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DIAGONAL SIMILARITY AND EQUIVALENCE FOR MATRICES OVER GROUPS WITH 0

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1. INTRODUCTION

In our previous paper [3], we have shown that there is a close connection between the cyclic products of matrices and diagonal similarity. In this paper we consider diagonal similarity for matrices, which may be infinite, and whose elements lie in a (possible non-commutative) group $\boldsymbol{G}$ with 0.

Let $\boldsymbol{H}$ be a subgroup of a group $\boldsymbol{G}$ and let A be an irreducible square matrix with entries in $\boldsymbol{G}^0$. In Theorem 3.4, we give necessary and sufficient conditions for the existence of a matrix B with entries in $\boldsymbol{H}^0$ which is diagonally similar to A. If $\boldsymbol{H}$ is a complete lattice ordered group whose positive cone $\boldsymbol{H}^+$ is normal in $\boldsymbol{G}$, we give necessary and sufficient conditions for the existence of a matrix B in $(\boldsymbol{H}^+)^0$ which is diagonally similar to A; see Theorems 4.1 and 4.2. Our Theorem 4.1 reduces to a result due to AFRIAT [1], [2] Theorem 2 and FIEDLER-PTÁK [4] Theorem 2.2 in the case when $\boldsymbol{G}$ and $\boldsymbol{H}$ are the additive group of reals, there is no absorbing (zero) element and A is a finite matrix.

Let A be a rectangular matrix, possible infinite, with entries in $\boldsymbol{G}^0$, such that each row and column has at least one element in $\boldsymbol{G}$. We construct a square matrix A^{lp} of larger size, which is always completely reducible, Corollary 5.4. In Theorem 5.6, we show that two rectangular matrices A and B are diagonally equivalent if and only if A^{lp} and B^{lp} are diagonally similar. Thus it is possible to derive theorems on diagonal equivalence for arbitrary rectangular matrices from theorems on the diagonal similarity of irreducible square matrices, e.g. Theorem 6.3. In particular, as a corollary to either Theorem 6.2 or Theorem 6.3 we obtain a slightly improved version of the remarkable result by LALLEMENT-PETRICH ([6], Theorem 1 (b) $\Leftrightarrow$ (e),

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or [7], Theorem 4.13) which motivated our construction of A^{lp}. Lallement-Petrich apparently were the first to prove a theorem relating products of elements of a matrix to diagonal equivalence, without an additional hypothesis on the matrix, such as a full indecomposability.

As an application in § 7 we derive a non-iterative algorithm for the optimal scaling of a real or complex matrix by diagonal similarity and give an example with details (Example 7.4). In the Appendix, we include an executable APL statement for the mapping $A \to A^{lp}$, with some examples.

2. PRELIMINARY DEFINITIONS

Definitions 2.1. By ω and τ we shall always denete sets and we shall assume $1 \in \omega \cap \tau$.

(i) A *path of length m* is a finite sequence $\beta = (i_1, i_2, \ldots, i_m)$ where m is a positive integer, $m > 1$, and $i_s \in \omega$ for $s = 1, 2, \ldots, m$. If $i = i_1$ and $j = i_m$, we say *β is a path from i to j*. We say a path $\beta = (i_1, i_2, \ldots, i_m)$ is a path without repetitions if $i_1, i_2, \ldots, i_{m-1}$ are distinct.

(ii) A *closed path* is a path $\beta = (i_1, i_2, \ldots, i_m)$ with $i_m = i_1$.

(iii) A *cycle* is a closed path without repetitions.

(iv) If $\beta = (i_1, i_2, \ldots, i_m)$ and $\alpha = (j_1, j_2, \ldots, j_n)$ are paths with $i_m = j_1$ then $\beta\alpha$ will denote the path $(i_1, i_2, \ldots, i_m, j_2, \ldots, j_n)$.

Definition 2.2. If G is a group, we shall always assume that $0 \notin G$ and define the semigroup $G^0 = G \cup \{0\}$, where $0g = 0 = g0$ for all $g \in G^0$.

Definition 2.3.

(i) If E is a set, let $E_{\omega,\tau}$ denote the set of $\omega \times \tau$ matrices with elements in E, viz, $A \in E_{\omega,\tau}$ if A is a function $(i, j) \to a_{ij}$ of $\omega \times \tau$ into E. We denote $E_{\omega,\omega}$ by E_ω.

(ii) Let G be a group and let $\beta = (i_1, i_2, \ldots, i_m)$ be a path. For $A \in G^0_\omega = (G^0)_\omega$ we define $\Pi_\beta(A) = \prod_{j=1}^{m-1} a_{i_j i_{j+1}}$. If β is a cycle, $\Pi_\beta(A)$ is said to be a *cyclic product*.

(iii) I β is a path such that $\Pi_\beta(A) \in G$, we call β a *non-zero* path for A.

(iv) Let $A \in G^0_\omega$. If for all $i, j \in \omega$, $i \neq j$, there exists a non-zero path from i to j, we call A *irreducible*.

Remark 2.4. Let $A \in G^0_\omega$. Let $\beta = (i_1, i_2, \ldots, i_m)$ be a non-zero closed path (cycle) for A. Then, for $1 \leq r < m$, $\gamma = (i_r, i_{r+1}, \ldots, i_{m-1}, i_1, \ldots, i_r)$ is also a non-zero closed path (cycle) for A.

Definitions 2.5.

(i) Let G be a group and let $B \in G_\omega^0$. Then B is called a *diagonal* matrix if for all $i, j \in \omega$, $b_{ij} \in G$ implies $i = j$. We write $B = \operatorname{diag}(b_{ii})$.

(ii) When G is a group, a diagonal matrix $B \in G_\omega^0$ is *non-singular* if $b_{ii} \in G$ for all $i \in \omega$.

(iii) If $A \in G_{\omega,\tau}^0$ and $B \in G_\omega^0$ is a diagonal matrix we define $C = BA \in G_{\omega,\tau}^0$ by $c_{ij} = b_{ii}a_{ij}$, for $i \in \omega$, $j \in \tau$. If $B \in G_\tau^0$ we define $C = AB \in G_{\omega,\tau}^0$ by $c_{ij} = a_{ij}b_{jj}$ for $i \in \omega$, $j \in \tau$.

