Diagonal Transformations of Graphs on Closed Surfaces

By

Seiya NEGAMI* and Atsuhiro NAKAMOTO†

0. Introduction

A *graph* is a topological space consisting of finitely many points, called *vertices,* and of simple arcs, called *edges,* each of which joins a pair of vertices. In particular, an edge is called a *self-loop* if both two ends coincide, and a pair of edges are called *multiple edges* if they join the same pair of vertices. Our graph is usually assumed to be *simple,* that is, to have neither self-loops nor multiple edges.

Our main object in this paper is an *embedding* of a graph G into or on a closed surface F^2. To specify how G is embedded on F^2, it is effective to regard an embedding as an injective continuous map $f: G \to F^2$. We however deal with G and $f(G)$ as the same one intuitively. Thus, $G=f(G)$ is a subset or a subspace of F^2. An embedding of G on F^2 is called a *2-cell embedding* if each component of F^2-G, called a *face* of G, is homeomorphic to the open 2-cell $\{(x, y)\in \boldsymbol{R}^2 : x^2+y^2<1\}$ and G is said to be *2-cell embedded* on F^2 in this case.

Let G be a graph 2-cell embedded on F^2 and choose two faces A and B of G so that they share an edge e. Then $A \cup B \cup e$ forms an open 2-cell region on F^2 with circular boundary and e is placed in this region as one of its diagonals. Roughly speaking, the *diagonal transformation* appearing in the title is to replace this diagonal e with another diagonal. In particular, we focus the diagonal transformations in triangulations and in quadrangulations and discuss the equivalence over such classes of graphs under diagonal transformations. This paper presents a survey and several observations for further research on this topic, with Negami's and Nakamoto's recent results in [9] and [6].

*Department of Mathematics, Faculty of Education, Yokohama National University, 156 Tokiwadai, Hodogaya-Ku, Yokohama 240, Japan. Email: negami@ms.ed.ynu.ac.jp
†Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-Ku, Yokohama 223, Japan. Email: nakamoto@comb.math.keio.ac.jp

Our terminology for graph theory is quite standard. For example, we use the notations K_n, $K_{m,n}$, C_n and so on to denote the *complete graph* with n vertices, the *complete bipartite graph* with partite sets of size m and n, the *cycle* of length n and et al. See [4] for topological graph theory.

1. Classical results on triangulations

A *triangulation* on a closed surface F^2 is a simple graph G embedded on F^2 so that (i) each face of G is bounded by a cycle of length 3 and that (ii) any two faces have at most one edge in common. Roughly speaking, a triangulation divides a closed surface into several triangles. In topology, a triangulation on a closed surface is regarded as a 2-dimensional simplicial complex together with faces, but our triangulation is only the 1-skeleton of such a complex.

The smallest example of such triangulations is the complete graph K_4 with four vertices on the sphere, which looks like the tetrahedron. The complete graph K_3 with three vertices on the sphere also divides the sphere into two triangular faces, but K_3 is not a triangulation under our definition since those faces share the three edges. Observe that if two triangular faces share two edges, then the other two edges, not shared, will form a pair of multiple edges if they don't coincide. This implies that if G is a simple graph embedded on a closed surface with Condition (i) but not (ii), then G has only two faces which meet each other along their whole boundary. Thus, we can define a triangulation on F^2 as a simple graph embedded on F^2 with only triangular faces, allowing the unique exception, K_3 on the sphere.

Now let G be a triangulation on a closed surface F^2 and consider an edge ac which two faces abc and acd share in G. A *diagonal flip* is to replace ac with another diagonal bd of the rectangle $abcd$, as shown in Figure 1. This is just a local deformation of G and the resulting graph G' also divides F^2 into

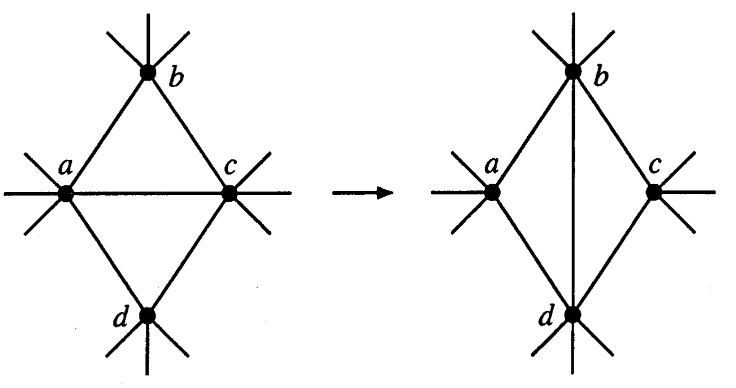


Figure 1: Diagonal flip

triangular faces. However, if there is an edge bd in G, the diagonal flip at ac yields multiple edges between b and d, and hence G' is not a triangulation. We forbit such a diagoanl flip not to break the simpleness of graphs.

The question is whether or not any two triangulations on a closed surface can be transformed into each other by diagonal flips. They are said to be *equivalent* under diagonal flips if they can. Of cource, they must have the same number of vertices and of edges if they are equivalent to each other. Note that triangulations with the same number of vertices have the same number of edges by Euler's formula.

