K}\big(\widetilde{F_{P,r}^G} \otimes_{kG} F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}\big)$$

is equal to

$$\frac{1}{|G|} \sum_{\substack{(\alpha,V,\beta)\in\Gamma_G(P',\gamma',Q'),\, g\in G_{p'} \\ (r',g)\in N_{G\times G}(\Delta(P',\alpha,V)) \\ (g,u')\in N_{G\times K}(\Delta(V,\beta,Q'))}} \tau_{\Delta(P',\alpha,V),(r',g)}^{G \times G}\big(\widetilde{F_{P,r}^G}\big) \tau_{\Delta(V,\beta,Q'),(g,u')}^{G \times K}\big(F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}\big).$$

By Remark 3.5, the evaluation $\tau_{\Delta(P',\alpha,V),(r',g)}^{G \times G}\big(\widetilde{F_{P,r}^G}\big)$ is non-zero if and only if $\big(\Delta(P',\alpha,V),(r',g)\big) =_{G\times G} \big(\Delta P,(r,r)\big)$ if and only if there exists $(g_1,g_2) \in G \times G$ such that $\big(\Delta({}^{g_1}P',i_{g_1}\alpha i_{g_2}^{-1},{}^{g_2}V),({}^{g_1}r',{}^{g_2}g)\big) = \big(\Delta P,(r,r)\big)$. Hence the above sum can be written as

$$\frac{1}{n|G|} \sum_{\substack{(g_1,g_2)\in G\times G,\\ {}^{g_1}r'=r \\ ({}^{g_2^{-1}}r,u')\in N_{G\times K}(\Delta(V,\beta,Q'))}} |C_G(P\langle r\rangle)| \tau_{\Delta({}^{g_2^{-1}}P,i_{g_2}^{-1}i_{g_1}\gamma',Q'),({}^{g_2^{-1}}r,u')}^{G \times K}\big(F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}\big),$$

where n is the number of pairs (g_1,g_2) which satisfy the conditions above when α,V,β and g are fixed. Now again by Corollary 2.6, the element

$$\tau_{\Delta({}^{g_2^{-1}}P,i_{g_2}^{-1}i_{g_1}\gamma',Q'),({}^{g_2^{-1}}r,u')}^{G \times K}\big(F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}\big)$$

is equal to

$$\frac{1}{|K|} \sum_{\substack{(\phi,Y,\psi)\in\Gamma_K({}^{g_2^{-1}}P,i_{g_2}^{-1}i_{g_1}\gamma',Q') \\ k\in K_{p'} \\ ({}^{g_2^{-1}}r,k)\in N_{G\times K}(\Delta({}^{g_2^{-1}}P,\phi,Y)) \\ (k,u')\in N_{K\times K}(\Delta(Y,\psi,Q'))}} \tau_{\Delta({}^{g_2^{-1}}P,\phi,Y),({}^{g_2^{-1}}r,k)}^{G \times K}\big(F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K}\big) \tau_{\Delta(Y,\psi,Q'),(k,u')}^{K \times K}\big(\widetilde{F_{Q,u}^K}\big).$$

Therefore, the element

$$\tau_{\Delta({}^{g_2^{-1}}P,i_{g_2}^{-1}i_{g_1}\gamma',Q'),({}^{g_2^{-1}}r,u')}^{G \times K}\big(F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}\big)$$

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is non-zero if and only if there exists $(\phi, Y, \psi) \in \Gamma_K({}^{g_2^{-1}}P, i_{g_2}^{-1}i_{g_1}\gamma', Q')$ such that

$$\left(\Delta({}^{g_2^{-1}}P, \phi, Y), ({}^{g_2^{-1}}r, k)\right) =_{G \times K} \left(\Delta(P, \gamma, Q), (r, u)\right) \tag{1}$$

and such that

$$\left(\Delta(Y, \psi, Q'), (k, u')\right) =_{K \times K} \left(\Delta Q, (u, u)\right).$$

The latter implies that there exists $(k_1, k_2) \in K \times K$ such that

$$\left(\Delta({}^{k_1}Y, i_{k_1}\psi i_{k_2}^{-1}, {}^{k_2}Q'), ({}^{k_1}k, {}^{k_2}u')\right) = \left(\Delta Q, (u, u)\right).$$

This, in particular, implies that $\psi = i_{k_1}^{-1}i_{k_2}$ and hence that $\phi = i_{g_2}^{-1}i_{g_1}\gamma' i_{k_2}^{-1}i_{k_1}$. Therefore the statement (1) is equivalent to

$$\left(\Delta({}^{g_2^{-1}g_1}P', i_{k_1}^{-1}i_{k_2}\gamma' i_{g_1}^{-1}i_{g_2}, {}^{k_1^{-1}k_2}Q'), ({}^{g_2^{-1}g_1}r', {}^{k_1^{-1}k_2}u')\right) =_{G \times K} \left(\Delta(P, \gamma, Q), (r, u)\right)$$

which is equivalent to

$${}^{(g_2^{-1}g_1, k_1^{-1}k_2)}\left(\Delta(P', \gamma', Q'), (r', u')\right) =_{G \times K} \left(\Delta(P, \gamma, Q), (r, u)\right).$$

This is clearly equivalent to

$$\left(\Delta(P', \gamma', Q'), (r', u')\right) =_{G \times K} \left(\Delta(P, \gamma, Q), (r, u)\right).$$

This shows that the element $\widetilde{F_{P,r}^G} \otimes_{kG} F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K}$ is a non-zero scalar multiple of the idempotent $F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K}$, i.e.,

$$\widetilde{F_{P,r}^G} \otimes_{kG} F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K} \otimes_{kK} \widetilde{F_{Q,u}^K} = \lambda \cdot F_{\Delta(P,\gamma,Q),(r,u)}^{G \times K}$$

for some non-zero $\lambda \in \mathbb{F}$. But multiplying by $\widetilde{F_{Q,u}^K}$ from the right, and by $\widetilde{F_{P,r}^G}$ from the left implies that $\lambda^2 = \lambda$. Hence $\lambda = 1$ and the result follows. $\square$

The following result will be used in Section 6.

3.7 Theorem *Let (P, r) be a pair of G and (Q, u) a pair of K. Then there exists a p-permutation (kG, kK)-bimodule M all of whose indecomposable direct summands have twisted diagonal vertices, with the property that*

$$\widetilde{F_{P,r}^G} \otimes_{kG} M \otimes_{kK} \widetilde{F_{Q,u}^K} \neq 0$$

if and only if $(\tilde{P}, \tilde{r}) \cong (\tilde{Q}, \tilde{u})$.

Proof Suppose that M is a diagonal p-permutation (kG, kK)-bimodule with the property that

$$\widetilde{F_{P,r}^G} \otimes_{kG} M \otimes_{kK} \widetilde{F_{Q,u}^K} \neq 0.$$

Then there exists a pair $(\Delta(U,\gamma,W),(s,t))$ of $G \times K$ such that

$$\tau^{G\times K}_{\Delta(U,\gamma,W),(s,t)}(\widetilde{F^G_{P,r}} \otimes_{kG} M \otimes_{kK} \widetilde{F^K_{Q,u}}) \neq 0\,.$$

By Corollary 2.6, there exists $(\alpha,V,\beta) \in \Gamma_G(U,\gamma,W)$ and $g \in G$ such that

$$\tau^{G\times G}_{\Delta(U,\alpha,V),(s,g)}(\widetilde{F^G_{P,r}}) \neq 0 \quad \text{and} \quad \tau^{G\times K}_{\Delta(V,\beta,W),(g,t)}(M \otimes_{kK} \widetilde{F^K_{Q,u}}) \neq 0\,.$$

Similarly, there exists $(\phi,Y,\psi) \in \Gamma_K(V,\beta,W)$ and $l \in K$ such that

$$\tau^{G\times K}_{\Delta(V,\phi,Y),(g,l)}(M) \neq 0 \quad \text{and} \quad \tau^{K\times K}_{\Delta(Y,\psi,W),(l,t)}(\widetilde{F^K_{Q,u}}) \neq 0\,.$$

Since $\tau^{G\times G}_{\Delta(U,\alpha,V),(s,g)}(\widetilde{F^G_{P,r}}) \neq 0$, Remark 3.5 implies that the pair (P,r) is G-conjugate to the pair (V,g). Similarly $\tau^{K\times K}_{\Delta(Y,\psi,W),(l,t)}(\widetilde{F^K_{Q,u}}) \neq 0$ implies that the pair (Q,u) is K-conjugate to the pair (Y,l). Moreover since $(\Delta(V,\phi,Y),(g,l))$ is a pair of $G \times K$, Lemma 3.3 implies that $(\tilde{V},\tilde{g}) \cong (\tilde{Y},\tilde{l})$. Hence we also have $(\tilde{P},\tilde{r}) \cong (\tilde{Q},\tilde{u})$, as desired.

Now assume that $(\tilde{P},\tilde{r}) \cong (\tilde{Q},\tilde{u})$. Then by Lemma 3.3, there exists a group isomorphism $\gamma : Q \to P$ such that $(\Delta(P,\gamma,Q),(r,u))$ is a pair of $G\times K$. By Lemma 3.6, $\widetilde{F^G_{P,r}} \otimes_{kG} F^{G\times K}_{\Delta(P,\gamma,Q),(r,u)} \otimes_{kK} \widetilde{F^K_{Q,u}}$ is nonzero and we are done. $\qquad\square$

4 Some Clifford theory

The results in this section will be used in Section 8. Let k be an algebraically closed field of positive characteristic p. Let also N be a finite group with a normal subgroup C such that N/C is a cyclic p'-group $\langle u \rangle$.

4.1 Theorem (i) *Any N-invariant projective indecomposable kC-module F can be extended to a projective indecomposable kN-module E. Moreover, $\mathrm{Ind}^N_C F \cong \bigoplus_{\lambda:\langle u\rangle \to k^\times} E_\lambda$ where $E_\lambda = E \otimes_k \mathrm{Inf}^N_{N/C} k_\lambda$.*

(ii) *If M is a projective indecomposable kN-module such that $\mathrm{Res}^N_C M$ admits an N-invariant indecomposable summand F, then $F = \mathrm{Res}^N_C M$, that is, $\mathrm{Res}^N_C M$ is indecomposable.*

Proof (i) Let F be an N-invariant indecomposable projective kC-module and set $S = F/J(F)$. By [F82, Theorem 2.14], we can extend S to a simple kN-module $\widehat{S}$, with projective cover E. For a group isomorphism $\lambda : \langle u \rangle \to k^\times$, we set $\widehat{S}_\lambda = \widehat{S} \otimes_k \mathrm{Inf}^N_{N/C} k_\lambda$. Then $\widehat{S}_\lambda$ is a simple kN-module with projective cover $E_\lambda = E \otimes_k \mathrm{Inf}^N_{N/C} k_\lambda$.

The simple kN-modules $\widehat{S}_\lambda$ are all distinct. Equivalently, if $\lambda : \langle u \rangle \to k^\times$ is a non-trivial character, then $\widehat{S}$ and $\widehat{S}_\lambda$ are not isomorphic. Indeed, if $\varphi : \widehat{S} \to \widehat{S}_\lambda$ is an isomorphism of kN-modules, then the restriction of φ to kC is an automorphism of S as a kC-module, hence it is a scalar multiple of the identity of S, by Schur's lemma. So there exists $\mu \in k^\times$ such that $\varphi(v) = \mu v$ for any $v \in S$. Now since $\varphi : \widehat{S} \to \widehat{S}_\lambda$ is a morphism of kN-modules, for any $x \in N$ and $v \in S$, we have

$$\varphi(x \cdot v) = \lambda(\overline{x}) x \cdot \varphi(v)\,,$$

where $\overline{x}$ is the image of x in $\langle u \rangle$. So we get that $\mu x \cdot v = \lambda(\overline{x})\mu x \cdot v$, hence $\lambda(\overline{x}) = 1$, so $\lambda = 1$.

Now $\operatorname{Hom}_{kN}(\operatorname{Ind}_C^N F, \widehat{S}_\lambda) \cong \operatorname{Hom}_{kC}(F, S)$ is one dimensional. Hence there is a non-zero morphism of kN-modules $\psi : \operatorname{Ind}_C^N F \to \widehat{S}_\lambda$, which is surjective since $\widehat{S}_\lambda$ is simple. Since $\operatorname{Ind}_C^N F$ is projective, it follows that ψ can be lifted to $\theta : \operatorname{Ind}_C^N F \to E_\lambda$, and θ is surjective, because E_λ is a projective cover of $\widehat{S}_\lambda$. Hence θ splits, and it follows that E_λ is a direct summand of $\operatorname{Ind}_C^N F$.

Since the modules $\widehat{S}_\lambda$ are all distinct, the modules E_λ are all distinct as well, and it follows that $\oplus_\lambda E_\lambda$ is a direct summand of $\operatorname{Ind}_C^N F$. Since $\langle u \rangle$ is a p'-group, this implies in particular that

$$|u| \dim_k E = \dim_k(\oplus_\lambda E_\lambda) \leqslant |N : C| \dim_k F = |u| \dim_k F,$$

so $\dim_k E \leqslant \dim_k F$.

But on the other hand, the surjection $E \to \widehat{S}$ restrict to a surjection of kC-modules $\operatorname{Res}_C^N E \to S$. Since $\operatorname{Res}_C^N E$ is projective, we get as above that the projective cover F of S is a direct summand of $\operatorname{Res}_C^N E$. In particular, $\dim_k F \leqslant \dim_k E$. It follows that $\dim_k E = \dim_k F$ and this proves (i).

(ii) Let M be a projective indecomposable kN-module such that $\operatorname{Res}_C^N M$ admits an N-invariant indecomposable summand F. Then as above, the simple kC-module $S = F/J(F)$ can be extended to a simple kN-module $\widehat{S}$ with projective cover E, and the simple kN-modules $\widehat{S}_\lambda$ are all distinct. Now

$$\operatorname{Hom}_{kN}(\operatorname{Ind}_C^N S, \widehat{S}_\lambda) \cong \operatorname{Hom}_{kC}(S, S)$$

is one dimensional, so there is a non-zero morphism $\operatorname{Ind}_C^N S \to \widehat{S}_\lambda$ which is surjective since $\widehat{S}_\lambda$ is simple. It follows that we have a surjective morphism of kN-modules

$$\sigma : \operatorname{Ind}_C^N S \to \bigoplus_\lambda \widehat{S}_\lambda.$$

But these two modules have the same dimension $|u| \dim_k S$, so σ is an isomorphism. In particular $\operatorname{Ind}_C^N S$ is semisimple.

Since F is a direct summand of $\operatorname{Res}_C^N M$, we get a non-zero morphism $\operatorname{Res}_C^N M \to S$, hence a non-zero morphism $M \to \operatorname{Ind}_C^N S$. The image L of this morphism is a semisimple quotient of M, which is projective and indecomposable. Hence L is simple, and isomorphic to one of the modules $\widehat{S}_\lambda$. Then $M \cong E_\lambda$ and $F = \operatorname{Res}_C^N M$. This proves (ii). $\qquad \square$

5 An equivalence of categories

Let $\mathcal{A}$ be an abelian category with arbitrary direct sums. Recall that an object P of $\mathcal{A}$ is called *compact* if for any family $(X_i)_{i \in I}$ of objects of $\mathcal{A}$, the natural morphism

$$\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{A}}(P, X_i) \to \operatorname{Hom}_{\mathcal{A}}(P, \bigoplus_{i \in I} X_i)$$

is an isomorphism. The following is well-known to specialists. We include the proof for the convenience of the reader.

5.1 Theorem *Let R be a commutative ring and $\mathcal{A}$ an R-linear abelian category with arbitrary direct sums. Let moreover $\mathcal{P}$ be a set of objects of $\mathcal{A}$ with the following properties:*

(i) If $P \in \mathcal{P}$, then P is projective and compact in $\mathcal{A}$.

(ii) The set $\mathcal{P}$ generates $\mathcal{A}$, i.e., for any object X of $\mathcal{A}$, there exists a family $(P_j)_{j \in J}$ of elements of $\mathcal{P}$, and an epimorphism $\oplus_{j \in J} P_j \twoheadrightarrow X$ in $\mathcal{A}$.

Let $\mathrm{Fun}_R(\mathcal{P}^{\mathrm{op}}, {}_R\mathsf{Mod})$ be the category of R-linear contravariant functors from the full subcategory $\mathcal{P}$ of $\mathcal{A}$ to the category of R-modules. Then the functor

$$\mathcal{H} : X \in \mathcal{A} \mapsto H_X = \mathrm{Hom}_{\mathcal{A}}(-, X) \in \mathrm{Fun}_R(\mathcal{P}^{\mathrm{op}}, {}_R\mathsf{Mod})$$

is an equivalence of R-linear abelian categories.

Proof We first show that $\mathcal{H}$ is fully faithful. Let X be an object of $\mathcal{A}$. By Condition (ii), there exists an exact sequence in $\mathcal{A}$ of the form

$$\oplus_{j \in J} Q_j \longrightarrow \oplus_{i \in I} P_i \longrightarrow X \longrightarrow 0 \,, \tag{2}$$

where $(Q_j)_{j \in J}$ and $(P_i)_{i \in I}$ are families of elements of $\mathcal{P}$. Now if $P \in \mathcal{P}$, then applying the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ gives an exact sequence of R-modules

$$\mathrm{Hom}_{\mathcal{A}}(P, \oplus_{j \in J} Q_j) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, \oplus_{i \in I} P_i) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X) \longrightarrow 0 \,,$$

since P is projective in $\mathcal{A}$. Since moreover P is compact in $\mathcal{A}$, this exact sequence is naturally isomorphic to the exact sequence

$$\oplus_{j \in J} \mathrm{Hom}_{\mathcal{A}}(P, Q_j) \longrightarrow \oplus_{i \in I} \mathrm{Hom}_{\mathcal{A}}(P, P_i) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X) \longrightarrow 0 \,.$$

In other words, we get an exact sequence in the category $\mathcal{F} = \mathrm{Fun}_R(\mathcal{P}^{\mathrm{op}}, {}_R\mathsf{Mod})$

$$\oplus_{j \in J} H_{Q_j} \longrightarrow \oplus_{i \in I} H_{P_i} \longrightarrow H_X \longrightarrow 0 \,.$$

Now for any object Y of $\mathcal{A}$, applying the functor $\mathrm{Hom}_{\mathcal{F}}(-, H_Y)$ to this sequence gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(\oplus_{i \in I} H_{P_i}, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(\oplus_{j \in J} H_{Q_j}, H_Y)$$

of R-modules. This sequence is naturally isomorphic to

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathcal{F}}(H_{P_i}, H_Y) \longrightarrow \prod_{j \in J} \mathrm{Hom}_{\mathcal{F}}(H_{Q_j}, H_Y) \,.$$

Now by the Yoneda lemma, for each $P \in \mathcal{P}$, we get a natural isomorphism $\mathrm{Hom}_{\mathcal{F}}(H_P, H_Y) \cong \mathrm{Hom}_{\mathcal{A}}(P, Y)$, so the previous sequence is isomorphic to

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y) \longrightarrow \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(P_i, H_Y) \longrightarrow \prod_{j \in J} \mathrm{Hom}_{\mathcal{A}}(Q_j, H_Y) \,,$$

or in other words to the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{i \in I} P_i, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{j \in J} Q_j, H_Y) \,. \tag{3}$$

Now applying the functor $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ to the exact sequence (2) gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{i \in I} P_i, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{j \in J} Q_j, H_Y) \,. \tag{4}$$

Moreover, the exact sequences (3) and (4) fit into a commutative diagram

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$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{i \in I} P_i, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{j \in J} Q_j, H_Y)$$
$$\downarrow \qquad\qquad\qquad \downarrow = \qquad\qquad\qquad \downarrow =$$
$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{i \in I} P_i, H_Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_{j \in J} Q_j, H_Y)$$

of R-modules. It follows that the vertical arrow $\mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(H_X, H_Y)$ is an isomorphism for any objects X and Y of $\mathcal{A}$. In other words, the functor $\mathcal{H}$ is fully faithful as was to be shown.