Definitions 2.6. Let $A \in G_{\omega,\tau}^0$.

(i) Let $\emptyset \neq \omega' \subseteq \omega$, $\emptyset \neq \tau' \subseteq \tau$. Then $B = A[\omega', \tau']$ denotes the matrix in $G_{\omega',\tau'}^0$ obtained by restricting A, to $\omega' \times \tau'$, viz $b_{ij} = a_{ij}$, for $(i, j) \in \omega' \times \tau'$.

(ii) We call $\{\omega_\kappa : \kappa \in K\}$ a partition of ω where K is an index set if $\omega_\kappa \neq \emptyset$, for $\kappa \in K$, $\omega_\kappa \cap \omega_{\kappa'} = \emptyset$, for $\kappa \neq \kappa'$, with κ and $\kappa' \in K$, and finally $\bigcup\{\omega_\kappa : \kappa \in K\} = \omega$.

(iii) Let $\{\omega_\kappa : \kappa \in K\}$ and $\{\tau_\kappa : \kappa \in K\}$ be partitions of ω and τ respectively. Then A is the *direct sum* of the $A[\omega_\kappa, \tau_\kappa]$, $\kappa \in K$ (we write $A = \oplus_{\kappa \in K} A[\omega_\kappa, \tau_\kappa]$) if $a_{ij} \in G$ implies there is a $\kappa \in K$ such that $i \in \omega_\kappa$ and $j \in \tau_\kappa$. If $K = \{1, 2\}$, we may write $A = A[\omega_1, \tau_1] \oplus A[\omega_2, \tau_2]$.

(iv) A matrix $A \in G_\omega^0$ is *chainable* if there are no partitions $\{\omega_1, \omega_2\}$ of ω, $\{\tau_1, \tau_2\}$ of τ such that $A = A[\omega_1, \tau_1] \oplus A[\omega_2, \tau_2]$.

Comment. The term chainable was introduced by SINKHORN-KNOPP [8]. It follows from Corollary 5.4(ii) that our definition is equivalent to that of [8].

Definitions 2.7. Let $A \in G_\omega^0$. Then A is *completely reducible* if there is a partition $\{\omega_\kappa : \kappa \in K\}$ of ω such that $A = \oplus_{\kappa \in K} A[\omega_\kappa, \omega_\kappa]$ and $A[\omega_\kappa, \omega_\kappa]$ is irreducible for $\kappa \in K$.

Remark 2.8. Let $A \in G^0$. It is well-known and easily proved that A is not irreducible if and only if there exists a partition $\{\omega_1, \omega_2\}$ of ω such that $A[\omega_1, \omega_2] = 0$.

3. DIAGONAL SIMILARITY FOR IRREDUCIBLE MATRICES

In § 3 and § 4, the results are trivial when A is the 1×1 matrix 0. In the proofs we therefore assume that A is not that matrix.

We begin by proving a lemma, which is related to [3], Lemma 2.4. By means of this lemma, we are able to replace conditions on products on closed paths by conditions on cyclic products, provided that we are considering a *normal* subgroup (or a subsemigroup invariant under conjugation).

Lemma 3.1. *Let $\boldsymbol{G}$ be a group and let $\boldsymbol{H}$ be a semigroup contained in $\boldsymbol{G}$ such that $x^{-1}\boldsymbol{H}x \in \boldsymbol{G}$ for all $x \in \boldsymbol{G}$. Let $A \in \boldsymbol{G}_\omega^0$. If $\Pi_\gamma(A) \in H^0$ for all cycles γ, then $\Pi_\beta(A) \in H^0$ for all closed paths β.*

Proof. Let $\beta = (i_1, i_2, \ldots, i_m)$ be a closed path. The proof is by induction on m. The result is true if $m = 2$. So, suppose that $m > 2$ and that the result is true for all closed paths $\alpha = (j_1, j_2, \ldots, j_p)$ with $2 \leqq p < m$. If β is a cycle there is no more to show. Otherwise, there exist integers q and r with $1 < q < r \leqq m$ such that $\gamma = (i_q, \ldots, i_r)$ is a cycle. Let $\delta = (i_1, \ldots, i_q)$, $\varepsilon = (i_r, \ldots, i_m)$. Then $\beta = \delta\gamma\varepsilon$. Let $x = \Pi_\delta(A)$, $y = \Pi_\gamma(A)$, $z = \Pi_\varepsilon(A)$. Then $\Pi_\beta(A) = xyz$. Since the inductive hypotheses hold for the closed path $\delta\varepsilon$, we have $\Pi_{\delta\varepsilon}(A) = xz \in \boldsymbol{H}$. Hence also $zx = x^{-1}(xz)\,x \in \boldsymbol{H}$. Thus $yzx \in \boldsymbol{H}$, whence $\Pi_\beta(A) = xyz = x(yzx)\,x^{-1} \in \boldsymbol{H}$. The lemma follows by induction.

Lemma 3.2. *Let $A \in \boldsymbol{G}_\omega^0$ be irreducible. Let $\boldsymbol{H}$ be the subgroup of $\boldsymbol{G}$ generated by all $\Pi_\beta(A)$, where β is a non-zero closed path. Then there exists a non-singular diagonal matrix $X \in \boldsymbol{G}_\omega^0$ such that $X^{-1}AX \in \boldsymbol{H}_\omega^0$.*

Proof. Since A is irreducible, there exist non-zero paths β_i from 1 to i and γ_i from i to 1, for all $i \in \omega$. Let $x_i = \Pi_{\beta_i}(A)$. Since $\beta_i\gamma_i$ is a non-zero closed path, we have $\Pi_{\beta_i}(A)\,\Pi_{\gamma_i}(A) = \Pi_{\beta_i\gamma_i}(A) = h_i \in \boldsymbol{H}$. Let $\Pi_{\beta_i}(A) = x_i$. Suppose that $a_{ij} \neq 0$. Then $x_i a_{ij} x_j^{-1} = \Pi_{\beta_i}(A)\, a_{ij} \Pi_{\gamma_j}(A)\, h_j^{-1} = \Pi_\delta(A)\, h_j^{-1}$, where δ is the non-zero closed path $\beta_i(i, j)\,\gamma_j$. Hence $\Pi_\delta(A) \in \boldsymbol{H}$, and so $x_i a_{ij} x_j^{-1} \in \boldsymbol{H}$. If $X = \operatorname{diag}(x_i) \in \boldsymbol{G}_\omega^0$, it follows that $XAX^{-1} \in \boldsymbol{H}_\omega^0$.