We now prove that $\mathcal{H}$ is essentially surjective. Let F be any object of $\mathcal{F} = \mathrm{Fun}_R(\mathcal{P}^{\mathrm{op}}, {}_R\mathsf{Mod})$. For every $P \in \mathcal{P}$ we choose a generating set s_P of $F(P)$ as an R-module. By the Yoneda lemma again, we have an isomorphism $\mathrm{Hom}_{\mathcal{F}}(H_P, F) \cong F(P)$ from which we get a morphism

$$\oplus_{P \in \mathcal{P}} \oplus_{s \in s_P} H_P \longrightarrow F, \tag{5}$$

which is an epimorphism as s_P generates $F(P)$ for any $P \in \mathcal{P}$. Now since $\mathcal{P}$ consists of compact objects of $\mathcal{A}$, we have an isomorphism

$$\oplus_{P \in \mathcal{P}} \oplus_{s \in s_P} \mathrm{Hom}_{\mathcal{A}}(-, P) \cong \mathrm{Hom}_{\mathcal{A}}(-, \oplus_{P \in \mathcal{P}} \oplus_{s \in s_P} P)$$

in $\mathcal{F}$, in other words an isomorphism $\oplus_{P \in \mathcal{P}} \oplus_{s \in s_P} H_P \cong H_X$ where $X = \oplus_{P \in \mathcal{P}} \oplus_{s \in s_P} P$.

So for any $F \in \mathcal{F}$, there exists $X \in \mathcal{A}$ and an epimorphism $H_X \twoheadrightarrow F$ in $\mathcal{F}$ as in (5). Then there exists $Y \in \mathcal{A}$ and an exact sequence

$$H_Y \longrightarrow H_X \longrightarrow F \longrightarrow 0$$

in $\mathcal{F}$. Since the functor $\mathcal{H}$ is fully faithful by the first part of the proof, the left arrow $H_Y \to H_X$ in this sequence is induced by a morphism $f : Y \to X$ in $\mathcal{A}$. Let Z be the cokernel of f in $\mathcal{A}$. The exact sequence

$$Y \xrightarrow{\ f\ } X \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$ gives an exact sequence $H_Y \longrightarrow H_X \longrightarrow H_Z \longrightarrow 0$ in $\mathcal{F}$, because $\mathcal{P}$ consists of projective objects of $\mathcal{A}$. It follows that H_Z is isomorphic to the cokernel of $H_Y \longrightarrow H_X$ in $\mathcal{F}$, that is $H_Z \cong F$. It follows that the functor $\mathcal{H}$ is essentially surjective, which completes the proof of the theorem. $\square$

6 Semisimplicity of the functor category

Recall that k denotes an algebraically closed field of positive characteristic p and $\mathbb{F}$ denotes an algebraically closed field of characteristic zero. We recall from the introduction the definition of diagonal p-permutation functors over $\mathbb{F}$.

6.1 Definition Let $\mathbb{F}pp_k^{\Delta}$ be the category with

- objects: finite groups
- $\mathrm{Mor}_{\mathbb{F}pp_k^{\Delta}}(G, H) = \mathbb{F} \otimes_{\mathbb{Z}} T^{\Delta}(H, G) = \mathbb{F}T^{\Delta}(H, G)$.

An $\mathbb{F}$-linear functor from $\mathbb{F}pp_k^\Delta$ to $_\mathbb{F}\mathsf{Mod}$ is called a *diagonal p-permutation functor* over $\mathbb{F}$. Diagonal p-permutation functors form an abelian category $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$.

Let G be a finite group and let (P, r) be a pair of G. Then the element $\widetilde{F_{P,r}^G}$ is an idempotent of $\mathbb{F}T^\Delta(G, G)$ which is the endomorphism algebra of the representable functor $\mathbb{F}T_G^\Delta$. Let $\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}$ denote the corresponding direct summand of the functor $\mathbb{F}T_G^\Delta$. For any finite group H, the evaluation $\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}(H)$ is given by

$$\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}(H) = \mathbb{F}T^\Delta(H, G)\widetilde{F_{P,r}^G} := \{X \otimes_{kG} \widetilde{F_{P,r}^G} : X \in \mathbb{F}T^\Delta(H, G)\}.$$

6.2 Lemma *Let G and K be finite groups. Let (P, r) be a pair of G and (Q, u) a pair of K. Let also $F \in \mathcal{F}_{\mathbb{F}pp_k}^\Delta$ be a diagonal p-permutation functor over $\mathbb{F}$.*

(i) *We have* $\mathrm{Hom}_{\mathcal{F}_{\mathbb{F}pp_k}^\Delta}(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, F) \equiv F(G)^{\widetilde{F_{P,r}^G}} := \{m \in F(G) : F(\widetilde{F_{P,r}^G})(m) = m\}.$

(ii) *We have* $\mathrm{Hom}_{\mathcal{F}_{\mathbb{F}pp_k}^\Delta}(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, \mathbb{F}T_K^\Delta) \equiv \widetilde{F_{P,r}^G}\mathbb{F}T^\Delta(G, K).$

(iii) *We have* $\mathrm{Hom}_{\mathcal{F}_{\mathbb{F}pp_k}^\Delta}(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, \mathbb{F}T_K^\Delta \widetilde{F_{Q,u}^K}) \equiv \widetilde{F_{P,r}^G}\mathbb{F}T^\Delta(G, K)\widetilde{F_{Q,u}^K}.$

Proof Let $\Phi \in \mathrm{Hom}_{\mathcal{F}_{\mathbb{F}pp_k}^\Delta}(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, F)$ be a natural transformation. One shows that Φ is completely determined by $\Phi_G(\widetilde{F_{P,r}^G}) =: m \in F(G)$. Moreover, we have $F(\widetilde{F_{P,r}^G})(m) = m$ and hence $m \in F(G)^{\widetilde{F_{P,r}^G}}$. One can also show that every $m \in F(G)^{\widetilde{F_{P,r}^G}}$ determines a natural transformation. This proves the statement (i).

Statements (ii) and (iii) are proved similarly. $\qquad\square$

6.3 Lemma *Let (P, r) be a pair of G. The functor $\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}$ is compact and projective in $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$.*

Proof The functor $\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}$ is a direct summand of the projective functor $\mathbb{F}T_G^\Delta$. Hence it is projective as well. Let $\{F_i\}$ be a set of objects of $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$. By Lemma 6.2(i) there is an isomorphism

$$\bigoplus_i \mathrm{Hom}(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, F_i) \to \mathrm{Hom}\big(\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}, \bigoplus_i F_i\big)$$

in $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$. Hence $\mathbb{F}T_G^\Delta \widetilde{F_{P,r}^G}$ is compact as well. $\qquad\square$

6.4 Proposition *Every functor in $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$ is a quotient of a direct sum of representable functors of the form $\mathbb{F}T_{Q\langle v\rangle}^\Delta \widetilde{F_{Q,v}^{Q\langle v\rangle}}$ where (Q, v) is a D^Δ-pair.*

Proof Since the representable functors generate the functor category $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$, it suffices to prove the statement for the representable functors. Let G be a finite group and consider the representable functor $\mathbb{F}T_G^\Delta$.

If the essential algebra $\mathbb{F}T^\Delta(G, G)/\big(\sum_{|H|<|G|} \mathbb{F}T^\Delta(G, H) \circ \mathbb{F}T^\Delta(H, G)\big)$ at G vanishes, then the identity morphism of the functor $\mathbb{F}T_G^\Delta$ can be written as a sum of morphisms $a_i \circ b_i$, where

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$b_i : G \to K_i$, $a_i : K_i \to G$ and $|K_i| < |G|$. Equivalently, the functor $\mathbb{F}T_G^\Delta$ is a quotient of the direct sum $\oplus_i \mathbb{F}T_{K_i}^\Delta$. By induction on the order of G, we can assume that the essential algebra at G does not vanish, i.e., $G = L\langle u \rangle$ for some D^Δ-pair (L, u) (see [BY20, Theorem 3.3]).

Now the identity map of $\mathbb{F}T_G^\Delta$ is equal to the sum of the idempotents $\widetilde{F_{P,r}^G}$ where (P, r) runs in a set of conjugacy classes of pairs of G. If $P\langle r \rangle \neq G$, then the idempotent $F_{P,r}^G$ is induced from a proper subgroup and hence $\widetilde{F_{P,r}^G}$ factors through this subgroup. If $P\langle r \rangle = G$, then $P = L$ and $\langle r \rangle$ is conjugate to $\langle u \rangle$. In particular, (P, r) is a D^Δ-pair. The result follows by induction on $|G|$. $\quad \blacksquare$

6.5 Notation Let $\mathcal{P}$ denote the full subcategory of $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$ consisting of the functors $\mathbb{F}T_{L\langle u \rangle}^\Delta \widetilde{F_{L,u}^{L\langle u \rangle}}$, where (L, u) runs through a set of isomorphism classes of D^Δ-pairs.

6.6 Corollary *The functor from $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$ to $\mathrm{Fun}_\mathbb{F}(\mathcal{P}^{op}, {}_\mathbb{F}\mathsf{Mod})$ sending a functor X to the representable functor $\mathrm{Hom}_{\mathcal{F}_{\mathbb{F}pp_k}^\Delta}(-, X)$ is an equivalence of categories.*

Proof This follows from Lemma 6.3, Proposition 6.4 and Theorem 5.1. $\quad \blacksquare$

Let (L, u) and (M, v) be D^Δ-pairs. By Lemma 6.2 we have

$$\mathrm{Hom}_\mathcal{P}\left(\mathbb{F}T_{L\langle u \rangle}^\Delta \widetilde{F_{L,u}^{L\langle u \rangle}}, \mathbb{F}T_{M\langle v \rangle}^\Delta \widetilde{F_{M,v}^{M\langle v \rangle}}\right) \equiv \widetilde{F_{L,u}^{L\langle u \rangle}} \mathbb{F}T^\Delta(L\langle u \rangle, M\langle v \rangle) \widetilde{F_{M,v}^{M\langle v \rangle}}.$$

Therefore the category $\mathcal{P}^{op}$ is isomorphic to the following category.

6.7 Definition Let $\mathcal{D}^\Delta$ denote the following category:

- objects are the isomorphism classes of D^Δ-pairs.

- $\mathrm{Hom}_{\mathcal{D}^\Delta}\big((L, u), (M, v)\big) = \widetilde{F_{M,v}^{M\langle v \rangle}} \mathbb{F}T^\Delta(L\langle u \rangle, M\langle v \rangle) \widetilde{F_{L,u}^{L\langle u \rangle}}$.

It follows from Theorem 3.7 that $\widetilde{F_{L,u}^{L\langle u \rangle}} \mathbb{F}T^\Delta(L\langle u \rangle, M\langle v \rangle) \widetilde{F_{M,v}^{M\langle v \rangle}}$ is non-zero if and only if (L, u) and (M, v) are isomorphic. Hence our next aim is to understand the structure of $\widetilde{F_{L,u}^{L\langle u \rangle}} \mathbb{F}T^\Delta(L\langle u \rangle, L\langle u \rangle) \widetilde{F_{L,u}^{L\langle u \rangle}}$. We start with some preliminary results.

6.8 Notation (i) Let $\mathrm{Aut}_u(L)$ denote the set of automorphisms f of L with the property that $f({}^u l) = {}^u f(l)$.

(ii) Let $\mathrm{Inn}(C_{L\langle u \rangle}(u))$ denote the normal subgroup of $\mathrm{Aut}_u(L)$ consisting of conjugations induced by the elements of $C_{L\langle u \rangle}(u)$.

(iii) Let $\mathrm{Aut}(L, u)$ denote the set of automorphisms of $L\langle u \rangle$ that sends u to a conjugate of u.

(iv) Let $\mathrm{Inn}(L\langle u \rangle)$ denote the normal subgroup of $\mathrm{Aut}(L, u)$ consisting of conjugations induced by elements of $L\langle u \rangle$.

(v) Set $\mathrm{Out}(L, u) := \mathrm{Aut}(L, u)/\mathrm{Inn}(L\langle u \rangle)$.

6.9 Remark For any $f \in \mathrm{Aut}_u(L)$ the pair $\big(\Delta(L, f, L), (u, u)\big)$ is a pair of $L\langle u \rangle$. Moreover, for any $f_1, f_2 \in \mathrm{Aut}_u(L)$ the pairs $\big(\Delta(L, f_1, L), (u, u)\big)$ and $\big(\Delta(L, f_2, L), (u, u)\big)$ are conjugate if and only if $f_1 f_2^{-1} \in \mathrm{Inn}(C_{L\langle u \rangle}(u))$.

We extend any $f \in \mathrm{Aut}_u(L)$ to $f \in \mathrm{Aut}(L\langle u \rangle)$ by defining $f(lu^i) := f(l)u^i$. This induces an embedding $\mathrm{Aut}_u(L) \hookrightarrow \mathrm{Aut}(L, u)$. We identify $\mathrm{Aut}_u(L)$ by its image in $\mathrm{Aut}(L, u)$.

6.10 Lemma *We have* $\mathrm{Out}(L, u) \cong \mathrm{Aut}_u(L)/\mathrm{Inn}(C_{L\langle u \rangle}(u))$.

Proof Let $\phi \in \mathrm{Aut}(L, u)$ be an automorphism and let $g \in L\langle u \rangle$ be an element with the property that $\phi(u) = {}^g u$. Set $f := i_{g^{-1}} \circ \phi$. Then $f \in \mathrm{Aut}_u(L)$ and $\phi = i_g \circ f \in \mathrm{Aut}(L, u)$. This shows that $\mathrm{Aut}(L, u) = \mathrm{Inn}(L\langle u \rangle)\mathrm{Aut}_u(L)$. We also have

$$\mathrm{Inn}(L\langle u \rangle) \cap \mathrm{Aut}_u(L) = \mathrm{Inn}(C_{L\langle u \rangle}(u)).$$

Hence the result follows from the second isomorphism theorem. $\qquad\square$

6.11 Lemma *Let* $(\Delta(P, \pi, L), (s, u))$ *be a pair of* $G \times L\langle u \rangle$, *let* $\gamma_1, \gamma_2 \in \mathrm{Aut}_u(L)$, *let* $x, y \in C_{L\langle u \rangle}(u)$ *and let* $z \in L\langle u \rangle$.

   (i) *The subgroups* $\Delta(P, \pi\gamma_1, L)$ *and* $\Delta(P, \pi i_z \gamma_1, L)$ *of* $G \times L\langle u \rangle$ *are conjugate.*

   (ii) *The pairs* $(\Delta(P, \pi\gamma_1, L), (s, u))$ *and* $(\Delta(P, \pi i_x \gamma_1, L), (s, u))$ *of* $G \times L\langle u \rangle$ *are conjugate.*

   (iii) *The pairs* $\big(\Delta(L, \gamma_2 i_x \gamma_1, L), (u, u)\big)$ *and* $\big(\Delta(L, \gamma_2 i_y \gamma_1, L), (u, u)\big)$ *of* $L\langle u \rangle \times L\langle u \rangle$ *are conjugate. In particular, the pairs* $\big(\Delta(L, \gamma_2 i_x \gamma_1, L), (u, u)\big)$ *and* $\big(\Delta(L, \gamma_2 \gamma_1, L), (u, u)\big)$ *are conjugate.*

Proof Let $a := \gamma_1^{-1}(z^{-1})$. Then we have

$$^{(1, a)}\Delta(P, \pi\gamma_1, L) = \Delta(P, \pi i_z \gamma_1, L)$$

which proves (i). The other parts are proved similarly. $\qquad\square$

6.12 Lemma *Let* G *be a finite group and* (L, u) *a* D^Δ-*pair. Let* $\big(\Delta(P, \phi, L), (s, u)\big)$ *be a pair of* $G \times L\langle u \rangle$ *and let* $\big(\Delta(L, \gamma, L), (u, u)\big)$ *be a pair of* $L\langle u \rangle \times L\langle u \rangle$. *Then*

$$F^{G \times L\langle u \rangle}_{\Delta(P, \phi, L), (s, u)} \otimes_{kL\langle u \rangle} F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L, \gamma, L), (u, u)} = \frac{1}{|Z(L\langle u \rangle)|} F^{G \times L\langle u \rangle}_{\Delta(P, \phi\gamma, L), (s, u)} \quad \text{in} \quad \mathbb{F}T^\Delta(G, L\langle u \rangle).$$

Proof Using Corollary 2.6 one shows that if $\big(\Delta(R, \sigma, L), (t, u^i)\big)$ is a pair of $G \times L\langle u \rangle$ with the property that

$$\tau^{G \times L\langle u \rangle}_{\Delta(R, \sigma, L), (t, u^i)}\big(F^{G \times L\langle u \rangle}_{\Delta(P, \phi, L), (s, u)} \otimes_{kL\langle u \rangle} F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L, \gamma, L), (u, u)}\big) \neq 0$$

then $\big(\Delta(R, \sigma, L), (t, u^i)\big)$ is conjugate to a pair of the form $\big(\Delta(P, \phi i_x \gamma, L), (s, u)\big)$ where $x \in C_{L\langle u \rangle}(u)$. Lemma 6.11 implies that in this case that the pair $\big(\Delta(R, \sigma, L), (t, u^i)\big)$ is conjugate to $\big(\Delta(P, \phi\gamma, L), (s, u)\big)$.

Now by Corollary 2.6 again, we have

$$\tau^{G \times L\langle u \rangle}_{\Delta(P, \phi\gamma, L), (s, u)}\big(F^{G \times L\langle u \rangle}_{\Delta(P, \phi, L), (s, u)} \otimes_{kL\langle u \rangle} F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L, \gamma, L), (u, u)}\big)$$

$$= \frac{1}{|L\langle u \rangle|} \sum_{\substack{(\alpha, L, \beta) \in \Gamma_{L\langle u \rangle}(P, \phi\gamma, L) \\ c \in \langle u \rangle \\ (s, c) \in N_{G \times L\langle u \rangle}(\Delta(P, \alpha, L)) \\ (c, u) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \beta, L))}} \tau^{G \times L\langle u \rangle}_{\Delta(P, \alpha, L), (s, c)}\big(F^{G \times L\langle u \rangle}_{\Delta(P, \phi, L), (s, u)}\big) \tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L, \beta, L), (c, u)}\big(F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L, \gamma, L), (u, u)}\big).$$

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The product

$$\tau^{G\times L\langle u\rangle}_{\Delta(P,\alpha,L),(s,c)}\big(F^{G\times L\langle u\rangle}_{\Delta(P,\phi,L),(s,u)}\big)\,\tau^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\beta,L),(c,u)}\big(F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u,u)}\big)$$

is non-zero if and only if there exist $g\in G$ and $l_1,l_2,l_3\in L\langle u\rangle$ such that

$$\big(\Delta(P,\alpha,L),(s,c)\big)=\,^{(g,l_1)}\big(\Delta(P,\phi,L),(s,u)\big)=\big(\Delta(\,^gP,i_g\phi i_{l_1}^{-1},L),(\,^gs,\,^{l_1}u)\big)$$

and

$$\big(\Delta(L,\beta,L),(c,u)\big)=\,^{(l_2,l_3)}\big(\Delta(L,\gamma,L),(u,u)\big)=\big(\Delta(L,i_{l_2}\gamma i_{l_3}^{-1},L),(\,^{l_2}u,\,^{l_3}u)\big)$$

hold. These conditions imply that

$$g\in N_G(P,s)\,,\quad l_3,l_2^{-1}l_1\in C_{L\langle u\rangle}(u)\,,\quad \alpha=i_g\phi i_{l_1}^{-1}\quad\text{and}\quad \beta=i_{l_2}\gamma i_{l_3}^{-1}\,.$$

Moreover, the condition that $\phi\gamma=\alpha\beta$ implies that

$$\phi\gamma=i_g\phi\gamma i_x$$

as maps from L to P, where $x=\gamma^{-1}(l_1^{-1}l_2)l_3^{-1}$.