Remarks and Examples 3.3. (i) It is clear from the proof of Lemma 3.2, that we may choose $X = \operatorname{diag}(x_i)$, where $x_i = \Pi_{\beta_i}(A)$, for any non-zero path β_i from 1 to i.

(ii) Let $\boldsymbol{G}$ be the free group with generators a, b. Let

$$A = \begin{bmatrix} a^2 & b \\ b & a^2 \end{bmatrix} \in \boldsymbol{G}_{\{1,2\}} .$$

Let $\boldsymbol{H}$ be defined as in Lemma 3.2. Then $\boldsymbol{H}$ is generated by a^2, b^2 and ba^2b. If $X = \operatorname{diag}(a^2, b)$ then

$$X^{-1}AX = \begin{bmatrix} a^2 & a^{-2}b^2 \\ a^2 & b^{-1}a^2b \end{bmatrix} \in \boldsymbol{H}_{\{1,2\}} .$$

(iii) We now show that in general there is no diagonal $X \in \boldsymbol{G}_\omega^0$ such that $XAX^{-1} \in \boldsymbol{K}_\omega^0$, where $\boldsymbol{K}$ is the group generated by the cyclic products $\Pi_\gamma(A)$. For let $\boldsymbol{G}$ and A be as in (ii). Then $\boldsymbol{K}$ is generated by a^2, b^2. We proceed by contradiction. For let $X = \operatorname{diag}(sb^q, tb^r) \in \boldsymbol{G}_{\{1,2\}}^0$ where either $s = 1$ or the canonical expression for s ends in a, and t satisfies the same conditions. Then

$$XAX^{-1} = \begin{bmatrix} sb^qa^2b^{-q}s^{-1} & sb^{q-r+1}t^{-1} \\ tb^{r-q+1}s^{-1} & tb^ra^2b^{-r}t^{-1} \end{bmatrix} .$$

By considering an off diagonal element we see that either r or q is odd. But by inspecting the diagonal elements we see that this is impossible.

Theorem 3.4. *Let $A \in G_\omega^0$ be irreducible. Let H be a normal subgroup of G. The following are equivalent:*

(i) *For all cycles γ, $\Pi_\gamma(A) \in H^0$.*

(ii) *There exists a non-singular diagonal matrix $X \in G^0$, such that $XAX^{-1} \in H_\omega^0$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 3.1, $\Pi_\beta(A) \in H$ for all non-zero closed paths β. The result follows from Lemma 3.2.

(ii) $\Rightarrow$ (i). Let $X = \operatorname{diag}(x_i) \in G_\omega^0$. Let γ be a non-zero cycle. Then for some $i \in \omega$, $x_i \Pi_\gamma(A) x_i^{-1} = \Pi_\gamma(X^{-1}AX) \in H$, and (i) follows.

For example, if $A \in G_\omega^0$ is an irreducible matrix whose entries are quaternions such that all cyclic products are real, then A is diagonally similar to a real matrix.

If ω is finite and G is commutative, our next theorem reduces to [3], Corollary 4.4, (1) $\Leftrightarrow$ (3).

Theorem 3.5. *Let $A, B \in G_\omega^0$ be irreducible. Then the following are equivalent:*

(i) *There exist $u_i \in G$, for $i \in \omega$, such that for every closed path β from i to i, $\Pi_\beta(B) = u_i^{-1}\Pi_\beta(A)\, u_i$.*

(ii) *There exists a non-singular diagonal $X \in G_\omega^0$ such that $XAX^{-1} = B$.*

Proof. (i) $\Rightarrow$ (ii). For $i \in \omega$, we define the paths β_i and γ_i as in the proof of Lemma 3.2. Since $\beta_i\gamma_i$ is a closed path from 1 to 1, it follows that

$$\Pi_{\beta_i}(B)\,\Pi_{\gamma_i}(B) = u_1^{-1}\Pi_{\beta_i}(A)\,\Pi_{\gamma_i}(A)\,u_1\,.$$

Let $x_i = \Pi_{\beta_i}(B)^{-1}\,u_1^{-1}\,\Pi_{\beta_i}(A)$. Hence $x_i^{-1} = \Pi_{\gamma_i}(A)\,u_1\,\Pi_{\gamma_i}(B)^{-1}$. Since $\beta_i(i,j)\,\gamma_j$ is a closed path it follows that

$$\Pi_{\beta_i}(B)\,b_{ij}\,\Pi_{\gamma_j}(B) = u_1^{-1}\,\Pi_{\beta_i}(A)\,a_{ij}\,\Pi_{\gamma_j}(A)\,u_1\,.$$

Hence $b_{ij} = x_i a_{ij} x_j^{-1}$, and (ii) follows.

(ii) $\Rightarrow$ (i). By straightforward computation of the path products the results follows with $u_i = x_i^{-1}$, $i \in \omega$. When G is commutative, we obtain a corollary, where we can replace closed paths by cycles in statement (i).

Corollary 3.6. *Let G be an abelian group and let $A, B \in G_\omega^0$ be irreducible. Then the following are equivalent:*

(i) *For all cycles γ, $\Pi_\gamma(B) = \Pi_\gamma(A)$.*

(ii) *There exists a non-singular diagonal $x \in G_\omega^0$ such that $XAX^{-1} = B$.*

4. DIAGONAL SIMILARITY OVER PARTIALLY ORDERED GROUPS

In this section we shall consider lattice ordered groups $\boldsymbol{H}$ (cf. FUCHS [5] p. 66). The partial order in $\boldsymbol{H}$ will be denoted by $\leqq$ and we write $\boldsymbol{H}^+ = \{h \in \boldsymbol{H} : h \geqq 1\}$. For the main result in this section we shall require a (conditionally) complete lattice ordered group $\boldsymbol{H}$ ([5] p. 89).