The number of quadruples (g,l_1,l_2,l_3) that satisfy these conditions is

$$|C_G(P,s)||C_{L\langle u\rangle}(u)|^2|L\langle u\rangle|\,,$$

where $C_G(P,s):=N_G(P,s)\cap C_G(P)$. However, when $(\alpha,L,\beta)\in\Gamma_{L\langle u\rangle}$ and $c\in\langle u\rangle$ are fixed there are $|C_{L\langle u\rangle}(u)|^2|Z(L\langle u\rangle)||C_G(P,s)|$ quadruples with these properties. Therefore, we have

$$\tau^{G\times L\langle u\rangle}_{\Delta(P,\phi\gamma,L),(s,u)}\big(F^{G\times L\langle u\rangle}_{\Delta(P,\phi,L),(s,u)}\otimes_{kL\langle u\rangle}F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u,u)}\big)=\frac{|C_G(P,s)||C_{L\langle u\rangle}(u)|^2|L\langle u\rangle|}{|L\langle u\rangle||C_{L\langle u\rangle}(u)|^2|Z(L\langle u\rangle)||C_G(P,s)|}$$

$$=\frac{1}{|Z(L\langle u\rangle)|}\,.$$

This completes the proof. $\qquad\qquad\square$

6.13 Proposition Let (L,u) be a D^Δ-pair. Then $\widetilde{F^{L\langle u\rangle}_{L,u}}\,\mathbb{F}T^\Delta(L\langle u\rangle,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$ is equal to the $\mathbb{F}$-algebra generated by the elements of the form $F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u,u)}$. In particular, the $\mathbb{F}$-dimension of $\widetilde{F^{L\langle u\rangle}_{L,u}}\,\mathbb{F}T^\Delta(L\langle u\rangle,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$ is equal to the cardinality of $\mathrm{Out}(L,u)$.

Proof Let $\big(\Delta(L,\gamma,L),(u^i,u^j)\big)$ be a pair of $L\langle u\rangle\times L\langle u\rangle$ such that

$$\widetilde{F^{L\langle u\rangle}_{L,u}}\,F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\,\widetilde{F^{L\langle u\rangle}_{L,u}}\neq 0\,.$$

Then there exists a pair $\big(\Delta(L,\varphi,L),(s,t)\big)$ of $L\langle u\rangle\times L\langle u\rangle$ such that

$$\tau^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\varphi,L),(s,t)}\big(\widetilde{F^{L\langle u\rangle}_{L,u}}\,F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\,\widetilde{F^{L\langle u\rangle}_{L,u}}\big)\neq 0\,.$$

By Corollary 2.6, there exists $(\alpha, L, \beta) \in \Gamma_{L\langle u \rangle}(L, \varphi, L)$ and $a \in \langle u \rangle$ such that $(s, a) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \alpha, L))$, $(a, t) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \beta, L))$ and

$$\tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\alpha,L),(s,a)}\big(\widetilde{F^{L\langle u \rangle}_{L,u}}\big) \cdot \tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\beta,L),(a,t)}\big(F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\widetilde{F^{L\langle u \rangle}_{L,u}}\big) \neq 0\,.$$

This implies, in particular, that

$$\big(\Delta(L, \alpha, L), (s, a)\big) =_{L\langle u \rangle \times L\langle u \rangle} \big(\Delta L, (u, u)\big)\,. \tag{6}$$

Moreover, applying Corollary 2.6 again, there exists $(\phi, L, \psi) \in \Gamma_{L\langle u \rangle}(L, \beta, L)$ and $b \in \langle u \rangle$ such that $(a, b) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \phi, L))$, $(b, t) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \psi, L))$ and

$$\tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\phi,L),(a,b)}\big(F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\big) \cdot \tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\psi,L),(b,t)}\big(\widetilde{F^{L\langle u \rangle}_{L,u}}\big) \neq 0\,.$$

Therefore

$$\big(\Delta(L, \phi, L), (a, b)\big) =_{L\langle u \rangle \times L\langle u \rangle} \big(\Delta(L, \gamma, L), (u^i, u^j)\big) \tag{7}$$

and

$$\big(\Delta(L, \psi, L), (b, t)\big) =_{L\langle u \rangle \times L\langle u \rangle} \big(\Delta L, (u, u)\big)\,. \tag{8}$$

Now using the conditions (6), (7) and (8) one shows that

$$\big(\Delta(L, \varphi, L), (s, t)\big) =_{L\langle u \rangle \times L\langle u \rangle} \big(\Delta(L, \gamma, L), (u^i, u^j)\big)\,.$$

This means that the algebra maps $\tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\varphi,L),(s,t)}$ and $\tau^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}$ are equal and hence we can replace the pair $\big(\Delta(L, \varphi, L), (s, t)\big)$ by the pair $\big(\Delta(L, \gamma, L), (u^i, u^j)\big)$. But then again the conditions above implies that there exists $l_1, l_2 \in L\langle u \rangle$ such that $u^i = {}^{l_1}u$ and $u^j = {}^{l_2}u$. Therefore we have

$${}^{(l_1^{-1}, l_2^{-1})}\big(\Delta(L, \gamma, L), (u^i, u^j)\big) = \big(\Delta(L, i_{l_1}^{-1}\gamma i_{l_2}, L), (u, u)\big)\,.$$

This shows that $\widetilde{F^{L\langle u \rangle}_{L,u}} F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\widetilde{F^{L\langle u \rangle}_{L,u}}$ is non-zero if and only if the pair $\big(\Delta(L, \gamma, L), (u^i, u^j)\big)$ is conjugate to a pair of the form $\big(\Delta(L, \gamma', L), (u, u)\big)$, and in that case $\widetilde{F^{L\langle u \rangle}_{L,u}} F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}\widetilde{F^{L\langle u \rangle}_{L,u}}$ is a scalar multiple of $F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u^i,u^j)}$. This proves the first claim. The second claim follows now from Remark 6.9. $\qquad\square$

6.14 Corollary *The map*

$$\widetilde{F^{L\langle u \rangle}_{L,u}}\mathbb{F}T^\Delta(L\langle u \rangle, L\langle u \rangle)\widetilde{F^{L\langle u \rangle}_{L,u}} \to \mathbb{F}\mathrm{Out}(L, u)$$
$$F^{L\langle u \rangle \times L\langle u \rangle}_{\Delta(L,\gamma,L),(u,u)} \mapsto \frac{1}{|Z(L\langle u \rangle)|}\gamma$$

is an algebra isomorphism.

Proof This follows from Lemma 6.12 and Proposition 6.13. □

6.15 Corollary *The category $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$ is semisimple. Moreover, the simple diagonal p-permutation functors, up to isomorphism, are parametrized by the isomorphism classes of triples (L, u, V) where (L, u) is a D^{Δ}-pair, and V is a simple $\mathbb{F}\mathrm{Out}(L, u)$-module.*

Proof By Corollary 6.6, the category $\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}$ is equivalent to $\mathrm{Fun}_{\mathbb{F}}(\mathcal{D}^{\Delta}, {}_{\mathbb{F}}\mathsf{Mod})$. By Theorem 3.7 the category $\mathcal{D}^{\Delta}$ is a product of the categories $\mathcal{D}^{\Delta}(L, u)$ where $\mathcal{D}^{\Delta}(L, u)$ is a category with one object, a D^{Δ}-pair (L, u) up to isomorphism, and hom set $\widetilde{F^{L\langle u\rangle}_{L,u}}\mathbb{F}T^{\Delta}(L\langle u\rangle, L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$. The result now follows from Corollary 6.14. □

7 More on simple functors

Let (L, u) be a D^{Δ}-pair and let V be a simple $\mathbb{F}\mathrm{Out}(L, u)$-module. Let $E_{L,u} := \widetilde{F^{L\langle u\rangle}_{L,u}}\mathbb{F}T^{\Delta}(L\langle u\rangle, L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$. We can consider V as an $E_{L,u}$-module via the isomorphism in Corollary 6.14. Let e_V denote a primitive idempotent of $\mathbb{F}\mathrm{Out}(L, u)$ such that V is isomorphic to $\mathbb{F}\mathrm{Out}(L, u)e_V$. Then the simple diagonal p-permutation functor $S_{L,u,V}$ that correspond to the triple (L, u, V) is $S_{L,u,V} = \mathbb{F}T^{\Delta}_{L,u}e_V$. More precisely, for any finite group G, we have

$$S_{L,u,V}(G) = \mathbb{F}T^{\Delta}(G, L\langle u\rangle)e_V = \{X \otimes_{kL\langle u\rangle} e_V | X \in \mathbb{F}T^{\Delta}(G, L\langle u\rangle)\}.$$

Our aim is to give a more precise description for the evaluation $S_{L,u,V}(G)$.

7.1 Let $M := \mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$. Note that M is an $(\mathbb{F}T^{\Delta}(G, G), E_{L,u})$-bimodule. Hence via the isomorphism in Corollary 6.14, M can be viewed as an $(\mathbb{F}T^{\Delta}(G, G), \mathbb{F}\mathrm{Out}(L, u))$-bimodule. Note that

$$\sum_{(P,s)\in[\mathcal{Q}^{\Delta}_{G,p}]} F^G_{P,s} = [k] \in \mathbb{F}T(G)$$

implies that the sum

$$\sum_{(P,s)\in[\mathcal{Q}^{\Delta}_{G,p}]} \widetilde{F^G_{P,s}} = [kG] \in \mathbb{F}T^{\Delta}(G, G)$$

is equal to the identity element of $\mathbb{F}T^{\Delta}(G, G)$. This means that we have the decomposition

$$M = \bigoplus_{(P,s)\in[\mathcal{Q}^{\Delta}_{G,p}]} \widetilde{F^G_{P,s}}M = \bigoplus_{(P,s)\in[\mathcal{Q}^{\Delta}_{G,p}]} \widetilde{F^G_{P,s}}\mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$$

as $\mathbb{F}$-vector spaces. Note that Theorem 3.7 implies that $\widetilde{F^G_{P,s}}M = 0$ unless $(\tilde{P}, \tilde{s}) \cong (L, u)$. Hence as $\mathbb{F}$-vector spaces, we have

$$M = \sum_{\substack{(P,s)\in[\mathcal{Q}^{\Delta}_{G,p}]\\(\tilde{P},\tilde{s})\cong(L,u)}} \widetilde{F^G_{P,s}}\mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}.$$

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7.2 (a) Let (P, s) be a pair of G with the property that $(\tilde{P}, \tilde{s}) \cong (L, u)$. Using Corollary 2.6 one shows that if $\big(\Delta(R, \gamma, L), (t, u^i)\big)$ is a pair of $G \times L\langle u\rangle$ with the property that

$$\widetilde{F_{P,s}^G} F_{\Delta(R,\gamma,L),(t,u^i)}^{G \times L\langle u\rangle} \widetilde{F_{L,u}^{L\langle u\rangle}} \neq 0$$

then the pair $\big(\Delta(R, \gamma, L), (t, u^i)\big)$ is conjugate to a pair of the form $\big(\Delta(P, \phi, L), (s, u)\big)$. But then Lemma 3.6 further implies that $\widetilde{F_{P,s}^G} \mathbb{F}T^\Delta(G, L\langle u\rangle) \widetilde{F_{L,u}^{L\langle u\rangle}}$ is generated by the elements of the form $F_{\Delta(P,\phi,L),(s,u)}^{G \times L\langle u\rangle}$.

(b) Fix an isomorphism $\phi_{P,s} : L \to P$ satisfying $\phi_{P,s}(^u l) = {}^s\phi_{P,s}(l)$ for all $l \in L$. Note that the existence of such an isomorphism follows from Lemma 3.3. For any $g \in N_G(P, s)$, the map $\phi_{P,s}^{-1} \circ i_g \circ \phi_{P,s}$ belongs to $\mathrm{Aut}_u(L)$. Therefore we have a group homomorphism

$$N_G(P, s) \to \mathrm{Out}(L, u) \tag{9}$$

that sends $g \in N_G(P, s)$ to the image of $\phi_{P,s}^{-1} \circ i_g \circ \phi_{P,s}$ in $\mathrm{Out}(L, u)$. This allows us to define an $\mathbb{F}N_G(P, s)$-module structure on any $\mathbb{F}\mathrm{Out}(L, u)$-module. Let N denote the image of $N_G(P, s)$ under this homomorphism and set

$$e_{P,s} := \frac{1}{|N|} \sum_{n \in N} n \,.$$

7.3 Proposition *Let G be a finite group and let (L, u) be a D^Δ-pair. The map*

$$\Psi : \widetilde{F_{P,s}^G} \mathbb{F}T^\Delta(G, L\langle u\rangle) \widetilde{F_{L,u}^{L\langle u\rangle}} \to e_{P,s}\mathbb{F}\mathrm{Out}(L, u)$$

$$F_{\Delta(P,\phi,L),(s,u)}^{G \times L\langle u\rangle} \mapsto e_{P,s}\overline{\phi_{P,s}^{-1}\phi}$$

is an isomorphism of right $\mathbb{F}\mathrm{Out}(L, u)$-modules where $\overline{}$ denotes the image in $\mathrm{Out}(L, u)$.

Proof First we show that the map Ψ is well-defined. Let $(\Delta(P, \phi, L), (s, u))$ and $(\Delta(P, \sigma, L), (s, u))$ be two conjugate pairs of $G \times L\langle u\rangle$. Then there exists $(g, l) \in G \times L\langle u\rangle$ such that

$$(\Delta(P, \phi, L), (s, u)) = {}^{(g,l)}\big(\Delta(P, \sigma, L), (s, u)\big) \,.$$

This means that $g \in N_G(P, s)$, $l \in C_{L\langle u\rangle}(u)$ and $\phi = i_g \sigma i_l^{-1}$. Therefore we have

$$\Psi(F_{\Delta(P,\phi,L),(s,u)}^{G \times L\langle u\rangle}) = e_{P,s}\overline{\phi_{P,s}^{-1}\phi} = e_{P,s}\overline{\phi_{P,s}^{-1}i_g\sigma i_l^{-1}}$$

$$= \frac{1}{|N|} \sum_{n \in N} n\overline{\phi_{P,s}^{-1}i_g\sigma i_l^{-1}}$$

$$= \frac{1}{|N|} \sum_{n \in N} n\overline{\phi_{P,s}^{-1}i_g\phi_{P,s}} \cdot \overline{\phi_{P,s}^{-1}\sigma i_l^{-1}}$$

$$= \frac{1}{|N|} \sum_{n \in N} n\overline{\phi_{P,s}^{-1}\sigma}$$

$$= \Psi(F_{\Delta(P,\sigma,L),(s,u)}^{G \times L\langle u\rangle}) \,.$$

This proves the well-definedness. The map Ψ is clearly surjective. For the injectivity, assume that

$$\Psi\left(\sum_\phi \lambda_\phi F^{G\times L\langle u\rangle}_{\Delta(P,\phi,L),(s,u)}\right) = 0 \tag{10}$$

where $\lambda_\phi \in \mathbb{F}$ and where the sum runs over a set of isomorphisms $\phi : L \to P$ such that the idempotents $F^{G\times L\langle u\rangle}_{\Delta(P,\phi,L),(s,u)}$ are all distinct. This implies that

$$\sum_\phi \lambda_\phi \left(\sum_{n\in N} n\overline{\phi^{-1}_{P,s}\phi}\right) = 0 \,. \tag{11}$$

Now if ϕ and σ are isomorphisms $L \to P$ that appear in (11), and if $m, n \in N$ such that

$$n\overline{\phi^{-1}_{P,s}\phi} = m\overline{\phi^{-1}_{P,s}\sigma}\,,$$

then there exists $g \in N_G(P,s)$ and $l \in C_{L\langle u\rangle}(u)$ such that

$$\phi^{-1}_{P,s} \circ i_g \circ \phi_{P,s} \circ \phi^{-1}_{P,s} \circ \phi = \phi^{-1}_{P,s} \circ \sigma \circ i_l$$

which implies that $i_g \circ \phi \circ i_l^{-1} = \sigma$. Therefore the pairs $(\Delta(P,\phi,L),(s,u))$ and $(\Delta(P,\sigma,L),(s,u))$ are conjugate and hence $F^{G\times L\langle u\rangle}_{\Delta(P,\phi,L),(s,u)} = F^{G\times L\langle u\rangle}_{\Delta(P,\sigma,L),(s,u)}$. Since the idempotents in (10) are chosen to be distinct, this implies that $\phi = \sigma$ and so $n = m$. This shows that the elements $n\overline{\phi^{-1}_{P,s}\phi} \in \mathrm{Out}(L,u)$ in (11) are distinct. Therefore we have $\lambda_\phi = 0$ for any ϕ in the sum. This shows that the map Ψ is injective and hence an isomorphism of $\mathbb{F}$-vector spaces. The right $\mathrm{Out}(L,u)$-module structure on $\widetilde{F^G_{P,s}}\mathbb{F}T^\Delta(G,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$ is given via the algebra isomorphism in Corollary 6.14 and hence Lemma 6.12 implies that the map Φ is also an $\mathbb{F}\mathrm{Out}(L,u)$-module homomorphism. $\qquad\square$

We define an $\mathbb{F}N_G(P,s)$-module structure on the simple $\mathbb{F}\mathrm{Out}(L,u)$-module V via the homomorphism in (9). Proposition 7.3 implies that the vector space $\widetilde{F^G_{P,s}}\mathbb{F}T^\Delta(G,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}e_V$ is isomorphic to the space of $N_G(P,s)$-fixed points $V^{N_G(P,s)}$ of V. We proved the following.

7.4 Corollary *For any finite group G, we have*

$$S_{L,u,V}(G) \cong \bigoplus_{\substack{(P,s)\in[\mathcal{Q}^\Delta_{G,p}] \\ (\tilde{P},\tilde{s})\cong(L,u)}} V^{N_G(P,s)} \,.$$

7.5 Remark For p-groups G and H, the $\mathbb{F}$-vector space $\mathbb{F}T^\Delta(G,H)$ is canonically isomorphic to $\mathbb{F}B^\Delta(G,H)$, the $\mathbb{F}$-linear extension of the Burnside group of bifree (G,H)-bisets. Moreover, through this isomorphism, the tensor product of bimodules becomes the composition of bisets. Hence diagonal p-permutation functors restricted to p-groups are precisely global Mackey functors restricted to p-groups (see [W93] for more information on global Mackey functors).