Theorem 4.1. *Let $\boldsymbol{G}$ be a group and let $\boldsymbol{H}$ be a subgroup of $\boldsymbol{G}$ such that*

(a) *$\boldsymbol{H}$ is a complete lattice ordered group.*

(b) *$x\boldsymbol{H}^+x^{-1} \subseteq \boldsymbol{H}^+$ for all $x \in \boldsymbol{G}$.*

Let $A \in \boldsymbol{G}^0_\omega$ be irreducible. Then the following are equivalent:

(i) *For all cycles γ, $\Pi_\gamma(A) \in (\boldsymbol{H}^+)^0$.*

(ii) *There exists a non-singular diagonal $X \in \boldsymbol{G}^0_\omega$ such that $XAX^{-1} \in (\boldsymbol{H}^+)^0_\omega$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 3.1, $\Pi_\beta(A) \in (\boldsymbol{H}^+)^0$, for all closed paths β. For each $i \in \omega$ let γ_i be a non-zero path from i to 1, and let P_i be the set of non-zero paths from 1 to i. Define $\Delta_i = \{\Pi_{\beta_i}(A)\, \Pi_{\gamma_i}(A) : \beta_i \in P_i\}$. For each pair $i, j \in \omega$ such that $a_{ij} \in \boldsymbol{G}$ we define $\Delta_{ij} = \{\Pi_{\beta_i}(A)\, a_{ij}\, \Pi_{\gamma_j}(A) : \beta_i \in P_i\}$. Since A is irreducible, $\emptyset \neq \Delta_{ij} \subseteq \Delta_j \subseteq \boldsymbol{H}^+$, where $\inf \Delta_{ij} \geqq \inf \Delta_j \geqq 1$. Let $w_{ij} = \inf \Delta_{ij}$, $z_j = \inf \Delta_j$. Let $\beta_i \in P_i$. Then

$$\Pi_{\beta_i}(A)\, a_{ij}\, \Pi_{\gamma_j}(A) = \Pi_{\beta_i}(A)\, \Pi_{\gamma_i}(A)\, \Pi_{\gamma_i}(A)^{-1}\, a_{ij}\, \Pi_{\gamma_j}(A)\,.$$

Let $\Pi_{\gamma_1}(A) = s_1$. It follows that $s_i^{-1}a_{ij}s_j \in \boldsymbol{H}$. Hence, taking the infinum over P_i, we obtain (cf. [5] p. 90)

$$w_{ij} = z_i s_i^{-1} a_{ij} s_j\,.$$

We deduce that

$$z_i(s_i^{-1}a_{ij}s_j) \geqq z_j\,.$$

Let $x_i = z_i s_i^{-1}$. Then $x_i a_{ij} x_i^{-1} \in \boldsymbol{H}^+$, and (ii) is proved.

(ii) $\Rightarrow$ (i). The proof is similar to the proof of Theorem 3.4, (ii) $\Rightarrow$ (i).

For example, if $A \in \boldsymbol{G}^0_\omega$ is a matrix whose entries are quaternions such that all cyclic products are in the real interval [0, 1]. Then A is diagonally similar to a matrix whose entries are in [0, 1].

When ω is finite we do not require the lattice ordered group $\boldsymbol{H}$ to be complete.

Theorem 4.2. *Let $\boldsymbol{G}$ be a group, and let $\boldsymbol{H}$ be a subgroup of $\boldsymbol{G}$ such that:*

(a) *$\boldsymbol{H}$ is a lattice ordered group.*

(b) *$x\boldsymbol{H}^+x^{-1} \in \boldsymbol{H}^+$ for all $x \in G$.*

Let $A \in G_\omega^0$ *be irreducible. Then the following are equivalent:*

(i) *For all cycles* γ, $\Pi_\gamma(A) \in (H^+)^0$.

(ii) *There exists a non-singular diagonal* $X \in G_\omega^0$ *suxh that* $XAX^{-1} \in (H^+)_\omega^0$.

Proof. We repeat the proof of Theorem 4.1, and we use the symbols defined there. We need only show that $\inf \Delta_{ij}$, and $\inf \Delta_i$ exist. Let $\beta_i \in P_i$ and suppose that $\beta_i = \delta\varepsilon\eta$, where ε is a closed path. Let $x = \Pi_\delta(A)$, $y = \Pi_\varepsilon(A)$, $z = \Pi_\eta(A)\, \Pi_{\gamma_i}(A)$. Then

$$\Pi_{\beta_i}(A)\, \Pi_{\gamma_i}(A) = xyz = xyx^{-1} \,.\, xz \geqq xz = \Pi_{\delta\eta}(A)\, \Pi_{\gamma_i}(A)\,.$$

It follows easily that

$$\Delta_i = \inf \{\Pi_{\beta_i}(A)\, \Pi_{\gamma_i}(A) : \beta_i \in P'_i\}$$

where P'_i is the set of non-zero paths from 1 to i without repetitions. This infimum exists since P'_i is a finite set. Similarly,

$$\Delta_{ij} = \inf \{\Pi_{\beta_i}(A)\, a_{ij}\, \Pi_{\gamma_j}(A) : \beta_i \in P'_i\}\,.$$

As an example let ω be finite and A a matrix of quaternions whose cyclic products are rational numbers in [0, 1]. Then A is similar to a matrix whose entries are rational in [0, 1].

5. THE CONNECTION BETWEEN DIAGONAL SIMILARITY AND DIAGONAL EQUIVALENCE

The following lemma is intuitively obvious, but since it is crucial to our argument we give a formal proof.

Lemma 5.1. *Let* $A \in G_{\omega,\tau}^0$ *have a non-zero element in each row and column. Then there is a partition* $\{\omega_\kappa : \kappa \in K\}$ *of* ω *and a partition* $\{\tau_\kappa : \kappa \in K\}$ *of* τ *such that*

$$A = \oplus_{k\in K} A[\omega_\kappa, \tau_\kappa]$$

and $A[\omega_\kappa, \tau_\kappa]$ *is chainable for all* $\kappa \in K$.