For a simple diagonal p-permutation functor $S_{L,1,V}$ and a p-group G, the isomorphism in Corollary 7.4 becomes

$$S_{L,1,V}(G) \cong \bigoplus_{P\cong L} V^{N_G(P)}$$

where P runs through the subgroups of G up to conjugation, and we recover the formula for the evaluations of simple global Mackey functors (see [W93], Theorem 2.6(ii)]).

8 Blocks as functors

In this section, G denotes a finite group and b a block idempotent of kG. For an arbitrary commutative ring of coefficients R, we define the *the block diagonal p-permutation functor* $RT_{G,b}^\Delta$ as

$$RT_{G,b}^\Delta : Rpp_k^\Delta \to {}_R\mathsf{Mod}$$
$$H \mapsto RT^\Delta(H, G) \otimes_{kG} kGb \,.$$

Our aim in this section is to describe the functor $\mathbb{F}T_{G,b}^\Delta$ in terms of the simple functors $S_{L,u,V}$. We first make a remark for the case of an arbitrary ring R.

8.1 Remark Let D be a defect group of b and let $i \in (kGb)^D$ be a source idempotent of b. The source algebra $ikGi$ of b is an interior D-algebra and for any finite group H we denote by $RT^\Delta(H, ikGi)$ the Grothendieck group of $(kH, ikGi)$-bimodules whose restriction to $H \times D$ lies in $RT^\Delta(H, D)$.

By [P81] the map sending a kGb-module M to the $ikGi$-module iM induces a Morita equivalence between ${}_{kGb}\mathsf{mod}$ and ${}_{ikGi}\mathsf{mod}$. Hence we have a Morita equivalence between ${}_{kH}\mathsf{mod}_{kGb}$ and ${}_{kH}\mathsf{mod}_{ikGi}$ given by a p-permutation bimodule in $RT^\Delta(kH, kG)$. It follows that the functor $RT_{G,b}^\Delta$ is isomorphic to the diagonal p-permutation functor $RT^\Delta(-, ikGi)$. This means in particular that the functor $RT_{G,b}^\Delta$ depends only on the source algebra of b. By [P88], the source algebra of b determines the local points on kGb and one of our aims in the rest of this section is to give a description of the multiplicities of the simple functors in $\mathbb{F}T_{G,b}^\Delta$ in terms of the local points on kGb (see Theorem 8.22).

8.2 Let (L, u) be a D^Δ-pair and V a simple $\mathbb{F}\mathrm{Out}(L, u)$-module. We set

$$\mathrm{Mult}(G, b, L, u, V) := kGb \otimes_{kG} \mathbb{F}T^\Delta(G, L\langle u \rangle) \otimes_{kL\langle u \rangle} \widetilde{F_{L,u}^{L\langle u \rangle}} e_V \,,$$

where e_V is an idempotent of $\mathbb{F}\mathrm{Out}(L, u)$ such that $V \cong \mathbb{F}\mathrm{Out}(L, u)e_V$.

By the Yoneda lemma we have

$$\mathrm{Hom}_{\mathcal{F}_{pp_k}^\Delta}\left(\mathbb{F}T_G^\Delta, S_{L,u,V}\right) \cong S_{L,u,V}(G) \,.$$

Therefore Schur's lemma implies that the multiplicity of the simple functor $S_{L,u,V}$ in the representable functor $\mathbb{F}T_G^\Delta$ is equal to

$$\dim_\mathbb{F}\left(S_{L,u,V}(G)\right) = \dim_\mathbb{F}\left(\mathbb{F}T^\Delta(G, L\langle u \rangle) \otimes_{kL\langle u \rangle} \widetilde{F_{L,u}^{L\langle u \rangle}} e_V\right) \,.$$

This implies that the multiplicity of $S_{L,u,V}$ in the functor $\mathbb{F}T_{G,b}^\Delta$ is equal to

$$\dim_\mathbb{F}\left(\mathrm{Mult}(G, b, L, u, V)\right) \,.$$

Our aim in this section is to give a description of $\mathrm{Mult}(G, b, L, u, V)$. First, we give a description of $kGb \otimes_{kG} \mathbb{F}T^\Delta(G, L\langle u \rangle) \otimes_{kL\langle u \rangle} \widetilde{F_{L,u}^{L\langle u \rangle}}$.

8.3 Let M be a $(kG, kL\langle u\rangle)$-bimodule and let $\Delta(P, \pi, L)$ be a twisted diagonal subgroup of $G \times L\langle u\rangle$. The Brauer quotient of $kGb \otimes_{kG} M \cong bM$ at $\Delta(P, \pi, L)$ is isomorphic to $\mathrm{Br}_P(b)\underbrace{M}[\Delta(P, \pi, L)]$. By [Br85], this implies, in particular, that any element of $b\mathbb{F}T^\Delta(G, L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}}$ is a linear combination of the elements of the form

$$m_{P,\pi,E} := M\left(\Delta(P, \pi, L), E\right)\widetilde{F_{L,u}^{L\langle u\rangle}},$$

where $\pi : L \to P \leqslant G$ is a group isomorphism, E is a projective indecomposable $kN_{G \times L\langle u\rangle}\left(\Delta(P, \pi, L)\right)/\Delta(P, \pi, L)$-module, and $M\left(\Delta(P, \pi, L), E\right)$ is the unique, up to isomorphism, indecomposable p-permutation $(kG, kL\langle u\rangle)$-bimodule whose Brauer quotient at $\Delta(P, \pi, L)$ is isomorphic to E.

8.4 (a) Let $\mathcal{P}(G, L, u)$ denote the set of pairs (P, π) where $P \leqslant G$ is a p-subgroup and $\pi : L \to P$ is a group isomorphism for which there exists $s \in G$ such that $(\Delta(P, \pi, L), (s, u))$ is a pair of $G \times L\langle u\rangle$. The group $G \times L\langle u\rangle$ acts on $\mathcal{P}(G, L, u)$ via

$$(g, t) \cdot (P, \pi) = ({}^gP, i_g \circ \pi \circ i_t^{-1})$$

for $g \in G$, $t \in L\langle u\rangle$ and $(P, \pi) \in \mathcal{P}(G, L, u)$. Two elements (P, π) and (Q, ρ) of $\mathcal{P}(G, L, u)$ lie in the same $G \times L\langle u\rangle$-orbit if and only if the subgroups $\Delta(P, \pi, L)$ and $\Delta(Q, \rho, L)$ of $G \times L\langle u\rangle$ are conjugate.

(b) The group $\mathrm{Aut}(L, u)$ also acts on $\mathcal{P}(G, L, u)$ via

$$(P, \pi) \cdot \gamma = (P, \pi\gamma)$$

for $(P, \pi) \in \mathcal{P}(G, L, u)$ and $\gamma \in \mathrm{Aut}(L, u)$. If also $g \in G$ and $t \in L\langle u\rangle$, we have

$$\left((g, t) \cdot (P, \pi)\right) \cdot \gamma = (g, \gamma^{-1}(t)) \cdot \left((P, \pi) \cdot \gamma\right)$$

(c) Let $[(G \times L\langle u\rangle)\backslash \mathcal{P}(G, L, u)]$ denote a set of representatives of $G \times L\langle u\rangle$-orbits of $\mathcal{P}(G, L, u)$. The group $\mathrm{Out}(L, u)$ acts on $[(G \times L\langle u\rangle)\backslash \mathcal{P}(G, L, u)]$ via

$$[(P, \pi)] \cdot \overline{\gamma} = [(P, \pi\gamma)]$$

for $[(P, \pi)] \in [(G \times L\langle u\rangle)\backslash \mathcal{P}(G, L, u)]$ and $\overline{\gamma} \in \mathrm{Out}(L, u)$, where $\gamma \in \mathrm{Aut}(L, u)$ is an automorphism with image $\overline{\gamma}$ in $\mathrm{Out}(L, u)$. Note that the class $[(P, \pi\gamma)]$ does not depend on the choice of γ.

(d) Let $\pi : L \to P$ be a group isomorphism and let $s \in G$ be an element with $\pi({}^ul) = {}^s\pi(l)$ for all $l \in L$. The p-part of the order of s is coprime to the order of u. Hence there are integers a and b with

$$a \cdot |u| + b \cdot |s|_p = 1.$$

Now setting $s' := s^{b|s|_p}$ and $s'' := s^{a|u|}$ we get that

$$i_s \circ \pi = i_{s'} \circ i_{s''} \circ \pi = i_{s'} \circ \pi \circ i_{u^{a|u|}} = i_{s'} \circ \pi,$$

since $u^{a|u|} = 1$. Hence the p'-element s' satisfies $i_{s'} \circ \pi = \pi \circ i_u$ which implies that $\pi \in \mathcal{P}(G, L, u)$.

8.5 Let $(P, \pi) \in \mathcal{P}(G, L, u)$ be a pair and let $\gamma \in \mathrm{Aut}(L, u)$.

(a) Set $\overline{N}_{P,\pi} := N_{G \times L\langle u \rangle}(\Delta(P, \pi, L))/\Delta(P, \pi, L)$. Since $(s, u) \in N_{G \times L\langle u \rangle}(\Delta(P, \pi, L))$ for some $s \in G$, the group homomorphism

$$\Phi : \overline{N}_{P,\pi} \to \langle u \rangle$$
$$\overline{(a, lu^i)} \mapsto u^i$$

is surjective. Also the map

$$\iota : C_G(P) \to \overline{N}_{P,\pi}$$
$$x \mapsto \overline{(x, 1)}$$

is an injective group homomorphism. One also shows that the kernel of Φ is equal to the image of ι. Therefore we have a short exact sequence

$$1 \to C_G(P) \to \overline{N}_{P,\pi} \to \langle u \rangle \to 1$$

of groups.

(b) Similarly, let $\overline{N}_\gamma = N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, \gamma, L))/(\Delta(L, \gamma, L)$. One shows that the map

$$\Phi : \overline{N}_\gamma \to \langle u \rangle$$
$$\overline{(a, lu^i)} \mapsto u^i$$

is well-defined and surjective. Also the map

$$\iota : Z(L) \to \overline{N}_\gamma$$
$$x \mapsto \overline{(x, 1)}$$

is an injective group homomorphism. One also shows that the kernel of Φ is equal to the image of ι. Therefore we have a short exact sequence

$$1 \to Z(L) \to \overline{N}_\gamma \to \langle u \rangle \to 1$$

of groups which is split since $\langle u \rangle$ is a p'-group and $Z(L)$ is a p-group. Therefore we have $\overline{N}_\gamma \cong Z(L)\langle u \rangle$.

(c) We have a group isomorphism

$$N_{G \times L\langle u \rangle}(\Delta(P, \pi\gamma, L)) \to N_{G \times L\langle u \rangle}(\Delta(P, \pi, L))$$
$$(s, t) \mapsto (s, \gamma(t))$$

which maps $\Delta(P, \pi\gamma, L)$ to $\Delta(P, \pi, L)$. Hence we have a group isomorphism

$$\overline{N}_{\pi\gamma} \to \overline{N}_\pi, \quad \overline{(s, t)} \mapsto \overline{(s, \gamma(t))}. \tag{12}$$

8.6 Lemma *Let $(P, \pi) \in \mathcal{P}(G, L, u)$ and let $\gamma \in \mathrm{Aut}(L, u)$. Let also V be a $k\overline{N}_{P,\pi}$-module and let W be a $k\langle u \rangle$-module. Consider the $k\overline{N}_{P,\pi\gamma}$-module $V \otimes_{kZ(L)} \left(\mathrm{Ind}_{\langle u \rangle}^{\overline{N}_\gamma} W \right)$ with the action*

$$\overline{(s, t)} \cdot (v \otimes w) := \overline{(s, \gamma(t))}v \otimes \overline{(\gamma(t), t)}w$$

for $\overline{(s,t)} \in \overline{N}_{P,\pi\gamma}$, $v \in V$ and $w \in \operatorname{Ind}_{\langle u \rangle}^{\overline{N}_\gamma} W$. *Consider also the $k\overline{N}_{P,\pi\gamma}$-module $V \otimes_k W$ with the action*

$$\overline{(s,t)} \cdot (v \otimes w) := \overline{(s,\gamma(t))}v \otimes u^i w$$

for $\overline{(s,t)} \in \overline{N}_{P,\pi\gamma}$, $v \otimes w \in V \otimes_k W$ where $t = lu^i$. Then the map

$$\Phi : V \otimes_{kZ(L)} \left(k\overline{N}_\gamma \otimes_{k\langle u \rangle} W \right) \to V \otimes_k W$$
$$v \otimes (lu^i \otimes w) \mapsto v\gamma(l) \otimes u^i w$$

is an isomorphism of $k\overline{N}_{P,\pi\gamma}$-modules, where we use the isomorphism $\overline{N}_\gamma \cong Z(L)\langle u \rangle$.

Proof Clearly, the map Φ is well-defined and surjective. Let $v \otimes (lu^i \otimes w) \in V \otimes_{kZ(L)} \left(k\overline{N} \otimes_{k\langle u \rangle} W \right)$ and $\overline{(s,t)} \in \overline{N}_{P,\pi\gamma}$. Write $t = l'u^j$ and note that the image of $(\gamma(t),t)$ in $\overline{N}_\gamma$ is $\overline{(\gamma(u^j),u^j)}$. Let also $l_0 \in Z(L)$ be the element with the property that $u^j l = l_0 u^j$. We have

$$
\begin{aligned}
\Phi\left(\overline{(s,t)} \cdot \left(v \otimes (lu^i \otimes w)\right)\right) &= \Phi\left(\overline{(s,\gamma(t))}v \otimes \left(\overline{(\gamma(t),t)} \cdot (lu^i \otimes w)\right)\right) \\
&= \Phi\left(\overline{(s,\gamma(t))}v \otimes \left(l_0 u^{i+j} \otimes w\right)\right) \\
&= \overline{((s,\gamma(t))}v)\gamma(l_0) \otimes u^{i+j}w \\
&= \overline{(1,\gamma(l_0^{-1}))}\,\overline{(s,\gamma(t))}v \otimes u^{i+j}w \\
&= \overline{(s,\gamma(l_0^{-1}t))}v \otimes u^{i+j}w \\
&= \overline{(s,\gamma(tl^{-1}))}v \otimes u^{i+j}w \\
&= \overline{(s,\gamma(t))(1,\gamma(l^{-1}))}v \otimes u^i w \\
&= \overline{(s,\gamma(t))}v\gamma(l) \otimes u^i w \\
&= \overline{(s,t)} \cdot \Phi\left(v \otimes (lu^i \otimes w)\right) .
\end{aligned}
$$

This shows that Φ is a $k\overline{N}_{P,\pi\gamma}$-module homomorphism. Since both sides have the same k-dimension, it follows that Φ is an isomorphism. □

8.7 Lemma *Let (L,u) be a D^Δ-pair and let $\gamma \in \operatorname{Aut}(L,u)$. We have*

$$F_{\Delta(L,\gamma,L),(u,u)}^{L\langle u \rangle \times L\langle u \rangle}[\Delta(L,\gamma,L)] = \frac{1}{|Z(L\langle u \rangle)|}\operatorname{Ind}_{\langle u \rangle}^{\overline{N}_\gamma}\left(F_{1,u}^{\langle u \rangle}\right)$$

in $\mathbb{F}T(\overline{N}_\gamma)$.

Proof Let $X \leqslant \Delta(L,\gamma,L)$ be a subgroup with the property that $X^{(u,u)} = X$. Let also $\lambda : \langle u \rangle \to k^\times$ be a group homomorphism. The indecomposable direct summands of

$$\operatorname{Ind}_{\langle X(u,u) \rangle}^{L\langle u \rangle \times L\langle u \rangle}\left(k_{\langle X(u,u) \rangle,\lambda}^{\langle \Delta(L,\gamma,L)(u,u) \rangle}\right)$$

have vertices contained in $\langle X(u,u) \rangle \cap {}^s\Delta(L,\gamma,L)$ for some s. Therefore the Brauer construction of such an induced module at $\Delta(L,\gamma,L)$ is zero if X is strictly contained in $\Delta(L,\gamma,L)$. It follows

by the primitive idempotent formula (see [D15, Proposition 2.7.8]) that

$$F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u,u)}[\Delta(L,\gamma,L)] =$$
$$\frac{|C_{\Delta(L,\gamma,L)}(u,u)|}{|C_{N_\gamma}(u,u)|}\sum_{\lambda:\langle u\rangle\to k^\times}\tilde\lambda(u^{-1})\left(\mathrm{Ind}^{L\langle u\rangle\times L\langle u\rangle}_{\langle\Delta(L,\gamma,L)(u,u)\rangle}\left(\mathrm{Inf}^{\langle\Delta(L,\gamma,L)(u,u)\rangle}_{\langle u\rangle}k_\lambda\right)\right)[\Delta(L,\gamma,L)],$$

where $\tilde\lambda$ is the Brauer character of k_λ. The classical formula for the Brauer construction of an induced module implies that

$$\left(\mathrm{Ind}^{L\langle u\rangle\times L\langle u\rangle}_{\langle\Delta(L,\gamma,L)(u,u)\rangle}\left(\mathrm{Inf}^{\langle\Delta(L,\gamma,L)(u,u)\rangle}_{\langle u\rangle}k_\lambda\right)\right)[\Delta(L,\gamma,L)] = \mathrm{Ind}^{N_\gamma}_{\langle\Delta(L,\gamma,L)(u,u)\rangle}\left(\mathrm{Inf}^{\langle\Delta(L,\gamma,L)(u,u)\rangle}_{\langle u\rangle}k_\lambda\right)$$

as kN_γ-modules. Therefore,

$$F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta(L,\gamma,L),(u,u)}[\Delta(L,\gamma,L)] = \frac{1}{|\langle u\rangle||Z(L\langle u\rangle)|}\sum_{\lambda:\langle u\rangle\to k^\times}\tilde\lambda(u^{-1})\mathrm{Ind}^{\overline{N}_\gamma}_{\langle u\rangle}(k_\lambda)$$
$$= \frac{1}{|Z(L\langle u\rangle)|}\mathrm{Ind}^{\overline{N}_\gamma}_{\langle u\rangle}\left(F^{\langle u\rangle}_{1,u}\right),$$

as desired. $\qquad\square$

8.8 Lemma *Let* $\Delta(P,\pi,L)$ *be a twisted diagonal subgroup of* $G\times L\langle u\rangle$ *and let* $m_{P,\pi,E}\in b\mathbb{F}T^\Delta(G,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$.

(i) *Let* $\Delta(Q,\rho,L)$ *be a twisted diagonal subgroup of* $G\times L\langle u\rangle$. *We have*

$$m_{P,\pi,E}[\Delta(Q,\rho,L)] = 0$$

unless $\Delta(Q,\rho,L)$ *is conjugate to* $\Delta(P,\pi,L)$.

(ii) *We have*

$$m_{P,\pi,E}\left[\Delta(P,\pi,L)\right] = E\otimes_k F^{\langle u\rangle}_{1,u} = \frac{1}{|\langle u\rangle|}\sum_{\lambda:\langle u\rangle\to k^\times}\tilde\lambda(u^{-1})E\otimes_k k_\lambda$$

in $\mathbb{F}\mathrm{Proj}(k\overline{N}_{P,\pi})$. *In particular, if* $m_{P,\pi,E}\left[\Delta(P,\pi,L)\right]$ *is non-zero, then* $(P,\pi)\in\mathcal{P}(G,L,u)$.