Proof. Let $i \in \omega$. For $k \in \omega$ we define $k \sim i$ if for all partitions $\{\omega'_\kappa : \kappa \in K'\}$, $\{\tau'_\kappa : \kappa \in K'\}$ of ω and τ respectively such that $A = \oplus_{\kappa\in K'} A[\omega'_\kappa, \tau'_\kappa]$ there is a $\kappa \in K$ such that $i \in \omega'_\kappa$, $k \in \omega'_\kappa$. It is easily seen that $\sim$ is an equivalence relation on ω. Let $\{\omega_\kappa : \kappa \in K\}$ be the set of equivalence classes. Let $\tau_\kappa = \{j \in \tau : \exists i \in \omega_\kappa, a_{ij} \neq 0\}$. Since each column of A has a non-zero element, it follows that $\bigcup\{\tau_\kappa : \kappa \in K\} = \tau$. Let $j \in \tau_\kappa \cap \tau_{\kappa'}$. Then there exists $i \in \omega_\kappa$, $i' \in \omega_{\kappa'}$ such that $a_{ij} \neq 0$, $a'_{i,j} \neq 0$. Let $\{\omega''_\lambda : \lambda \in L\}$, $\{\tau''_\lambda : \lambda \in L\}$ be partitions of ω and τ respectively such that

$$A = \oplus_{\lambda\in L} A[\omega''_\lambda, \tau''_\lambda]\,.$$

Suppose $j \in \tau''_\lambda$. Then it follows that both $i \in \omega''_\lambda$, $i' \in \omega''_\lambda$. Hence $i \sim i'$, whence $\kappa = \kappa'$ and it follows that $\{\tau_\kappa : \kappa \in K\}$ is a partition of τ. Clearly $A = \oplus_{\kappa \in K'} A[\omega_\kappa, \tau_\kappa]$. Let $\lambda \in \in K$. Since any partition of ω_λ for which A is a direct sum, can be extended to a partition of ω for which A is a direct sum by uniting it with $\{\omega_\kappa : \kappa \in K \setminus \{\lambda\}\}$, it follows that $A[\omega_\lambda, \tau_\lambda]$ is chainable.

Definition 5.2. (of A^{lp}) Let $A \in G^0_{\omega,\tau}$. Let $\sigma = \{(i, j) \in \omega \times \tau : a_{ij} \neq 0\}$. We define the matrix $B = A^{lp} \in G^0_\sigma$ by

$$b_{(i,j)(k,l)} = a_{ij}^{-1} a_{il}\,.$$

Observe that if $A = 0$ then A^{lp} is empty (i.e. the 0×0 matrix). To avoid trivial exceptions we assume that the 0×0 matrix is irreducible and chainable. In the proofs of the results in § 5 and § 6 it is possible to assume that $A \in G^0_\omega$ has a least one element in each row and column and then one may check that the results hold without that restriction.

Theorem 5.3. *Let $\{\omega_\kappa : \kappa \in K\}$, $\{\tau_\kappa : \tau_\kappa \in K\}$ be partitions of ω and τ respectively and put $A_\kappa = A[\omega_\kappa, \tau_\kappa] \in G^0_{\omega_\kappa,\tau_\kappa}$. If*

(1) $$A = \oplus_{k \in K} A_\kappa$$

and

(2) $$A_\kappa \quad \text{is chainable for} \quad \kappa \in K$$

then

(3) $$A^{lp} = \oplus_{\kappa \in K} (A_\kappa)^{lp}$$

and

(4) $$(A_\kappa)^{lp} \quad \text{is irreducible for} \quad \kappa \in K\,.$$

Proof. Let $B = A^{lp}$, and let $(A_\kappa)^{lp} \in G^0_{\sigma_\kappa}$, when $\sigma_\kappa \subseteq \omega_\kappa \times \tau_\kappa$. Let $(i, j) \in \sigma_\kappa$, $(k, l) \in \sigma_{\kappa'}$, $\kappa \neq \kappa'$. Since $i \in \omega_\kappa$, $l \in \tau_{\kappa'}$, we have $b_{ij} = a_{ij}^{-1} a_{il} = 0$. Hence $B = = \oplus_{k \in K} (A_\kappa)^{lp}$. So we need only prove that if A is chainable then A^{lp} is irreducible.

Suppose $B = A^{lp} \in G^0_\sigma$ is reducible. Then there exists a partition $\{\sigma_1, \sigma_2\}$ of σ such that $B[\sigma_1, \sigma_2] = 0$.

For $\kappa = 1, 2$, define

$$\omega_\kappa = \{i \in \omega : \exists j \in \tau,\ (i, j) \in \sigma_\kappa\}\,, \quad \tau_\kappa = \{j \in \tau : \exists i \in \omega,\ (i, j) \in \sigma_\kappa\}\,.$$

Since $\{\sigma_1, \sigma_2\}$ is partition of σ, it follows that $\omega_\kappa, \tau_\kappa$, $\kappa = 1, 2$ are non-empty and $\omega_1 \cup \omega_2 = \omega$, $\tau_1 \cup \tau_2 = \tau$. We must still prove that $\omega_1 \cap \omega_2 = \emptyset$ and $\tau_1 \cap \tau_2 = \emptyset$. Let $i \in \omega_1 \cap \omega_2$. Then for some $j, l \in \tau$, $(i, j) \in \sigma_1$, $(i, l) \in \sigma_2$. Hence $b_{(i,j)(i,l)} = = a_{ij}^{-1} a_{il} \neq 0$, which is impossible since $B[\sigma_1, \sigma_2] = 0$. Hence $\omega_1 \cap \omega_2 = \emptyset$.

Let $j \in \tau_1 \cap \tau_2$. Then for some $i, k \in \omega$, we have $(i, j) \in \sigma_1$, $(k, j) \in \sigma_2$. Hence $b_{(i,j)(k,j)} = a_{ij}^{-1} a_{ij} = 1$, which is impossible. Hence $\tau_1 \cap \tau_2 = \emptyset$, and the result is proved.

Corollary 5.4. (i) *For all* $A \in G_\omega^0$, A^{lp} *is completely reducible.*

(ii) If $A \in G_\omega^0$, *then* A^{lp} *is irreducible if and only if* A *is chainable.*

Corollary 5.5. *Let* $\omega = \{1, 2, \ldots, n\}$ *and let* $A \in G_\omega^0$. *Then the following are equivalent:*

(i) *There exist permutation matrices* P, Q *such that*

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1, A_2 *are rectangular.*

(ii) *There exists a permutation matrix* R *such that*

$$RA^{lp}R^{-1} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1, B_2 *are square.*

In the Appendix, we give some examples of A^{lp} for some matrices A which may be helpful for the understanding of the proofs of the results in the rest of this section. If $A \in G_{\omega,\tau}^0$ we denote the *support matrix* of A by A^*, viz $A^* \in G_{\omega,\tau}^0$ and $a_{i,j}^* = 1$ if $a_{ij} \in G$, $a_{ij}^* = 0$ if $a_{ij} = 0$, for $(i, j) \in \omega \times \tau$.

Theorem 5.6. *Let* $A, C \in G_{\omega,\tau}^0$ *and suppose that* $A^* = C^*$. *Let* $A^{lp} \in G_\sigma^0$. *Then the following are equivalent:*

(i) *There exist non-singular diagonal matrices* $Y \in G_\omega^0$, $Z \in G_\tau^0$ *such that* $C = YAZ$.

(ii) *There exists a non-singular diagonal* $X \in G_\sigma^0$ *such that* $XA^{lp}X^{-1} = C^{lp}$.

Proof. For a non-sigular diagonal $Z \in G_\tau^0$ define $\tilde{Z} = \operatorname{diag}(\tilde{z}_{ij}) \in G_\sigma^0$ by $z_{ij} = z_j$, for $(i, j) \in \sigma$.

(i) $\Rightarrow$ (*ii*). If $C = YAZ$, then $C^{lp} = \tilde{Z}^{-1} A^{lp} \tilde{Z}$, by straightforward computation.

(ii) $\Rightarrow$ (i). Let $XA^{lp}X^{-1} = C^{lp}$, where $X = \operatorname{diag}(x_{ij}) \in G_\sigma^0$. Let $B = A^{lp}$, $F = C^{lp}$. Let $i, k \in \omega, j \in \tau$. Since $b_{(i,j)(k,j)} = a_{ij}^{-1} a_{ij} = 1$ and $f_{(i,j)(k,j)} = c_{ij}^{-1} c_{ij} = 1$, it follows that $x_{(i,j)} = x_{(k,j)}$. Hence we may define $z_j = x_{(i,j)}^{-1}$, for $j \in \tau$. Now let $i \in \omega, j, l \in \tau$. Then $z_j^{-1} a_{ij}^{-1} a_{il} z_l = c_{ij}^{-1} c_{il}$ whence $c_{ij} z_j^{-1} a_{ij}^{-1} = c_{il} z_l^{-1} a_{il}$. Hence we may define $y_i = c_{ij} z_j^{-1} a_{ij}^{-1}$, for $i \in \omega$. Then $y_i a_{ij} z_j = c_{ij}$ and the result follows.

Lemma 5.7. *Let $\boldsymbol{H}$ be a subgroup of G. Let $A \in \boldsymbol{G}^0_{\omega,\tau}$ and $A^{lp} \in \boldsymbol{G}^0_\sigma$. Then the following are equivalent*:

(i) *There exist a non-singular diagonal $\gamma \in G^0_\omega$ such that $\gamma A \in \boldsymbol{H}^0_{\omega,\tau}$.*

(ii) $A^{lp} \in \boldsymbol{H}^0_\sigma$.

Proof. (i) $\Rightarrow$ (ii). Let $B = A^{lp}$. Then let $(i,j), (k,l) \in \sigma$. If $\gamma = \text{diag}(y_i) \in \boldsymbol{G}^0_\omega$, then $h_{(i,j)(k,l)} = a_{ij}a_{il} = a_{ij}^{-1}y_i^{-1} \cdot y_i a_{il} \in \boldsymbol{H}^0$.

(ii) $\Rightarrow$ (i). For each $i \in \omega$ there is a $j \in \tau$ such that $a_{ij} \in \boldsymbol{G}$. Let $y_i = a_{ij}^{-1} \in \boldsymbol{G}$ at $l \in \tau$. Then $y_i a_{il} = b_{(i,j)(i,l)} \in \boldsymbol{H}^0$. If $\gamma = \text{diag}(y_i)$, it follows that $\gamma A \in \boldsymbol{H}^0_{\omega,\tau}$.

Lemma 5.8. *Let $\boldsymbol{H}$ be a subgroup of G and let $A \in \boldsymbol{G}^0_{\omega,\tau}$ be chainable. Let $A^{lp} \in \boldsymbol{G}^0_\sigma$. Then the following are equivalent*:

(i) *There exist non-singular diagonal $Y \in \boldsymbol{G}^0_\omega$, $Z \in \boldsymbol{G}^0_\tau$ such that $YAZ \in \boldsymbol{H}^0_{\omega,\tau}$.*

(ii) *There exists a non-singular diagonal $X \in \boldsymbol{G}^0_\sigma$ such that $XA^{lp}X^{-1} \in \boldsymbol{H}^0_\sigma$.*

Proof. (i) $\Rightarrow$ (ii). Immediate by Theorem 5.6.

(ii) $\Rightarrow$ (i). Let $XA^{lp}X^{-1} \in \boldsymbol{H}^0_\sigma$ and let $B = A^{lp}$. Since $b_{(i,j)(k,j)} = 1$ for $(i,j)\,(k,j) \in \sigma$ it follows that $x_{ij}x_{kj}^{-1} \in \boldsymbol{H}$. Hence for $i \in \omega, j \in \tau$ there exist $z_j^{-1} \in G$ and $h_{ij} \in H$ such that $x_{ij} = h_{ij}z_j^{-1}$. Define $\tilde{Z}$ as ie the proof of Theorem 5.6. Then $\tilde{Z}A^{lp}\tilde{Z}^{-1} \in H^0_\sigma$. If $C = AZ$, $C^{lp} = \tilde{Z}A^{lp}\tilde{Z}^{-1} \in \boldsymbol{H}^0_\sigma$, whence by Lemma 5.7 $YAZ = YC \in \boldsymbol{H}^0_{\omega,\tau}$ for, some non-singular diagonal $Y \in \boldsymbol{G}^0_\omega$.