Proof (i) By Proposition 2.4, the Brauer quotient of the element

$$m_{P,\pi,E}\in b\mathbb{F}T^\Delta(G,L\langle u\rangle)\widetilde{F^{L\langle u\rangle}_{L,u}}$$

at $\Delta(Q,\rho,L)$ is equal to

$$\bigoplus_{\theta:=(\alpha,V,\beta)}\mathrm{Ind}^{N_{G\times L\langle u\rangle}(\Delta(Q,\rho,L))}_{X(\theta)*Y(\theta)}\left(M\left(\Delta(P,\pi,L),E\right)[\Delta(Q,\alpha,V)]\otimes_{kZ(L)}\widetilde{F^{L\langle u\rangle}_{L,u}}[\Delta(V,\beta,L)]\right),$$

in $\mathbb{F}T\left(N_{G\times L\langle u\rangle}(\Delta(Q,\rho,L))\right)$, where $(\alpha,V,\beta)\in\tilde\Gamma_{L\langle u\rangle}(Q,\rho,L)$, $X(\theta) = N_{G\times L\langle u\rangle}(\Delta(Q,\alpha,V))$ and $Y(\theta) = N_{L\langle u\rangle\times L\langle u\rangle}(\Delta(V,\beta,L))$. By Remark 3.5, we have $\widetilde{F^{L\langle u\rangle}_{L,u}} = |Z(L\langle u\rangle|\cdot F^{L\langle u\rangle\times L\langle u\rangle}_{\Delta L,(u,u)}$ which

implies that the Brauer quotient $\widetilde{F^{L\langle u\rangle}_{L,u}}[\Delta(V,\beta,L)]$ is zero unless the group $\Delta(V,\beta,L)$ is $(L\langle u\rangle \times L\langle u\rangle)$-conjugate to ΔL. This implies that $V = L$ and the map β is an inner automorphism of L. Therefore, up to the action of $N_{G\times L\langle u\rangle}(\Delta(Q,\rho,L)) \times L\langle u\rangle$, we can assume that β is equal to identity and hence $\alpha = \rho$. This shows that

$$m_{P,\pi,E}[\Delta(Q,\rho,L)] = 0$$

unless $\Delta(Q,\rho,L)$ is conjugate to $\Delta(P,\pi,L)$.

(ii) We use the calculations in the proof of part (i). One shows that for $\theta = (\pi,L,\mathrm{id})$, we have

$$N_{G\times L\langle u\rangle}(\Delta(P,\pi,L)) = X(\theta) * Y(\theta).$$

Therefore, the calculations above, together with Lemma 8.6 and Lemma 8.7, imply that

$$
\begin{aligned}
m_{P,\pi,E}[\Delta(P,\pi,L)] &= M(\Delta(P,\pi,L),E)\,[\Delta(P,\pi,L)] \otimes_{kZ(L)} \widetilde{F^{L\langle u\rangle}_{L,u}}[\Delta(L)] \\
&= E \otimes_{kZ(L)} \mathrm{Ind}^{\overline{N}}_{\langle u\rangle}\left(F^{\langle u\rangle}_{1,u}\right) \\
&= E \otimes_k F^{\langle u\rangle}_{1,u}
\end{aligned}
$$

in $\mathbb{F}T(\overline{N}_\pi)$. For the second assertion, note that if the Brauer construction $m_{P,\pi,E}[\Delta(P,\pi,L)]$ is non-zero, then its Brauer character is non-zero at some p'-element $(s,t) \in N_{G\times L\langle u\rangle}(\Delta(P,\pi,L))$. This implies that the Brauer character of the module $F^{\langle u\rangle}_{1,u}$ is non-zero at $t \in \langle u\rangle$ which in turn implies that $t = u$. This shows that $(\Delta(P,\pi,L),(s,u))$ is a pair of $G \times L\langle u\rangle$. $\qquad\square$

8.9 (a) Let $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi},u\right)$ denote the subgroup consisting of elements ω of the group $\mathbb{F}\mathrm{Proj}(k\overline{N}_{P,\pi})$ of projective $k\overline{N}_{P,\pi}$-modules such that

$$
\begin{cases}
\mathrm{Br}_P(b)w = w \\
\omega_\lambda = \tilde{\lambda}(u)\omega \text{ for any } \lambda : \langle u\rangle \to k^\times.
\end{cases}
$$

(b) Let $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi},u\right)^\sharp$ denote the subgroup of $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi},u\right)$ consisting of linear combinations of projective indecomposable $k\overline{N}_{P,\pi}$-modules E with isotypic restriction to $C_G(P)$, i.e., $\mathrm{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}E$ has an indecomposable direct summand with the inertia group $\overline{N}_{P,\pi}$. By Theorem 4.1 this is equivalent to requiring that $\mathrm{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}E$ is indecomposable.

8.10 Lemma *Elements of* $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi},u\right)$ *are linear combinations of sums of the type*

$$S_E := \sum_{\lambda:\langle u\rangle\to k^\times} \tilde{\lambda}(u^{-1})E_\lambda$$

where E is a projective indecomposable $k\overline{N}_{P,\pi}$-module.

Proof Let $v \in \mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi}, u\right)$ be an arbitrary element and let E be a projective inde-composable $k\overline{N}_{P,\pi}$-module appearing in v with a nonzero coefficient x. Since $v_\lambda = \tilde{\lambda}(u)v$ for any $\lambda : \langle u \rangle \to k^\times$, it follows that the coefficient of E_λ in v is $\tilde{\lambda}(u^{-1}) \cdot x$. Hence the sum S_E appears in v with the coefficient x. $\square$

Let $(P, \pi) \in \mathcal{P}(G, L, u)$ and $m_{P,\pi,E} \in b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$. Since $bm_{P,\pi,E} = m_{P,\pi,E}$, it follows that $\mathrm{Br}_P(b)$ acts as the identity on E, where we identify $C_G(P)$ with its image in $\overline{N}_{P,\pi}$. Moreover, by Lemma 8.8, we have

$$m_{P,\pi,E}[\Delta(P, \pi, L)] = \frac{1}{|\langle u \rangle|} \sum_{\lambda : \langle u \rangle \to k^\times} \tilde{\lambda}(u^{-1}) E_\lambda \,.$$

This implies that $m_{P,\pi,E}[\Delta(P, \pi, L)]$ lies in $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi}, u\right)$.

Let F be an indecomposable direct summand of the restriction of E to $C_G(P)$, and let T be its inertial subgroup in $\overline{N}_{P,\pi}$. By Clifford theory, we have $E \cong \mathrm{Ind}_T^{\overline{N}_{P,\pi}} W$ for some indecomposable direct summand W of $\mathrm{Ind}_{C_G(P)}^T F$. The isomorphism

$$(\mathrm{Ind}_T^{\overline{N}_{P,\pi}} W) \otimes_k k_\lambda \to \mathrm{Ind}_T^{\overline{N}_{P,\pi}}(W \otimes_k k_\lambda)$$
$$(n \otimes w) \otimes k_0 \mapsto n \otimes (w \otimes n^{-1} \cdot k_0)$$

implies that we have

$$\sum_{\lambda : \langle u \rangle \to k^\times} \tilde{\lambda}(u^{-1}) E \otimes_k k_\lambda = \mathrm{Ind}_T^{\overline{N}_{P,\pi}} \left(\sum_{\lambda : \langle u \rangle \to k^\times} \tilde{\lambda}(u^{-1}) W \otimes_k k_\lambda \right) \,.$$

The module $W \otimes_k k_\lambda$ depends only on the restriction of λ to the image $\langle u^i \rangle$ of T in $\langle u \rangle$. Let $\lambda_0 : \langle u \rangle \to k^\times$ be given. If $\langle u^i \rangle$ is a proper subgroup of $\langle u \rangle$, then the element $u^{-1}\langle u^i \rangle \in \langle u \rangle/\langle u^i \rangle$ is not identity and hence the sum

$$\sum_{\substack{\lambda : \langle u \rangle \to k^\times \\ \lambda|_{\langle u^i \rangle} = \lambda_0|_{\langle u^i \rangle}}} \tilde{\lambda}(u^{-1}) = \sum_{\substack{\lambda : \langle u \rangle \to k^\times \\ \langle u^i \rangle \leqslant \ker(\lambda\lambda_0^{-1})}} \tilde{\lambda}(u^{-1}) = \sum_{\substack{\lambda : \langle u \rangle \to k^\times \\ \langle u^i \rangle \leqslant \ker(\lambda\lambda_0^{-1})}} \tilde{\lambda_0}(u^{-1})\widetilde{\lambda\lambda_0^{-1}}(u^{-1})$$
$$= \sum_{\overline{\lambda} : \langle u \rangle/\langle u^i \rangle \to k^\times} \tilde{\lambda_0}(u^{-1})\mathrm{Inf}_{\langle u \rangle/\langle u^i \rangle}^{\langle u \rangle}\overline{\tilde{\lambda}}(u^{-1})$$
$$= \sum_{\overline{\lambda} : \langle u \rangle/\langle u^i \rangle \to k^\times} \tilde{\lambda_0}(u^{-1})\overline{\tilde{\lambda}}(u^{-1}\langle u^i \rangle)$$

is zero. Therefore, $m_{P,\pi,E}[\Delta(P, \pi, L)] = 0$ unless $T = \overline{N}_{P,\pi}$. This proves the following.

8.11 Lemma Let $(P, \pi) \in \mathcal{P}(G, L, u)$ and $m_{P,\pi,E} \in b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$. We have $m_{P,\pi,E}[\Delta(P, \pi, L)] = 0$ unless the restriction of E to $C_G(P)$ is isotypic. In particular, $m_{P,\pi,E}[\Delta(P, \pi, L)]$ lies in $\mathbb{F}\mathrm{Proj}\left(k\mathrm{Br}_P(b)\overline{N}_{P,\pi}, u\right)^\sharp$.

8.12 Theorem *The map*

$$\Phi : b\mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}} \rightarrow \bigoplus_{(P,\pi)\in[(G\times L\langle u\rangle)\backslash\mathcal{P}(G,L,u)]} \mathbb{F}\text{Proj}\left(k\text{Br}_P(b)\overline{N}_{P,\pi}, u\right)^{\sharp}$$

sending an element v to the sequence of its Brauer quotients $v[\Delta(P, \pi, L)]$, for (P, π) in a set of representatives of $G \times L\langle u\rangle$-orbits of $\mathcal{P}(G, L, u)$, is an isomorphism of $\mathbb{F}$-vector spaces.

Proof An element in the kernel of Φ has all its Brauer quotients equal to zero, so it is zero. Hence Φ is injective. By Lemma 8.10, an element of $\mathbb{F}\text{Proj}\left(k\text{Br}_P(b)\overline{N}_{P,\pi}, u\right)^{\sharp}$ is a linear combination of the elements S_E, where E is a projective indecomposable $k\overline{N}_{P,\pi}$-module with isotypic restriction to $C_G(P)$. The Brauer quotient $m_{P,\pi,E}[\Delta(Q, \rho, L)]$ is equal to zero if $\Delta(Q, \rho, L)$ is not conjugate to $\Delta(P, \pi, L)$, and $m_{P,\pi,E}[\Delta(P, \pi, L)]$ is equal to a nonzero scalar multiple of S_E. Hence Φ is also surjective. $\square$

8.13 We define $\mathcal{Z} = \mathcal{Z}(G, L, u)$ as the set of triples (P, π, E) where

- P is a p-subgroup of G.

- $\pi : L \rightarrow P$ is a group isomorphism such that there exists a p'-element $s \in G$ with $\pi(^{u}l) = {}^{s}\pi(l)$ for all $l \in L$.

- E is a projective indecomposable $k\text{Br}_P(b)\overline{N}_{P,\pi}$-module such that $\text{Res}^{\overline{N}_{P,\pi}}_{C_G(P)} E$ is indecomposable.

With the notation above this means that $(P, \pi) \in \mathcal{P}(G, L, u)$.

(b) The group $G \times L\langle u\rangle$ acts on $\mathcal{Z}$ by

$$(g, t) \cdot (P, \pi, E) := ({}^{g}P, i_g\pi i_{t^{-1}}, {}^{(g,t)}E)$$

for $(g, t) \in G \times L\langle u\rangle$ and $(P, \pi, E) \in \mathcal{Z}$. Here ${}^{(g,t)}E$ is the $k\overline{N}_{{}^{g}P, i_g\pi i_{t^{-1}}}$-module equal to E as a k-vector space and on which $\overline{(a, b)} \in \overline{N}_{{}^{g}P, i_g\pi i_{t^{-1}}}$ acts by

$$\overline{(a, b)} \cdot {}^{(g,t)}e := \overline{(a^{g}, b^{t})}e\,.$$

To show that the action is well-defined, we first show that there exists a p'-element $a \in G$ such that $(i_g\pi i_{t^{-1}})(^{u}l) = {}^{a}((i_g\pi i_{t^{-1}})(l))$ for all $l \in L$. By 8.13(a), it suffices to show that there exists $a \in G$ with this property. In other words, we need to show that $\overline{(a, u)} \in \overline{N}_{{}^{g}P, i_g\pi i_{t^{-1}}}$ for some $a \in G$. This is equivalent to the condition that $\overline{(a^{g}, u^{t})} \in \overline{N}_{P,\pi}$. Now one shows that $u^{t} = l_0 \cdot u$ for some $l_0 \in L$. Therefore for any $l \in L$ we have

$$\pi(^{u^{t}}l) = \pi(l_0)\pi(^{u}l)\pi(l_0^{-1}) = \pi(l_0)s\pi(l)s^{-1}\pi(l_0^{-1})\,.$$

Hence the element $a = g\pi(l_0)sg^{-1} \in G$ satisfies the desired condition.

Also since the action on ${}^{(g,t)}E$ is induced from the action on E, it follows that ${}^{(g,t)}E$ is a projective indecomposable $k\text{Br}_{{}^{g}P}(b)\overline{N}_{{}^{g}P, i_g\pi i_{t^{-1}}}$-module whose restriction to $kC_G(^{g}P)$ is also indecomposable.

(c) The group $\mathrm{Aut}(L,u)$ acts on $\mathcal{Z}$ by

$$\varphi \cdot (P,\pi,E) := (P,\pi\varphi^{-1}, {}^{\varphi}E)$$

for $\varphi \in \mathrm{Aut}(L,u)$ and $(P,\pi,E) \in \mathcal{Z}$, where ${}^{\varphi}E$ is the $k\overline{N}_{P,\pi\varphi^{-1}}$-module equal to E as the k-vector space on which $\overline{(a,b)} \in \overline{N}_{P,\pi\varphi^{-1}}$ acts via

$$\overline{(a,b)} \cdot {}^{\varphi}e := \overline{(a,\varphi^{-1}(b))}e\,.$$

To show that this action is well-defined, we first need to show that $\overline{(a,u)} \in \overline{N}_{P,\pi\varphi^{-1}}$ for some $a \in G$. Since φ maps u to a conjugate of u, the existence of a is similar to Part(b).

Also, since $\mathrm{Res}_{C_G(P)}^{\overline{N}_{P,\pi\varphi^{-1}}}({}^{\varphi}E) = \mathrm{Res}_{C_G(P)}^{\overline{N}_{P,\pi}}E$, it follows that the action is well-defined.

(d) The group $\widehat{\langle u \rangle} = \mathrm{Hom}(\langle u\rangle, k^{\times})$ acts $\mathcal{Z}$ on the right by

$$(P,\pi,E)\cdot\lambda := (P,\pi,E_\lambda)$$

where E_λ is the $k\overline{N}_{P,\pi}$-module equal to E as the k-vector space on which $\overline{(a,b)} \in \overline{N}_{P,\pi}$ acts by

$$\overline{(a,b)} \cdot e_\lambda := \tilde{\lambda}(b)\overline{(a,b)}e\,.$$

Here $\tilde{\lambda} : L\langle u\rangle \to \langle u\rangle \to k^{\times}$, $l \cdot u \mapsto \lambda(u)$ is the composition map.

Since E_λ is projective indecomposable $k\mathrm{Br}_P(b)\overline{N}_{P,\pi}$-module and since $\mathrm{Res}_{C_G(P)}^{\overline{N}_{P,\pi}}E_\lambda = \mathrm{Res}_{C_G(P)}^{\overline{N}_{P,\pi}}E$ is indecomposable, it follows that this action is well-defined.

(e) The group $\mathrm{Aut}(L,u)$ acts on $G \times L\langle u\rangle$ by $\varphi \cdot (g,t) := (g,\varphi(t))$ for $\varphi \in \mathrm{Aut}(L,u)$ and $(g,t) \in G \times L\langle u\rangle$. We set $S := (G \times L\langle u\rangle) \rtimes \mathrm{Aut}(L,u)$ using this action. Let $((g,t),\varphi) \in (G \times L\langle u\rangle) \rtimes \mathrm{Aut}(L,u)$ and $(P,\pi,E) \in \mathcal{Z}$. Then for any $l \in L$, we have

$$(i_g\pi i_{t^{-1}}\varphi^{-1})(l) = i_g\pi\left(t^{-1}\varphi^{-1}(l)t\right) = i_g\pi\left(\varphi^{-1}(\varphi(t^{-1}))\varphi^{-1}(l)\varphi^{-1}(\varphi(t))\right)$$
$$= i_g\pi\left(\varphi^{-1}(\varphi(t^{-1})l\varphi(t))\right) = \left(i_g\pi\varphi^{-1}i_{\varphi(t^{-1})}\right)(l)\,.$$

Moreover, for $\overline{(a,b)} \in \overline{N}^{\,g}_{P,i_g\pi i_{t^{-1}}\varphi^{-1}} = \overline{N}^{\,g}_{P,i_g\pi\varphi^{-1}i_{\varphi(t^{-1})}}$ and $e \in E$, we have

$$\overline{(a,b)} \cdot {}^{\varphi}\left({}^{(g,t)}e\right) = \overline{(a,\varphi^{-1}(b))} \cdot {}^{(g,t)}e = \overline{(a^g,\varphi^{-1}(b)^t)}e$$
$$= \overline{(a^g,\varphi^{-1}(b^{\varphi(t)}))}e = \overline{(a^g,b^{\varphi(t)})} \cdot {}^{\varphi}e$$
$$= \overline{(a,b)} \cdot {}^{(g,\varphi(t))}\left({}^{\varphi}e\right)\,.$$

These show that

$$\varphi \cdot ((g,t) \cdot (P,\pi,E)) = (g,\varphi(t)) \cdot (\varphi \cdot (P,\pi,E))\,.$$

Therefore the group S acts on $\mathcal{Z}$.

(f) Let $(g,t) \in G \times L\langle u\rangle$, $\varphi \in \mathrm{Aut}(L,u)$, $\lambda \in \widehat{\langle u\rangle}$ and $(P,\pi,E) \in \mathcal{Z}$.

For any $b = l_1 u^j \in L\langle u \rangle$, we have $b^t = l_2 u^j$ for some $l_2 \in L$, and hence $\tilde{\lambda}(b) = \tilde{\lambda}(b^t)$. Therefore for any $\overline{(a,b)} \in \overline{N}_{{}^g P, i_g \pi i_{t-1}}$ and $e \in E$, we have

$$\overline{(a,b)} \cdot ({}^{(g,t)}e)_\lambda = \tilde{\lambda}(b)\overline{(a,b)} \cdot {}^{(g,t)}e = \tilde{\lambda}(b)\overline{(a^g, b^t)}e = \tilde{\lambda}(b^t)\overline{(a^g, b^t)}e$$
$$= \overline{(a^g, b^t)} \cdot e_\lambda = \overline{(a,b)} \cdot {}^{(g,t)}(e_\lambda)\,.$$

This means that

$$((g,t) \cdot (P, \pi, E)) \cdot \lambda = (g,t) \cdot ((P, \pi, E) \cdot \lambda)\,,$$

i.e., the actions of $G \times L\langle u \rangle$ and $\widehat{\langle u \rangle}$ on $\mathcal{Z}$ commute.