6. APPLICATIONS OF A^{lp}

Theorem 6.1. *Let $A \in \boldsymbol{G}^0_{\omega,\tau}$. Let $\boldsymbol{H}$ be the subgroup of $\boldsymbol{G}$ generated by all $\Pi_\beta(A^{lp})$ where β is a non-zero path for A^{lp}. Then there exist non-singular diagonal $Y \in \boldsymbol{G}^0_\omega$ and $Z \in \boldsymbol{G}^0_\tau$ such that $YAZ \in \boldsymbol{H}^0_\omega$.*

Proof. Let $A^{lp} \in \boldsymbol{G}^0_\sigma$. Since A^{lp} is the direct sum of irreducible matrices, we may apply Lemma 3.2 to the irreducible blocks of B. Hence there exists a non-singular diagonal $X \in \boldsymbol{G}^0_\sigma$ such that $XA^{lp}X^{-1} \in \boldsymbol{H}^0_\sigma$. By Lemma 5.8, the result follows.

To obtain results in the other direction, we again need the hypothesis that $\boldsymbol{H}$ is a normal subgroup of $\boldsymbol{G}$.

Theorem 6.2. *Let $A \in \boldsymbol{G}^0_{\omega,\tau}$ and let $A^{lp} \in \boldsymbol{G}^0_\sigma$. Let H be a normal subgroup of G. The following are equivalent*:

(i) *For all cycles γ, $\Pi_\gamma(A^{lp}) \in \boldsymbol{H}^0$.*

(ii) *There exist non-singular diagonal matrices $Y \in \boldsymbol{G}^0_\omega$, $Z \in \boldsymbol{G}^0_\tau$ such that $YAZ \in \boldsymbol{H}^0_\omega$.*

Proof. Again, we need only consider the case when A is chainable and A^{lp} is irreducible. By Theorem 3.4 (i) is equivalent to

(ii)′ There exists and $X \in \boldsymbol{G}_\sigma^0$ such that $XA^{lp}X^{-1} \in \boldsymbol{H}_\sigma^0$, and by Lemma 5.8, (ii)′ is equivalent to (ii).

Theorem 6.3. *Let $A, C \in \boldsymbol{G}_{\omega,\tau}^0$, and suppose that $A^* = C^*$. Let $A^{lp} \in \boldsymbol{G}_\sigma^0$. Then the following are equivalent:*

(i) *If $(i, j) \in \sigma$ then there is a $u_{ij} \in \boldsymbol{G}$ such that for every closed path β from (i, j) to (i, j), $\Pi_\beta(C^{lp}) = u_{ij}^{-1}\, \Pi_\beta(A^{lp})\, u_{ij}$.*

(ii) *There exist non-singular diagonal $Y \in \boldsymbol{G}_\omega^0$ and $Z \in \boldsymbol{G}_\tau^0$ such that $C = YAZ$.*

Proof. Immediate by Theorem 3.5 and Theorem 5.6.

Corollary 6.4. *Let $A \in \boldsymbol{G}_{\omega,\tau}^0$. Then the following are equivalent:*

(i) *For all cycles γ, $\Pi_\gamma(A^{lp}) \in \{0, 1\}$.*

(i)′ *For all closed paths β, $\Pi_\beta(A^{lp}) \in \{0, 1\}$.*

(ii) *There exist non-singular diagonal $Y \in G^0$ and $Z \in \boldsymbol{G}_\tau^0$ such that $YAZ \in \{0, 1\}_{\omega,\tau}$.*

Proof 1. Put $C = A^*$ in Theorem 6.3.

Proof 2. Put $\boldsymbol{H} = \{1\}$ in Theorem 6.2.

Corollary 6.4 is the theorem by Lallement-Petrich [6], [7], mentioned in our introduction.

Corollary 6.5. *Let $A, C \in \boldsymbol{G}^0$ and suppose that $A^* = C^*$ and a_{ii} is in the center of G for $i \in \omega$. Let $A^{lp} \in \boldsymbol{G}_\sigma^0$. Then the following are equivalent:*

(i) (a) *$a_{ii} = c_{ii}$, for $i \in \omega$,*

(b) *if $(i, j) \in \sigma$ then there is a $u_{ij} \in \boldsymbol{G}$ such that for every closed path β from (i, j) to (i, j),*

$$\Pi_\beta(C^{lp}) = u_{ij}^{-1}\, \Pi_\beta(A^{lp})\, u_{ij}\,.$$

(ii) *There exists a non-singular diagonal $X \in \boldsymbol{G}_\omega^0$ such that $C = XAX^{-1}$.*

7. APPLICATIONS

Let $\mathbf{C}$ be the complex field, and $\mathbf{R}$ the real field. Also ω will be a finite set in this section.

Definition 7.1.

(i) Let $A \in \mathbf{C}_\omega$. Let Γ be the set of non-zero cycles for A. If $\gamma = (i_1, \ldots, i_m) \in \Gamma$, we put $|\gamma| = m - 1$ and $\bar{\gamma} = \{i_1, i_2, \ldots, i_{m-1}\}$.