Since φ maps u to a conjugate of u, one has that $\tilde{\lambda}(b) = \tilde{\lambda}(\varphi^{-1}(b))$ for any $b \in L\langle u \rangle$. Therefore calculations similar to above show that

$$(\varphi \cdot (P, \pi, E)) \cdot \lambda = \varphi \cdot ((P, \pi, E) \cdot \lambda)\,,$$

i.e., the actions of $\mathrm{Aut}(L, u)$ and $\widehat{\langle u \rangle}$ on $\mathcal{Z}$ commute. These imply that $\mathcal{Z}$ is an $(S, \widehat{\langle u \rangle})$-biset. Note that since $E_\lambda = E$ if and only if $\lambda = 1$, it follows that $\mathcal{Z}$ is free on the right.

(g) We have a map from $\mathcal{Z}$ to $b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$ sending (P, π, E) to $m_{P,\pi,E}$. This extends to a linear map

$$\Theta : \mathbb{F}\mathcal{Z} \to b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}\,.$$

8.14 Lemma *Let $(g,t) \in G \times L\langle u \rangle$, $\varphi \in \mathrm{Aut}(L, u)$, $\lambda \in \widehat{\langle u \rangle}$ and $z \in \mathcal{Z}$. Then*
(i) $\Theta((g,t)z) = \Theta(z)$.
(ii) $\Theta(z\lambda) = \lambda(u)\Theta(z)$.
(iii) $\Theta(\varphi z) = \Theta(z)\varphi^{-1}$.

Proof (i) Let $z = (P, \pi, E) \in \mathcal{Z}$. We have

$$\Theta\left((g,t)(P, \pi, E)\right) = \Theta(({}^g P, i_g \pi i_{t-1}, {}^{(g,t)}E)) = m_{{}^g P, i_g \pi i_{t-1}, {}^{(g,t)}E} = m_{P,\pi,E} \in \mathbb{F}T^\Delta(G, L\langle u \rangle)\,.$$

Hence (i) follows.

(ii) We have

$$\Theta((P, \pi, E)\lambda) = \Theta(P, \pi, E_\lambda) = m_{P,\pi,E_\lambda} = \lambda(u)m_{P,\pi,E} = \lambda(u)\Theta(P, \pi, E)\,.$$

(iii) Finally, we have

$$\Theta(\varphi(P, \pi, E)) = \Theta(P, \pi\varphi^{-1}, {}^\varphi E) = m_{P,\pi\varphi^{-1}, {}^\varphi E} = m_{P,\pi,E}\varphi = \Theta(P, \pi, E)\varphi\,.$$

$\square$

8.15 Let $\Omega = [\mathcal{Z}/\widehat{\langle u \rangle}]$ be a set of representatives of the right orbits of $\widehat{\langle u \rangle}$ on $\mathcal{Z}$. As $\mathcal{Z}$ is an $(S, \widehat{\langle u \rangle})$-biset, we can choose Ω to be left invariant by the action of S. Then, since $G \times L\langle u \rangle$ is a normal subgroup of S, the set $\Sigma = (G \times L\langle u \rangle)\backslash\Omega$ is a left $\mathrm{Aut}(L, u)$-set. Now by Lemma 8.14(i), the map Θ induces a map

$$\overline{\Theta} : \mathbb{F}\Sigma \to b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$$

sending the orbit $(G \times L\langle u \rangle)(P, \pi, E)$, for $(P, \pi, E) \in \Omega$, to $m_{P,\pi,E}$.

(a) Every element in $b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$ is a linear combination of the elements $m_{P,\pi,E}$. Moreover, if (P, π, E) and (P', π', E') are in the same $((G \times L\langle u \rangle) \times \widehat{\langle u \rangle})$-orbit, then the elements $m_{P,\pi,E}$ and $m_{P',\pi',E'}$ differ by a constant. Therefore, the map $\overline{\Theta}$ is surjective.

Now assume that $\alpha_1 m_{P_1,\pi_1,E_1} + \cdots + \alpha_l m_{P_l,\pi_l,E_l} = 0$ for pairwise distinct elements (P_i, π_i, E_i) of Σ and $\alpha_i \in \mathbb{F}$. Fix $i \in \{1, \cdots, l\}$. Then the (P_i, π_i)-component of the image of the sum $\alpha_1 m_{P_1,\pi_1,E_1} + \cdots + \alpha_l m_{P_l,\pi_l,E_l}$ under the isomorphism in Theorem 8.12 is also zero. This component is equal to

$$\sum_j \alpha_j S_{E_j} \tag{13}$$

up to a non-zero scalar, where the sum runs over $j \in \{1, \cdots, l\}$ with the property that $(P_j, \pi_j) = (P_i, \pi_i)$. Now if α_i is non-zero, then since the sum in (13) is zero there exists an index $j \neq i$ such that $E_i = (E_j)_\lambda$ for some $\lambda \in \widehat{\langle u \rangle}$. But this implies that (P_i, π_i, E_i) and (P_j, π_j, E_j) are in the same $(G \times L\langle u \rangle) \times \widehat{\langle u \rangle}$-orbit. This is a contradiction. Therefore we conclude that the map $\overline{\Theta}$ is injective and hence an isomorphism.

(b) By Lemma 8.14(iii), we have $\overline{\Theta}(\varphi f) = \overline{\Theta}(f)\varphi^{-1}$ for any $\varphi \in \mathrm{Aut}(L, u)$ and $f \in \mathbb{F}\Sigma$. Together with Part(a), this implies that the map $\overline{\Theta} : \mathbb{F}\Sigma \to b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}}$ is an isomorphism of right $\mathbb{F}\mathrm{Aut}(L, u)$-modules.

8.16 Let $\mathcal{Y} = \mathcal{Y}(G, L, u)$ be the set of triples (P, π, F) where

- P is a p-subgroup of G.

- $\pi : L \to P$ is a group isomorphism such that there exists a p'-element $s \in G$ with $\pi({}^u l) = {}^s\pi(l)$ for all $l \in L$.

- F is an u-invariant projective indecomposable $k\mathrm{Br}_P(b)C_G(P)$-module.

(a) The group $G \times L\langle u \rangle$ acts on $\mathcal{Y}$ by

$$(g, t) \cdot (P, \pi, F) := ({}^g P, i_g \pi i_{t^{-1}}, {}^g F)$$

for $(g, t) \in G \times L\langle u \rangle$ and $(P, \pi, F) \in \mathcal{Y}$. Here ${}^g F$ is the $kC_G({}^g P)$-module equal to F as a k-vector space and on which $c \in C_G({}^g P)$ acts by

$$c \cdot {}^g f := c^g f.$$

To show that this action is well-defined, we only need to show that ${}^g F$ is u-invariant projective indecomposable $k\mathrm{Br}_{{}^g P}(b)C_G({}^g P)$-module. Since F is projective indecomposable, it follows that ${}^g F$ is also projective indecomposable. Moreover, by Theorem 4.1 the module F extends to a projective

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indecomposable $k\overline{N}_{P,\pi}$-module E. By 8.13(b) the module $^{(g,t)}E$ is a projective indecomposable $k\overline{N}_{^gP, i_g\pi i_{t^{-1}}}$-module whose restriction to $C_G(^gP)$ is indecomposable. But the restriction of $^{(g,t)}E$ to $C_G(^gP)$ is ^gF and hence it is u-invariant.

(b) The group $\text{Aut}(L,u)$ acts on $\mathcal{Y}$ by

$$\varphi \cdot (P,\pi,F) := (P, \pi\varphi^{-1}, F)$$

for $\varphi \in \text{Aut}(L,u)$ and $(P,\pi,F) \in \mathcal{Y}$. The well-definedness of this action is proved similar to Part(a), using 8.13(c).

(c) Let $((g,t),\varphi) \in S = (G \times L\langle u\rangle) \rtimes \text{Aut}(L,u)$ and $(P,\pi,E) \in \mathcal{Z}$. Then, as before, for any $l \in L$, we have

$$(i_g\pi i_{t^{-1}}\varphi^{-1})(l) = \left(i_g\pi\varphi^{-1}i_{\varphi(t^{-1})}\right)(l)\,.$$

Moreover, for $c \in C_G(^gP)$ and $f \in F$, we have

$$c \cdot {}^{\varphi}\left({}^{(g,t)}f\right) = c^g f = c \cdot {}^{(g,\varphi(t))}\left({}^{\varphi}f\right)\,.$$

These show that

$$\varphi \cdot ((g,t) \cdot (P,\pi,F)) = (g,\varphi(t)) \cdot (\varphi \cdot (P,\pi,F))\,.$$

Therefore the group S acts on $\mathcal{Y}$.

8.17 (a) We view $\mathcal{Y}$ as an $(S, \widehat{\langle u\rangle})$-biset with trivial right action. Consider the map

$$\Psi : \mathcal{Z} \to \mathcal{Y}$$

$$(P,\pi,E) \mapsto (P,\pi, \text{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}E)\,.$$

Note that the map is well-defined. Now let $((g,t),\varphi) \in S$, $\lambda \in \widehat{\langle u\rangle}$ and $(P,\pi,E) \in \mathcal{Z}$. Then we have

$$\Psi\left(((g,t),\varphi)\cdot(P,\pi,E)\right) = \Psi\left({}^gP, i_g\pi\varphi^{-1}i_{t^{-1}}, {}^{(g,t)}({}^{\varphi}E)\right)$$

$$= \left({}^gP, i_g\pi\varphi^{-1}i_{t^{-1}}, \text{Res}^{\overline{N}_{^gP,i_g\pi\varphi^{-1}i_{t^{-1}}}}_{C_G(^gP)}\left({}^{(g,t)}({}^{\varphi}E)\right)\right)$$

$$= \left({}^gP, i_g\pi\varphi^{-1}i_{t^{-1}}, {}^{(g,t)}\left(\text{Res}^{\overline{N}_{P,\pi\varphi^{-1}}}_{C_G(P)}({}^{\varphi}E)\right)\right)$$

$$= ((g,t),\varphi)\cdot\left(P,\pi,\text{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}E\right)$$

$$= ((g,t),\varphi)\cdot\Psi(P,\pi,E)\,.$$

Hence Ψ is a map of S-sets. Moreover,

$$\Psi\left((P,\pi,E)\cdot\lambda\right) = \Psi(P,\pi,E_\lambda) = (P,\pi,\text{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}E_\lambda) = (P,\pi,\text{Res}^{\overline{N}_{P,\pi}}_{C_G(P)}) = \Psi(P,\pi,E)$$

$$= \Psi(P,\pi,E)\cdot\lambda\,.$$

Hence Ψ is also a map of right $\widehat{\langle u \rangle}$-sets and therefore a map of $(S, \widehat{\langle u \rangle})$-bisets.

(b) The map Ψ induces a map $\overline{\Psi} : \Omega = [\mathcal{Z}/\widehat{\langle u \rangle}] \to \mathcal{Y}$ of S-sets. The map $\overline{\Psi}$ is an isomorphism by Theorem 4.1. Indeed, given $(P, \pi) \in \mathcal{P}(G, L, u)$, any $\langle u \rangle$-invariant projective indecomposable $kC_G(P)$-module F extends to a projective indecomposable $k\overline{N}_{P,\pi}$-module E and any such extension differ by an element λ of $\widehat{\langle u \rangle}$.

Now $\overline{\Psi}$ induces an isomorphism of left $\mathrm{Aut}(L, u)$-sets

$$\psi : (G \times L\langle u \rangle) \setminus \Omega \to \Xi := (G \times L\langle u \rangle) \setminus \mathcal{Y}$$

and hence 8.15(b) implies that

$$b\mathbb{F}T^{\Delta}(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \mathbb{F}\Xi \tag{14}$$

as right $\mathbb{F}\mathrm{Aut}(L, u)$-modules where now $\mathrm{Aut}(L, u)$ acts on the right on Ξ.

(c) Let $\mathcal{U}$ be a set of representatives of $\mathrm{Aut}(L, u)$-orbits on Ξ. For $(P, \pi, F) \in \mathcal{Y}$, we write $\overline{(P, \pi, F)} := (G \times L\langle u \rangle)(P, \pi, F)$. Also for $\overline{(P, \pi, F)} \in \Xi$, we denote by $\mathrm{Aut}(L, u)_{\overline{(P,\pi,F)}}$ the stabilizer of $\overline{(P, \pi, F)}$ in $\mathrm{Aut}(L, u)$. The isomorphism in (14) can be written as

$$b\mathbb{F}T^{\Delta}(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{\overline{(P,\pi,F)} \in \mathcal{U}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi,F)}}}^{\mathrm{Aut}(L,u)} \mathbb{F}. \tag{15}$$

Note that $\overline{(P, \pi, F)}$ and $\overline{(P', \pi', F')}$ lie in the same $\mathrm{Aut}(L, u)$-orbit if and only if there exist $\varphi \in \mathrm{Aut}(L, u)$ and $(g, t) \in G \times L\langle u \rangle$ such that

$$(P', \pi'\varphi, F') = ({}^gP, i_g\pi i_{t^{-1}}, {}^gF).$$

Furthermore, the element $\varphi \in \mathrm{Aut}(L, u)$ belongs to $\mathrm{Aut}(L, u)_{\overline{(P,\pi,F)}}$ if and only if there exists $(g, t) \in G \times L\langle u \rangle$ such that

$$(P, \pi\varphi, F) = ({}^gP, i_g\pi i_{t^{-1}}, {}^gF),$$

i.e., $g \in N_G(P)$, $\varphi\pi = i_g\pi i_{t^{-1}}$ and ${}^gF = F$.

8.18 (a) Let $\mathcal{V}$ be a set of representatives of $\mathrm{Aut}(L, u)$-orbits on $(G \times L\langle u \rangle) \setminus \mathcal{P}(G, L, u)$. For $(P, \pi) \in \mathcal{P}(G, L, u)$, let $\overline{(P, \pi)}$ denote the orbit $(G \times L\langle u \rangle)(P, \pi)$ and let $\mathrm{Aut}(L, u)_{\overline{(P,\pi)}}$ denote its stabilizer in $\mathrm{Aut}(L, u)$. Then for $(P, \pi, F) \in \mathcal{Y}$, the stabilizer $\mathrm{Aut}(L, u)_{\overline{(P,\pi,F)}}$ is a subgroup $\mathrm{Aut}(L, u)_{\overline{(P,\pi)}}$, and we can rewrite (15) as

$$b\mathbb{F}T^{\Delta}(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{\overline{(P,\pi)} \in \mathcal{V}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}}^{\mathrm{Aut}(L,u)} \left(\bigoplus_{F \in \mathcal{W}_{\overline{(P,\pi)}}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi,F)}}}^{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}} \mathbb{F} \right), \tag{16}$$

where $\mathcal{W}_{\overline{(P,\pi)}}$ is a set of representatives of $\mathrm{Aut}(L, u)_{\overline{(P,\pi)}}$-orbits on the set of u-invariant projective indecomposable $k\mathrm{Br}_P(b)C_G(P)$-modules. Note that we have

$$\bigoplus_{F \in \mathcal{W}_{\overline{(P,\pi)}}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi,F)}}}^{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}} \mathbb{F} \cong \mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u)$$

as right $\mathbb{F}\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$-modules where the action of $\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$ is given as follows: Let $F \in \mathcal{W}_{\overline{(P,\pi)}}$ and let $\varphi \in \mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$. Then there exists $(x,t) \in G \times L\langle u\rangle$ such that

$$(\,^x P, i_x \pi i_{t^{-1}}) = (P, \pi\varphi)\,.$$

This is equivalent to the existence of an element $g \in G$ such that

$$(\,^g P, i_g \pi) = (P, \pi\varphi)\,.$$

Indeed, since $\pi \in \mathcal{P}(G,L,u)$, there exists $s \in G$ such that $\pi(\,^u l) = \,^s \pi(l)$ for all $l \in L$. Now if $t = l_0 \cdot u^i$, then $g := xs^{-i}\pi(l_0^{-1})$ satisfies the desired equation. Moreover, one can show that the element g is well-defined up to multiplication by an element of $C_G(P)$. Now since we have

$$(P, \pi, F) \cdot \varphi = (P, \pi\varphi, F) = (\,^g P, i_g \pi, F) = g \cdot (P, \pi,\, ^{g^{-1}} F)\,,$$

the element φ maps F to $\,^{g^{-1}} F$.

(b) All these imply that we have an isomorphism

$$b\mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}} \cong \bigoplus_{\overline{(P,\pi)}\in\mathcal{V}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u)\,, \qquad (17)$$

of right $\mathbb{F}\mathrm{Aut}(L,u)$-modules. Arguments in Part(a) implies that the orbits of G and of $G \times L\langle u\rangle$ on the set $\mathcal{P}(G,L,u)$ are the same. Hence we have a bijection

$$\mathcal{D} := [G \backslash \mathcal{P}(G,L,u)/\mathrm{Aut}(L,u)] \to \mathcal{V} = [(G \times L\langle u\rangle) \backslash \mathcal{P}(G,L,u)/\mathrm{Aut}(L,u)]$$

and the isomorphism in (17) can be written as

$$b\mathbb{F}T^{\Delta}(G, L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}} \cong \bigoplus_{\overline{(P,\pi)}\in\mathcal{D}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u)\,. \qquad (18)$$

8.19 Let $\hat{\mathcal{F}}_b$ denote the category whose objects are b-Brauer pairs (P,e) and whose morphisms from (P,e) to (Q,f) are group homomorphisms $\psi: P \to Q$ for which there exists $g \in G$ such that $\psi(x) = \,^g x$ for any $x \in P$ and such that $\,^g(P,e) \leqslant (Q,f)$ (see, for instance, [L18, Section 6.3] for more details on Brauer pairs).

Let $\hat{\mathcal{P}}_b(G,L,u)$ be the set of triples (P,e,π) where

- $(P,e) \in \hat{\mathcal{F}}_b$

- $\pi: L \to P$ is a group isomorphism such that $\pi i_u \pi^{-1} \in \mathrm{Aut}_{\hat{\mathcal{F}}_b}(P,e)$.

(a) The set $\hat{\mathcal{P}}_b(G,L,u)$ is a $(G, \mathrm{Aut}(L,u))$-biset via

$$g \cdot (P,e,\pi) \cdot \varphi = (\,^g P, \,^g e, i_g \pi\varphi)$$

for $g \in G$, $(P,e,\pi) \in \hat{\mathcal{P}}_b(G,L,u)$ and $\varphi \in \mathrm{Aut}(L,u)$. Indeed, we have $(\,^g P, \,^g e) \in \hat{\mathcal{P}}_b(G,L,u)$ and one shows that

$$i_g \pi \varphi i_u \varphi^{-1} \pi^{-1} i_{g^{-1}} = i_{s'}$$

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where $s' = g\pi(l_0)s \in N_G(\,^gP,\,^ge)$, $l_0 \in L$ with $\pi(u) = l_0 \cdot u$ and $s \in N_G(P,e)$ such that $\pi i_u \pi^{-1} = i_s$.

(b) Let $\overline{(P,\pi)} \in \mathcal{D}$. We have

$$\mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u) \cong \bigoplus_{\substack{e \in \mathrm{Bl}(C_G(P)) \\ \mathrm{Br}_P(b)e = e}} \mathbb{F}\mathrm{Proj}(keC_G(P), u)$$

as right $\mathbb{F}\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$-modules. Note that the group $\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$ permutes the summands on the right-hand side as follows: Let $\varphi \in \mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$ and let $e \in \mathrm{Bl}(C_G(P))$ with $\mathrm{Br}_P(b)e = e$. Then there exists $g \in N_G(P)$ such that $i_g\pi = \pi\varphi$ and by 8.18(a), φ sends an u-invariant projective indecomposable $keC_G(P)$-module S to $\,^{g^{-1}}S$. Hence φ maps the b-Brauer pair (P,e) to $(P,\,^{g^{-1}}e)$.

Let $\mathrm{Bl}(C_G(P), u)$ denote the set of block idempotents e of $C_G(P)$ for which there exists an indecomposable projective u-invariant $keC_G(P)$-module. Let $[\mathrm{Bl}(C_G(P), u)]$ denote a set of representatives of $\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$-orbits of $\mathrm{Bl}(C_G(P), u)$. Then as right $\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$-modules, we have

$$\mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u) \cong \bigoplus_{e \in [\mathrm{Bl}(C_G(P), u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,e,\pi)}}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(keC_G(P), u)$$

where $\mathrm{Aut}(L,u)_{\overline{(P,e,\pi)}}$ is the stabilizer in $\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$ of e.

Note that $\varphi \in \mathrm{Aut}(L,u)_{\overline{(P,\pi)}}$ fixes e if and only if there exists $g \in N_G(P)$ with $i_g\pi = \pi\varphi$ and $\,^ge = e$. Hence $\mathrm{Aut}(L,u)_{\overline{(P,e,\pi)}}$ is equal to the stabilizer $\mathrm{Aut}(L,u)_{P,e,\pi}$ of the G-orbit of (P,e) in $\mathrm{Aut}(L,u)$. These imply that we have

$$b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{(P,e,\pi) \in [G \backslash \hat{\mathcal{P}}_b(G,L,u)/\mathrm{Aut}(L,u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{P,e,\pi}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(keC_G(P), u) \qquad (19)$$

as right $\mathbb{F}\mathrm{Aut}(L,u)$-modules.

8.20 Let $\mathcal{F}_b$ denote the fusion system of kGb with respect to a maximal b-Brauer pair (D, e_D). For each subgroup $P \leqslant D$ let e_P denote the unique block of $kC_G(P)$ with $(P, e_P) \leqslant (D, e_D)$. Note that every G-orbit in $\hat{\mathcal{F}}_b$ contains an element in $\mathcal{F}_b$.

For $(P, e_P) \in \mathcal{F}_b$, let $\mathcal{P}_{(P, e_P)}(L, u)$ denote the set of group isomorphisms $\pi : L \to P$ with $\pi i_u \pi^{-1} \in \mathrm{Aut}_{\mathcal{F}_b}(P, e_P)$. The set $\mathcal{P}_{(P, e_P)}(L, u)$ is an $(N_G(P, e_P), \mathrm{Aut}(L, u))$-biset via

$$g \cdot \pi \cdot \varphi = i_g \pi \varphi$$

for $g \in N_G(P, e_P)$, $\pi \in \mathcal{P}_{(P, e_P)}(L, u)$ and $\varphi \in \mathrm{Aut}(L, u)$. Let $[\mathcal{P}_{(P, e_P)}(L, u)]$ denote a set of representatives of $N_G(P, e_P) \times \mathrm{Aut}(L, u)$-orbits of $\mathcal{P}_{(P, e_P)}(L, u)$. Then the isomorphism in (19) can be written as

$$b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{(P,e_P) \in [\mathcal{F}_b]} \bigoplus_{\pi \in [\mathcal{P}_{(P,e_P)}(L,u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{P,e,\pi}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(keC_G(P), u). \qquad (20)$$

8.21 Let $\mathcal{L}_b(G, L, u)$ denote the set of pairs (P_γ, π) where

- P_γ is a local pointed point group on kGb,

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- $\pi : L \to P$ is a group isomorphism such that $\pi i_u \pi^{-1} = \mathrm{Res}(i_s)$ for some $s \in N_G(P_\gamma)$.

The set $\mathcal{L}_b(G, L, u)$ is a $(G, \mathrm{Aut}(L, u))$-biset via

$$g \cdot (P_\gamma, \pi) \cdot \varphi = ({}^g P_{g\gamma}, i_g \pi \varphi)$$

for $g \in G$ and $\varphi \in \mathrm{Aut}(L, u)$. For $(P_\gamma, \pi) \in \mathcal{L}_b(G, L, u)$, we write $\mathrm{Aut}(L, u)_{(P_\gamma, \pi)}$ for the stabilizer of the G-orbit of (P_γ, π) in $\mathrm{Aut}(L, u)$.

Let $\overline{(P, \pi)} \in \mathcal{D}$. A projective indecomposable $k\mathrm{Br}_P(b)C_G(P)$-module S determines a conjugacy class of a primitive idempotent of $k\mathrm{Br}_P(b)C_G(P)$ and hence via the Brauer morphism

$$\mathrm{Br}_P : (kGb)^P \to kC_G(P)$$

a local point γ of P on kGb. This in fact induces a bijection between the sets of local points of P on kGb and projective indecomposable $k\mathrm{Br}_P(b)C_G(P)$-modules. See for instance [T95, Corollary 37.6] for more details. This bijection induces an isomorphism of right $\mathrm{Aut}(L, u)_{\overline{(P, \pi)}}$-modules

$$\mathbb{F}\mathrm{Proj}(k\mathrm{Br}_P(b)C_G(P), u) \cong \mathbb{F}\mathcal{L}_{(P, \pi)}$$

where $\mathcal{L}_{(P, \pi)}$ is the set of local points P_γ with $\pi i_u \pi^{-1} = i_s$ for some $s \in N_G(P_\gamma)$. The action of $\varphi \in \mathrm{Aut}(L, u)_{\overline{(P, \pi)}}$ on $\mathbb{F}\mathcal{L}_{(P, \pi)}$ is given as follows: There exists $g \in G$ such that $({}^g P, i_g \pi) = (P, \pi \varphi)$. Then φ maps $P_\gamma \in \mathcal{L}_{(P, \pi)}$ to $P_{g^{-1}\gamma}$. Hence the isomorphism in (18) implies that

$$b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{\overline{(P, \pi)} \in \mathcal{D}} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}}^{\mathrm{Aut}(L,u)} \left(\bigoplus_{P_\gamma \in [\mathcal{L}_{(P,\pi)}/\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}_\gamma}}^{\mathrm{Aut}(L,u)_{\overline{(P,\pi)}}} \mathbb{F} \right)$$

as right $\mathbb{F}\mathrm{Aut}(L, u)$-modules where $\mathrm{Aut}(L, u)_{\overline{(P, \pi)}_\gamma}$ is the stabilizer of P_γ in $\mathrm{Aut}(L, u)_{\overline{(P, \pi)}}$. But one shows that $\mathrm{Aut}(L, u)_{\overline{(P, \pi)}_\gamma}$ is equal to $\mathrm{Aut}(L, u)_{(P_\gamma, \pi)}$ and it follows that

$$b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} \cong \bigoplus_{(P_\gamma, \pi) \in [G \backslash \mathcal{L}_b(G, L, u)/\mathrm{Aut}(L, u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{(P_\gamma,\pi)}}^{\mathrm{Aut}(L,u)} \mathbb{F} \tag{21}$$

as right $\mathbb{F}\mathrm{Aut}(L, u)$-modules.

8.22 Theorem *Let $S_{L,u,V}$ be a simple diagonal p-permutation functor and let b be a block idempotent of kG. The multiplicity of $S_{L,u,V}$ in $\mathbb{F}T_{G,b}^\Delta$ is equal to the $\mathbb{F}$-dimensions of any of the following vector spaces.*

(a) $\bigoplus_{\overline{(P, \pi, F)} \in \mathcal{U}} V^{\mathrm{Aut}(L,u)_{\overline{(P,\pi,F)}}}$

(b) $\bigoplus_{(P, e_P) \in [\mathcal{F}_b]} \bigoplus_{\pi \in [\mathcal{P}_{(P,e_P)}(L, u)]} \mathbb{F}\mathrm{Proj}(keC_G(P), u) \otimes_{\mathrm{Aut}(L,u)_{P,e,\pi}} V$

(c) $\bigoplus_{(P_\gamma, \pi) \in [G \backslash \mathcal{L}_b(G, L, u)/\mathrm{Aut}(L, u)]} V^{\mathrm{Aut}(L,u)_{(P_\gamma,\pi)}}$

Proof Recall from 8.2 that the multiplicity of $S_{L,u,V}$ in $\mathbb{F}T_{G,b}^\Delta$ is equal to the $\mathbb{F}$-dimension of $b\mathbb{F}T^\Delta(G, L\langle u \rangle)\widetilde{F_{L,u}^{L\langle u \rangle}} e_V$ where e_V is an idempotent of $\mathbb{F}\mathrm{Out}(L, u)$ such that $V \cong \mathbb{F}\mathrm{Out}(L, u)e_V$. Hence part (a) follows from the isomorphism in (15), part (b) follows from the isomorphism in (20), and part (c) follows from the isomorphism in (21). $\square$

8.23 Corollary (i) *The multiplicity of the simple functor $S_{1,1,\mathbb{F}}$ in the functor $\mathbb{F}T_{G,b}^{\Delta}$ is equal to the number of isomorphism classes of simple kGb-modules.*

(ii) *The multiplicity of the simple functor $S_{L,u,V}$ at $\mathbb{F}T_{G,b}^{\Delta}$ is zero unless L is isomorphic to a subgroup of a defect group of b.*

Proof Both statements follow immediately from Theorem 8.22. □

9   Nilpotent blocks

Nilpotent blocks were introduced by Broué and Puig in [BrP80]. In this section we give a characterization of nilpotent blocks in terms of diagonal p-permutation functors. Let G be a finite group.

9.1 Definition ([BrP80]) A block idempotent b of kG is called *nilpotent* if for any b-Brauer pair (P,e) the quotient $N_G(P,e)/C_G(P)$ is a p-group.

9.2 Theorem *Let b be a block idempotent of kG with a defect group D. The following are equivalent.*

(i) *The block b is nilpotent.*

(ii) *If $S_{L,u,V}$ is a simple summand of $\mathbb{F}T_{G,b}^{\Delta}$, then $u = 1$.*

(iii) *The functor $\mathbb{F}T_{G,b}^{\Delta}$ is isomorphic to the functor $\mathbb{F}T_D^{\Delta}$.*

Proof (i) $\Rightarrow$ (ii): Suppose that b is nilpotent and that $S_{L,u,V}$ is a simple summand of $\mathbb{F}T_{G,b}^{\Delta}$. Then by Theorem 8.22(b), there exists a triple $(P,e,\pi) \in \hat{\mathcal{P}}_b(G,L,u)$ such that

$$\mathbb{F}\mathrm{Proj}(keC_G(P),u) \neq 0.$$

Let $s \in N_G(P,e)$ be an element with the property $\pi i_u \pi^{-1} = i_s : P \to P$. By 8.4(d), there exists a p'-element $s' \in N_G(P,e)$ with $\pi i_u \pi^{-1} = i_{s'}$. Since the block idempotent b is nilpotent, the quotient $N_G(P,e)/C_G(P)$ is a p-group. It follows that $s' \in C_G(P)$ and hence

$$\pi(\,^u l) = \,^{s'}\pi(l) = \pi(l)$$

for any $l \in L$. Since (L,u) is a D^{Δ}-pair, this means that $u = 1$. Hence (i) implies (ii).

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Then for any D^{Δ}-pair (L,u) with $u \neq 1$, by Theorem 8.22(a), the set $\mathcal{Y}(G,L,u)$ is empty. This is equivalent to the following statement:

$(B)_{G,b}$: If P is a p-subgroup of G and if $s \in N_G(P)$ induces a non-trivial p'-automorphism of P, then there is no s-invariant simple $k\mathrm{Br}_P(b)C_G(P)$-module.

Indeed, if $s \in N_G(P)$ induces a nontrivial p'-automorphism u of P, setting $L = P$, $\pi = \mathrm{id}$, then (L,u) is a D^{Δ}-pair and $(P,\pi) \in \mathcal{P}(G,L,u)$. Hence if S is an s-invariant simple $k\mathrm{Br}_P(b)C_G(P)$-module, then $(P,\pi,\hat{S}) \in \mathcal{Y}(G,L,u)$ where $\hat{S}$ is a projective cover of S. This contradicts our assumption.

Now we claim that the statement $(B)_{G,b}$ is equivalent to the following statement:

$(C)_{G,b}$: If (P,e) is a b-Brauer pair and if $s \in N_G(P,e)$ induces a nontrivial p'-automorphism of P, then there is no s-invariant simple $kC_G(P)e$-module.

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Indeed, $(B)_{G,b} \Rightarrow (C)_{G,b}$ is clear. Now assume that $(C)_{G,b}$ holds and suppose that $s \in N_G(P)$ induces a non-trivial p'-automorphism of P and S is an s-invariant simple $k\mathrm{Br}_P(b)C_G(P)$-module. Then S belongs to a unique block e of $kC_G(P)$. Since S is s-invariant, it follows that e is also s-invariant. This is a contradiction.

We will prove that the statement $(C)_{G,b}$ implies that the block b is nilpotent. We use induction on the order of G.

If G is the trivial group, then the block $kG = k$ is obviously nilpotent. Now assume that the statement $(C)_{H,c}$ implies that the block c is nilpotent whenever c is a block idempotent of a group H with $|H| < |G|$, and assume that the statement $(C)_{G,b}$ holds.

Let (P, e) be a b-Brauer pair. We will show that $N_G(P, e)/C_G(P)$ is a p-group. Set $H = C_G(P)$. If $H = G$, then the quotient $N_G(P, e)/C_G(P)$ is trivial hence a p-group.

So we can assume that $|H| < |G|$. Let (Q, f) be an e-Brauer pair of H. Then one can show that the pair (QP, f) is a b-Brauer pair of G. Let $s \in N_H(Q, f)$ be an element which induces a nontrivial p'-automorphism u of Q. Then $s \in C_G(P) \cap N_G(Q, f) \subseteq N_G(QP, f)$ induces a nontrivial p'-automorphism of of QP. Since $(C)_{G,b}$ holds and since (QP, f) is a b-Brauer pair, it follows that there is no s-invariant simple $kC_G(QP)f$-module. Therefore there is no s-invariant simple $kC_H(Q)f$-module. This proves that the statement $(C)_{H,e}$ holds. Since $|H| < |G|$, by induction hypothesis, the block e of $kH = kC_G(P)$ is nilpotent. So there is a unique simple module S of $kC_G(P)e$. Now if $t \in N_G(P, e)$ induces a nontrivial p'-automorphism v of P, then S is invariant by t and hence $v = 1$ since $(C)_{G,b}$ holds. In other words $t \in C_G(P)$ and so the quotient $N_G(P, e)/C_G(P)$ is a p-group. This shows that (ii) implies (i).

(iii) $\Rightarrow$ (ii): This is clear.

(i) $\Rightarrow$ (iii): Assume that b is nilpotent. By the first part of the proof, if $S_{L,u,V}$ is a simple summand of $\mathbb{F}T_{G,b}^\Delta$, then $u = 1$. So let $S_{L,1,V}$ be a simple functor. We will show that

$$b\mathbb{F}T^\Delta(G, L)\widetilde{F_{L,1}^L} \cong \mathbb{F}T^\Delta(D, L)\widetilde{F_{L,1}^L}$$

as right $\mathbb{F}\mathrm{Aut}(L)$-modules using the isomorphism in (20). Since b is nilpotent, for any b-Brauer pair (P, e_P), the block idempotent $e_P \in kC_G(P)$ is nilpotent. So there is a unique simple $ke_PC_G(P)$-module, and hence $\mathbb{F}\mathrm{Proj}(ke_PC_G(P), 1) \cong \mathbb{F}$. Moreover, $\mathcal{F}_b = \mathcal{F}_D$, $\mathcal{P}(P, e_P)(L, 1) = \mathrm{Isom}(L, P)$ and $N_G(P, e_P) = N_D(P)$. The result follows. $\square$

10 Functorial equivalence of blocks

In this section G and H denote finite groups. We come back to the case of an arbitrary commutative ring R of coefficients.

10.1 Definition Let b be a block idempotent of kG and let c be a block idempotent of kH. We say that the pairs (G, b) and (H, c) are *functorially equivalent* over R, if the corresponding diagonal p-permutation functors $RT_{G,b}^\Delta$ and $RT_{H,c}^\Delta$ are isomorphic in $\mathcal{F}_{Rpp_k}^\Delta$.

10.2 Lemma Let (G, b) and (H, c) be as in Definition 10.1.

(i) (G, b) and (H, c) are functorially equivalent over R if and only if there exists $\omega \in bRT^\Delta(G, H)c$ and $\sigma \in cRT^\Delta(H, G)b$ such that

$$\omega \cdot_G \sigma = [kGb] \quad \text{in} \quad bRT^\Delta(G, G)b \quad \text{and} \quad \sigma \cdot_H \omega = [kHc] \quad \text{in} \quad cRT^\Delta(H, H)c.$$

(ii) *If kGb and kHc are p-permutation equivalent, then (G, b) and (H, c) are functorially equivalent over R.*

Proof The first statement follows from the Yoneda lemma. The second statement follows from the first one, and from the definition of p-permutation equivalence in [BP20]. $\square$

10.3 Remark It follows that functorial equivalence over $\mathbb{Z}$ is almost the same notion as p-permutation equivalence, which only requires in addition that σ be the opposite of ω in (i).