(ii) If $A \in \mathbf{C}_\omega$, we put

$$\|A\| = \max\{|a_{ij}| : i, j \in \omega\}.$$

Theorem 7.2. *Let $A \in \mathbf{C}_\omega$ be irreducible. Then*

$$\min\{\|XAX^{-1}\| : \mathbf{C}_\omega \text{ is non-singular diagonal}\} = \max\{|\Pi_\gamma(A)^{1/|\gamma|} : \gamma \in \Gamma\}.$$

Proof. If A is the 1×1 matrix 0, there is no more to prove so assume A is not this matrix. Further, without loss of generality we assume that $a_{ij} \geqq 0$, for $i, j \in \omega$. Let

$$r(A) = \max\{|\Pi_\gamma(A)|^{1/|\gamma|} : \gamma \in \Gamma\}.$$

Since A is irreducible, $r(A) > 0$. Hence there exists a non-zero cycle $\beta = (i_1, \ldots, i_m)$ such that $r(A)^{|\beta|} = \Pi_\beta(A)$. Let $B = r(A)^{-1} A$. Then $\Pi_\beta(B) = 1$ and $\Pi_\gamma(B) \leqq 1$, for $\gamma \in \Gamma$. By Theorem 4.2 there exists a non-singular $X = \operatorname{diag}(x_i) \in \mathbf{R}_\omega$, $x_i > 0$, such that for $U = XBX^{-1}$, we have $0 \leqq u_{ij} \leqq 1$, for $i, j \in \omega$. Since $\Pi_\beta(U) = \Pi_\beta(B) = 1$, we have $u_{kl} = 1$, where $k = i_1$, $l = i_2$.

Since $XAX^{-1} = r(A)\, U$, we have

$$XAX^{-1} = r(A).$$

We still need to show that if $Y = \operatorname{diag}(y_i) \in \mathbf{R}_\omega$, $y_i > 0$, is non-singular, then

$$r(A) \leqq YAY^{-1}.$$

So let $V = Y^{-1}AY$. Then $\Pi_\beta(V) = \Pi_\beta(A) = r(A)^{|\beta|}$, whence we can find k, l such that $v_{kl} \geqq r(A)$, where $k = i_r$, $l = i_{r+1}$ for some $1 \leqq r < m$.

Remark 7.3. It is easy to see that in Theorem 7.2 the assumption that A is irreducible may be considerably relaxed and that the theorem still holds if A has a non-zero cycle. Indeed, for all $A \in \mathbf{C}_\omega$

$$\inf\{\|XAX^{-1}\| : X \text{ non-singular diagonal}\} = \max\{|\Pi_\gamma(A)|^{1/|\gamma|} : \gamma \in \Gamma\}$$

where the right hand side is defined to be 0 when $\Gamma = \emptyset$.

Example 7.4. Let

$$A = \begin{bmatrix} .01 & 1 & 0 \\ 0 & .03 & 2.5 \\ .05 & .008 & 0 \end{bmatrix}.$$

The non-zero cycles are

$$(1, 1),\ (2, 2),\ (2, 3, 2),\ (1, 2, 3, 1) = \beta, \quad \text{say}.$$

An easy computation shows that

$$r(A) = \Pi_\beta(A)^{1/|\beta|} = .5$$

where $r(A)$ is defined as in the proof of Theorem 7.2. Thus, by Theorem 7.2, there exists $X = \operatorname{diag}(x_1, x_2, x_3)$ such that the elements of XAX^{-1} lie in the interval $[0, .5]$. Furthermore, by the same theorem, the interval is minimal.

We have proved much more than an existence theorem. By Theorem 4.2 we can actually find an X. For, $2A$ satisfies the hypotheses of that theorem (where $\boldsymbol{H} = \boldsymbol{G}$ is the multiplicative group of positive reals ordered in the opposite manner from the usual one). We use the notation of the proof of Theorem 4.2. In fact if $\beta_1 = (1, 2, 3, 1)$, $\beta_2 = (1, 2)$, $\beta_3 = (1, 2, 3)$ then $P'_i = \{\beta_i\}$, $i = 1, 2, 3$. Hence $x_1 = \Pi_{\beta_1}(2A) = 1$, $x_2 = \Pi_{\beta_2}(2A) = 2$, $x_3 = \Pi_{\beta_3}(2A) = 10$. (Observe that it is unnecessary to compute the $\Pi_\gamma(2A)$ for the cycles γ). We compute

$$XAX^{-1} = \begin{pmatrix} .01 & .5 & 0 \\ 0 & .03 & .5 \\ .5 & .04 & 0 \end{pmatrix}.$$

In order to state the additive analogue of Theorem 7.2, we define $\Sigma_\beta(A) = a_{i_1 i_2} + \ldots + a_{i_{m-1} i_m}$, for any cycle $\beta = (i_1, \ldots, i_m)$.

Theorem 7.5. *Let $A \in \mathbf{R}_\omega$, (ω finite) and let*

$$s(A) = \max_{\{\gamma \in \Gamma\}} \frac{\Sigma_\gamma(A)}{|\gamma|}.$$

Then

$$s(A) = \min_{\{x_i \in \mathbf{R} : i \in \omega\}} \left(\max_{(i,j) \in \omega \times \omega} (x_i + a_{ij} - x_j) \right).$$

Theorem 7.5 is a sharpened version of the theorem of Afriat [1], [2] and Fiedler-Pták [4], which was mentioned in the introduction.

APPENDIX. APL STATEMENT AND EXAMPLES

Suppose A is a finite real matrix: The following APL program may be used to compute an approximation A^{lp}. This approximation will be precise when there are no round off errors in the division.

```
    ∇ALP[□] ∇
    ∇T ← ALP A; J; K: X
[1] ∇T ← X[(K × ⌊J ÷ K) ∘ . + 1 + (K ← 1 ↓ ⍴A) | J ← J − 1] ÷
      ÷ X[J ∘ . + 0 × J ← (X ≠ 0)/⍳⍴X ←, A]
```

∇

C

1	2	5
4	0	0
10	0	0

ALP C

1	2	5	1	1
0.5	1	2.5	0.5	0.5
0.2	0.4	1	0.2	0.2
1	0	0	1	1
1	0	0	1	1

A

1	0	5
4	2	0
0	0	10

ALP A

1	5	1	0	5
0.2	1	0.2	0	1
1	0	1	0.5	0
2	0	2	1	0
0	1	0	0	1

B

1	0	5
4	2	0
10	0	0

ALP B

1	5	1	0	1
0.2	1	0.2	0	0.2
1	0	1	0.5	1
2	0	2	1	2
1	0	1	0	1

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