10.4 Proposition *Let b be a block idempotent of kG and c a block idempotent of kH.*

(i) *If (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$, then we have $l(kGb) = l(kHc)$.*

(ii) *If (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$, then b and c have isomorphic defect groups.*

(iii) *If b has defect zero, then the functor $\mathbb{F}T_{G,b}^{\Delta}$ is isomorphic to the simple functor $S_{1,1,\mathbb{F}}$. In particular, all pairs (G, b), where b is a block of defect zero of kG, are functorially equivalent over $\mathbb{F}$.*

(iv) *More generally, for any p-group D, all pairs (G, b), where b is a nilpotent block of kG with defect isomorphic to D are functorially equivalent over $\mathbb{F}$.*

Proof The first statement follows from Corollary 8.23(i). For the second statement, let D be a defect group of b. The multiplicity of the simple functor $S_{D,1,\mathbb{F}}$ at $\mathbb{F}T_{G,b}^{\Delta}$ is non-zero by Theorem 8.22. Hence it is also nonzero at $\mathbb{F}T_{H,c}^{\Delta}$. By Corollary 8.23(ii), it follows that D is isomorphic to a subgroup of a defect group of the block c. Similarly, one can show that a defect group of c is isomorphic to a subgroup of a defect group of b whence (ii) holds. For the third statement assume that b has defect zero. Then by Corollary 8.23, the functor $\mathbb{F}T_{G,b}^{\Delta}$ is isomorphic to $l(kGb)S_{1,1,\mathbb{F}} = S_{1,1,\mathbb{F}}$. The last statement follows from Theorem 9.2. $\square$

10.5 Theorem *Let b be a block idempotent of kG and c a block idempotent of kH. The following are equivalent:*

(i) *(G, b) and (H, c) are functorially equivalent over $\mathbb{F}$.*

(ii) *For any D^{Δ}-pair (L, u) and any $\varphi \in \mathrm{Aut}(L, u)$, one has*

$$|(G \setminus \mathcal{L}_b(G, L, u))^{\varphi}| = |(H \setminus \mathcal{L}_c(H, L, u))^{\varphi}|.$$

Proof (i) $\Rightarrow$ (ii): Suppose that (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$, and let (L, u) be a D^{Δ}-pair. For any simple $\mathbb{F}\mathrm{Out}(L, u)$-module V, we have, by Theorem 8.22(iii),

$$\sum_{(P_{\gamma}, \pi) \in [G \setminus \mathcal{L}_b(G,L,u)/\mathrm{Aut}(L,u)]} \dim_{\mathbb{F}}(V^{\mathrm{Aut}(L,u)_{(P_{\gamma}, \pi)}}) = \sum_{(Q_{\delta}, \rho) \in [H \setminus \mathcal{L}_c(H,L,u)/\mathrm{Aut}(L,u)]} \dim_{\mathbb{F}}(V^{\mathrm{Aut}(L,u)_{(Q_{\delta}, \rho)}}).$$

Note that since $\mathrm{Inn}(L, u)$ acts trivially on the G-orbits of $\mathcal{L}_b(G, L, u)$, this means that we have

$$\sum_{(P_{\gamma}, \pi) \in [G \setminus \mathcal{L}_b(G,L,u)/\mathrm{Aut}(L,u)]} \dim_{\mathbb{F}}(V^{\mathrm{Out}(L,u)_{(P_{\gamma}, \pi)}}) = \sum_{(Q_{\delta}, \rho) \in [H \setminus \mathcal{L}_c(H,L,u)/\mathrm{Aut}(L,u)]} \dim_{\mathbb{F}}(V^{\mathrm{Out}(L,u)_{(Q_{\delta}, \rho)}})$$

which implies that

$$\sum_{(P_\gamma,\pi)\in[G\backslash\mathcal{L}_b(G,L,u)/\mathrm{Aut}(L,u)]}\langle\chi_V,1\rangle_{\mathrm{Out}(L,u)_{(P_\gamma,\pi)}}=\sum_{(Q_\delta,\rho)\in[H\backslash\mathcal{L}_c(H,L,u)/\mathrm{Aut}(L,u)]}\langle\chi_V,1\rangle_{\mathrm{Out}(L,u)_{(Q_\delta,\rho)}}.$$

Since $\mathbb{F}\mathrm{Out}(L,u)$ is semisimple, it follows that this equality holds for any class function on $\mathrm{Out}(L,u)$. So let $\mathcal{C}$ be the conjugacy class of $\overline{\varphi}$ in $\mathrm{Out}(L,u)$ and $\chi_\mathcal{C}$ denote the characteristic function of $\mathcal{C}$. We have

$$\sum_{(P_\gamma,\pi)\in[G\backslash\mathcal{L}_b(G,L,u)/\mathrm{Aut}(L,u)]}\langle\chi_\mathcal{C},1\rangle_{\mathrm{Out}(L,u)_{(P_\gamma,\pi)}}$$

$$=\sum_{(P_\gamma,\pi)\in[G\backslash\mathcal{L}_b(G,L,u)/\mathrm{Aut}(L,u)]}\frac{1}{|\mathrm{Out}(L,u)_{(P_\gamma,\pi)}|}\sum_{\overline{\psi}\in\mathrm{Out}(L,u)_{(P_\gamma,\pi)}}\chi_\mathcal{C}(\overline{\psi})$$

$$=\sum_{(P_\gamma,\pi)\in[G\backslash\mathcal{L}_b(G,L,u)]}\frac{1}{|\mathrm{Out}(L,u)|}\sum_{\overline{\psi}\in\mathrm{Out}(L,u)_{(P_\gamma,\pi)}\cap\mathcal{C}}1$$

$$=\frac{1}{|\mathrm{Out}(L,u)|}\sum_{\overline{\psi}\in\mathcal{C}}\sum_{(P_\gamma,\pi)\in[(G\backslash\mathcal{L}_b(G,L,u))^{\overline{\psi}}]}1$$

$$=\frac{1}{|\mathrm{Out}(L,u)|}|\mathcal{C}||[(G\backslash\mathcal{L}_b(G,L,u))^{\overline{\varphi}}]|$$

Similarly, we have

$$\sum_{(Q_\delta,\rho)\in[H\backslash\mathcal{L}_c(H,L,u)/\mathrm{Aut}(L,u)]}\langle\chi_\mathcal{C},1\rangle_{\mathrm{Out}(L,u)_{(Q_\delta,\rho)}}=\frac{1}{|\mathrm{Out}(L,u)|}|\mathcal{C}||[(H\backslash\mathcal{L}_c(H,L,u))^{\overline{\varphi}}]|.$$

Therefore, we have $|[(G\backslash\mathcal{L}_b(G,L,u))^{\overline{\varphi}}]|=|[(H\backslash\mathcal{L}_c(H,L,u))^{\overline{\varphi}}]|$ which implies (ii).

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Then the permutation $\mathbb{F}\mathrm{Aut}(L,u)$-modules $\mathbb{F}[G\backslash\mathcal{L}_b(G,L,u)]$ and $\mathbb{F}[H\backslash\mathcal{L}_c(H,L,u)]$ have the same character and hence they are isomorphic. The result then follows from Theorem 8.22(iii). □

10.6 Theorem *Let D be a finite p-group. Then there is only a finite number of pairs (G,b), where G is a finite group, and b is a block idempotent of kG with defect D, up to functorial equivalence over $\mathbb{F}$.*

Proof We know that the functor $\mathbb{F}T_{G,b}^\Delta$ splits as a direct sum of simple functors $S_{L,u,V}$. Using Theorem 8.22 (b), we will show that only a finite number (depending only on D) of simple functors $S_{L,u,V}$, can appear in $\mathbb{F}T_{G,b}^\Delta$, and that the multiplicity of $S_{L,u,V}$ as a summand of $\mathbb{F}T_{G,b}^\Delta$ is bounded by a constant depending only on D. This will imply that, up to isomorphism, there is only a finite number of possibilities for the functor $\mathbb{F}T_{G,b}^\Delta$, once the defect D of b is fixed.

Recall that the simple functors $S_{L,u,V}$ are parametrized by triples (L,u,V), where L is a finite p-group, u is a faithful p'-automorphism of L, and V is a simple $\mathbb{F}\mathrm{Out}(L,u)$-module, where $\mathrm{Out}(L,u)$ is a quotient of $\mathrm{Aut}(L,u)$, itself a subgroup of the automorphism group $\mathrm{Aut}(L)$ of L. By Theorem 8.22 (b), the multiplicity of $S_{L,u,V}$ as a summand of $\mathbb{F}T_{G,b}^\Delta$ is equal to the $\mathbb{F}$-dimension of

$$m_{l,u,V}(b)=\bigoplus_{(P,e_P)\in[\mathcal{F}_b]}\bigoplus_{\pi\in[\mathcal{P}_{(P,e_P)}(L,u)]}\mathbb{F}\mathrm{Proj}(keC_G(P),u)\otimes_{\mathrm{Aut}(L,u)_{P,e,\pi}}V,$$

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where $\mathcal{F}_b$ is the fusion system of b and $[\mathcal{P}_{(P,e_P)}(L,u)]$ is a set of $N_G(P,e_P) \times \mathrm{Aut}(L,u)$-orbits of the set $\mathcal{P}_{(P,e_P)}(L,u)$ of group isomorphisms $\pi : L \to P$ such that $\pi \circ i_u \circ \pi^{-1}$ is an automorphism of (P,e) in the fusion system $\mathcal{F}_b$. Moreover $\mathrm{Proj}\big(keC_G(P),u\big)$ is a subgroup of the group of projective $keC_G(P)$-modules.

It follows that if $S_{L,u,V}$ appears in $\mathbb{F}T_{G,b}^\Delta$ with non zero multiplicity, then L is isomorphic to a subgroup of D. Hence there is only a finite number of such groups L, up to isomorphism. For each L, there is only a finite number - at most $|\mathrm{Aut}(L)|$ - of faithful p'-automorphisms u of L, and for each such u, there is only a finite number - at most $|\mathrm{Aut}(L)|$ again - of simple $\mathbb{F}\mathrm{Aut}(L,u)$-modules, up to isomorphism. Hence the number of simple summands of $\mathbb{F}T_{G,b}^\Delta$ is bounded by a number c_D depending only on D.

Now for such a summand $S_{L,u,V}$, the dimension of the $\mathbb{F}$-vector space $\mathbb{F}\mathrm{Proj}\big(keC_G(P),u\big) \otimes_{\mathrm{Aut}(L,u)_{P,e,\pi}} V$ is less than or equal to $\dim \mathbb{F}\mathrm{Proj}\big(keC_G(P)\big) \dim V$, and moreover $\dim V \leqslant |\mathrm{Out}(L,u)| \leqslant |\mathrm{Aut}(L)|$. Now $\dim \mathbb{F}\mathrm{Proj}\big(keC_G(P)\big)$ is equal to the number $l\big(keC_G(P)\big)$ of simple $keC_G(P)$-modules, which is equal to the number $l\big(kePC_G(P)\big)$ of simple $kePC_G(P)$-modules, as P acts trivially on each simple $kePC_G(P)$-module. Now e is a block idempotent of $kPC_G(P)$, and by Corollary 4.5 of [AB79], we can assume that $P \leqslant D$, and that e has defect $PC_D(P)$, which is a subgroup of D. By Theorem 1 of [BF59], the number of ordinary irreducible characters in a block c of a finite group H is at most $\frac{1}{4}p^{2d} + 1$, where p^d is the order of the defect group of c. This number is smaller than $2p^{2d}$. It follows that $l\big(keC_G(P)\big)$, which is at most equal to the number of ordinary irreducible characters in the block e of $PC_G(P)$, is smaller than $2|PC_D(P)|^2 \leqslant 2|D|^2$.

Finally, we get that

$$ m_{L,u,V}(b) \leqslant n_{D,L} 2|D|^2 |\mathrm{Aut}(L)|^2, $$

where $n_{D,L}$ is the number of subgroups of D isomorphic to L. So there is a constant m_D, depending only on D, such that $m_{L,u,V}(b) \leqslant m_D$ (for example $m_D = n_D 2|D|^2 M_D$, where n_D is the number of subgroups of D, and M_D is the sup of $|\mathrm{Aut}(L)|$ over subgroups L of D). This bound only depends on D, as was to be shown. This completes the proof. $\square$

11 Blocks with abelian defect groups

Let b be a block idempotent of kG with an abelian defect group D. Let c be a block idempotent of kH which in Brauer correspondence with b where $H = N_G(D)$. Let (L,u) be a D^Δ-pair with the property that the multiplicity of $S_{L,u,V}$ in $\mathbb{F}T_{G,b}^\Delta$ is non-zero for some $V \in \mathbb{F}\mathrm{Out}(L,u)$-mod. Then L is also abelian.

Let (D,e_D) be a maximal b-Brauer pair and note that (D,e_D) is also a maximal c-Brauer pair. For every $P \leqslant D$, let $e_P \in \mathrm{Bl}(kC_G(P))$ and $f_P \in \mathrm{Bl}(kC_H(P))$ such that $(P,e_P) \leqslant_G (D,e_D)$ and $(P,f_P) \leqslant_H (D,e_D)$. One can show that the block idempotents e_P and f_P are Brauer correspondents. Let $\mathcal{F}_b$ be the fusion system of b associated to (D,e_D) and let $\mathcal{F}_c$ be the fusion system of c associated to (D,e_D).

Recall that by (20), we have isomorphisms

$$ b\mathbb{F}T^\Delta(G,L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}} \cong \bigoplus_{(P,e_P)\in[\mathcal{F}_b]} \bigoplus_{\pi\in[\mathcal{P}_{(P,e_P)}^G(L,u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{P,e_P,\pi}}^{\mathrm{Aut}(L,u)} \mathbb{F}\mathrm{Proj}(ke_PC_G(P),u) $$

and

$$c\mathbb{F}T^{\Delta}(H, L\langle u\rangle)\widetilde{F_{L,u}^{L\langle u\rangle}} \cong \bigoplus_{(P,f_P)\in[\mathcal{F}_c]}\bigoplus_{\pi\in[\mathcal{P}_{(P,f_P)}^{H}(L,u)]} \mathrm{Ind}_{\mathrm{Aut}(L,u)_{P,f_P,\pi}}^{\mathrm{Aut}(L,u)}\mathbb{F}\mathrm{Proj}(kf_PC_H(P), u)$$

of right $\mathbb{F}\mathrm{Aut}(L, u)$-modules.

(a) Let $(P, e_P) \in \mathcal{F}_b$. Let also $s \in N_G(P, e_P)$. Then since D is abelian, we have $N_G(P, e_P) \subseteq N_G(D, e_D)C_G(P)$. Hence we have $s = ht$ for some $h \in N_G(D, e_D) \leqslant H$ and $t \in C_G(P)$. Thus $i_s = i_h : P \to P$ and $h \in N_H(P, e_P)$.

(b) We have $(P, e_P) \equiv_{\mathcal{F}_b} (Q, e_Q)$ if and only if $(P, f_P) \equiv_{\mathcal{F}_c} (Q, f_Q)$. Indeed, let $\phi : (P, e_P) \to (Q, e_Q)$ be an isomorphism in $\mathcal{F}_b$. Then by Alperin's fusion theorem, there exists an automorphism $\psi : (D, e_D) \to (D, e_D)$ such that $\psi|_P = \phi$. There exists an element $g \in G$ such that $\psi(x) = {}^gx$ for any $x \in D$ and such that ${}^g(D, e_D) = (D, e_D)$. This implies in particular that $g \in N_G(D, e_D) \leqslant H$. This shows that $\psi \in \mathcal{F}_c$ and ${}^gf_P = {}^g(\mathrm{Br}_D^{C_G(P)}(e_P)) = \mathrm{Br}_D^{C_G(Q)}({}^ge_P) = f_Q$. Hence $\psi|_P : (P, f_P) \to (Q, f_Q)$ is an isomorphism in $\mathcal{F}_c$.

Conversely, let $\phi : (P, f_P) \to (Q, f_Q)$ be an isomorphism in $\mathcal{F}_c$. Then again by Alperin's fusion theorem, there exists $\psi : (D, e_D) \to (D, e_D)$ such that $\psi|_P = \phi$. Let $h \in N_H(D, e_D)$ with $\psi = i_h$. We have $f_Q = {}^hf_P = {}^h(\mathrm{Br}_D^{C_G(P)}(e_P)) = \mathrm{Br}_D^{C_G(Q)}({}^he_P)$ which implies that ${}^he_P$ and f_Q are Brauer correspondents. Hence ${}^he_P = e_Q$ and $\psi|_P : (P, e_P) \to (Q, e_Q)$ is an isomorphism in $\mathcal{F}_b$.

(c) We have $\mathcal{P}_{(P,e_P)}^{G}(L, u) = \mathcal{P}_{(P,f_P)}^{H}(L, u)$ for any $P \leqslant D$. Let $\pi \in \mathcal{P}_{(P,e_P)}^{G}(L, u)$. Then by definition $\pi i_u \pi^{-1} \in \mathrm{Aut}_{\mathcal{F}_b}(P, e_P)$. So there exists $s \in N_G(P, e_P)$ such that $\pi i_u \pi^{-1} = i_s$. By part (a), there exists $h \in N_H(P, e_P)$ with $i_s = i_h$. But then $h \in N_H(P, f_P)$ and hence $\pi \in \mathcal{P}_{(P,e_P)}^{H}(L, u)$ as desired.

Conversely, let $\pi \in \mathcal{P}_{(P,f_P)}^{H}(L, u)$. Then $\pi i_u \pi^{-1} = i_h$ for some $h \in N_H(P, f_P)$. By part (a) we can choose $h \in N_H(D, e_D)$. Then one shows that ${}^he_P = e_P$ which implies that $h \in N_H(P, e_P) \subseteq N_G(P, e_P)$. Therefore, $\pi \in \mathcal{P}_{(P,e_P)}^{G}(L, u)$.

(d) Let $P \leqslant D$ and let $\pi, \rho \in \mathcal{P}_{(P,f_P)}^{H}(L, u) = \mathcal{P}_{(P,e_P)}^{G}(L, u)$. Then $\rho \in N_G(P, e_P) \cdot \pi \cdot \mathrm{Aut}(L, u)$ if and only if $\rho \in N_H(P, f_P) \cdot \pi \cdot \mathrm{Aut}(L, u)$. Indeed, suppose that for some $g \in N_G(P, e_P)$ and $\varphi \in \mathrm{Aut}(L, u)$ we have $\rho = i_g\pi\varphi$. By part (a), there exists $h \in N_H(P, e_P)$ with $i_g = i_h$. But then $h \in N_H(P, f_P)$ and $\rho = i_h\pi\varphi$ which proves the claim. Converse is proved similarly.

(e) Let $P \leqslant D$ and let $\pi \in [\mathcal{P}_{(P,e_P)}^{G}(L, u)] = [\mathcal{P}_{(P,f_P)}^{H}(L, u)]$. We have $\mathrm{Aut}(L, u)_{P,e_P,\pi}^{G} = \mathrm{Aut}(L, u)_{P,f_P,\pi}^{H}$. Indeed, let $\varphi \in \mathrm{Aut}(L, u)_{P,e,\pi}^{G}$. Then there exists $g \in N_G(P, e_P)$ such that $\pi\varphi\pi^{-1} = i_g$. As above this means that there exists $h \in N_H(P, f_P)$ such that $\pi\varphi\pi^{-1} = i_g = i_h$. This proves that $\varphi \in \mathrm{Aut}(L, u)_{P,f_P,\pi}^{H}$.

Conversely, if $\varphi \in \mathrm{Aut}(L, u)_{P,f_P,\pi}^{H}$, then there exists $h \in N_H(P, f_P)$ such that $\pi\varphi\pi^{-1} = i_h$. Again as above, we can choose $h \in N_H(D, e_D)$ and hence it follows that $h \in N_H(P, e_P) \subseteq N_G(P, e_P)$. Therefore, $\varphi \in \mathrm{Aut}(L, u)_{P,e,\pi}^{G}$.

All these imply the following.

11.1 Theorem *Let b be a block idempotent of kG with an abelian defect group D. Let c be a block idempotent of kH which is in Brauer correspondence with b where $H = N_G(D)$. Let (D, e_D) be a maximal b-Brauer pair. For every $P \leqslant D$, let $e_P \in \mathrm{Bl}(kC_G(P))$ and $f_P \in \mathrm{Bl}(kC_H(P))$ such that $(P, e_P) \leqslant_G (D, e_D)$ and $(P, f_P) \leqslant_H (D, e_D)$.*

(a) The multiplicity of $S_{D,u,V}$ in $\mathbb{F}T_{G,b}^{\Delta}$ and $\mathbb{F}T_{H,c}^{\Delta}$ are the same.

(b) *If for every $P \leqslant D$ and $s \in N_H(P, f_P)_{p'}$ we have an isomorphism*

$$\mathbb{F}\mathrm{Proj}(ke_P C_G(P), s) \cong \mathbb{F}\mathrm{Proj}(kf_P C_H(P), s)$$

of $\mathbb{F}N_H(P, f_P)$-modules, then (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$.

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