The following theorem is the origin of this topic and was proved by Wagner in his paper [16] written in German. An article introducing this theorem can be found in Ore's book [11] on Four Color Problem. His proof is very easy and constructive as shown below, (The meanings of "up to ambient isotopy" and "up to homeomorphism" will be explained later in the next setction.)

THEOREM 1 (Wagner, 1936). *Any two triangulations on the sphere with the same number of vertices are equivalent to each other under diagonal flips, up to ambient isotopy.*

Proof. Let G be any triangulation of the sphere. We represent it as a plane graph, so that there is a big triangle u_0vw of G and the triangular region with boundary u_0vw contains the whole of G. Let $v, x, y, z, \ldots, w$ be the neighbors of u_0, lying clockwise in this order. First suppose that u_0 has degree at least 4. (If $\deg u_0=4$, then $z=w$.) If y is not adjacent to v, then we replace u_0x with yv to decrease the degree of u_0. Otherwise, the edge yv is placed outside the rectangle u_0vxy and the cycle u_0vy separates x and z. Thus, x is not adjacent to z and hence u_0y can be replaced with xz by a diagonal flip. Repeating these deformations, we can reduce the degree of u_0 to 3.

Now suppose that u_0 has precisely three neighbors v, u_1, w and consider the subgraph G_1 of G bounded by the triangle u_1vw. By the same arguments, we can deform G_1 by diagonal flips inside u_1vw so that u_1 has degree 3 in G_1 and hence degree 4 in G afterward. Repeating these arguments with subgraphs $G_1, G_2, \cdots$, we get finally the triangulation which consists of the triangle u_0vw and the path $u_1u_2\cdots u_m$ of vertices of degree 4 adjacent to both v and w. This is the standard form of the spherical triangulations, as shown in Figure 2. Therefore, any two triangulations of the sphere with the same number of vertices can be transformed into each other by diagonal flips via this standard form. ■

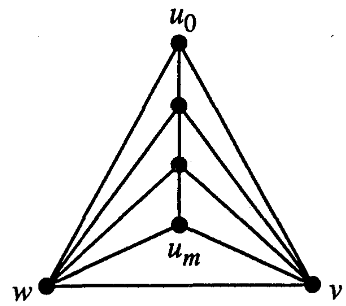


Figure 2: The standart form Δ_m of spherical triangulations

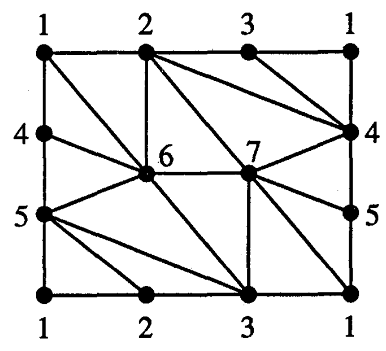


Figure 3: The unique embedding of K_7 on the torus

The following three theorems have been proved by Dewdney in [3] and by Negami and Watanabe in [10], as natural generalizations of Wagner's theorem for other surfaces.

THEOREM 2 (Dewdney, 1973). *Any two triangulations on the torus with the same number of vertices are equivalent to each other under diagonal flips, up to homeomorphism.*

THEOREM 3 (Negami and Watanabe, 1990). *Any two triangulations on the projective plane with the same number of vertices are equivalent to each other under diagonal flips, up to ambient isotopy.*

THEOREM 4 (Negami and Watanabe, 1990). *Any two triangulations on the Klein bottle with the same number of vertices are equivalent to each other under diagonal flips, up to homeomorphism.*

As shown in the proof of Wagner's theorem, we have the "standard form" of the spherical triangulations with notation Δ_m and all the triangulations with $m+3$ vertices on the sphere can be generated from Δ_m by diagonal flips. Similarly to the spherical case, we can choose suitably some standard forms for the torus, the projective plane and the Klein bottle.

For example, the unique embedding of K_7 on the torus, given in Figure 3, is a unique minimal triangulation of the torus. Here a triangulation of a closed surface F^2 is said to be *minimal* if it has the fewest vertices among all the possible triangulations on F^2. Add Δ_m $(m \geq 0)$ to one face of K_7 arbitrarily chosen. The resulting graph, denoted by $K_7+\Delta_m$, is obviously a triangulation of the torus which has precisely $7+m$ vertices. (Define $K_7+\Delta_0$ formally as K_7.) If we believe Theorem 2, any triangulation with $7+m$ vertices on the torus will be obtained from $K_7+\Delta_m$ by diagonal flips. Thus, we may call $K_7+\Delta_m$ the standard form of the toroidal triangulations.

Similarly, we have the unique embedding of K_6 on the projective plane, as the unique minimal triangulation of the projective plane (see the left in the Figure 10) and call $K_6+\Delta_m$ the standard form of the projective-planar triangulations with $6+m$ vertices. On the other hand, there are precisely three minimal triangulations on the Klein bottle, shown in Figure 4. Negami and Watanabe have chosen the first one from the three as the standard form of the Klein-bottlal triangulations in [10], but we can choose any one from the other

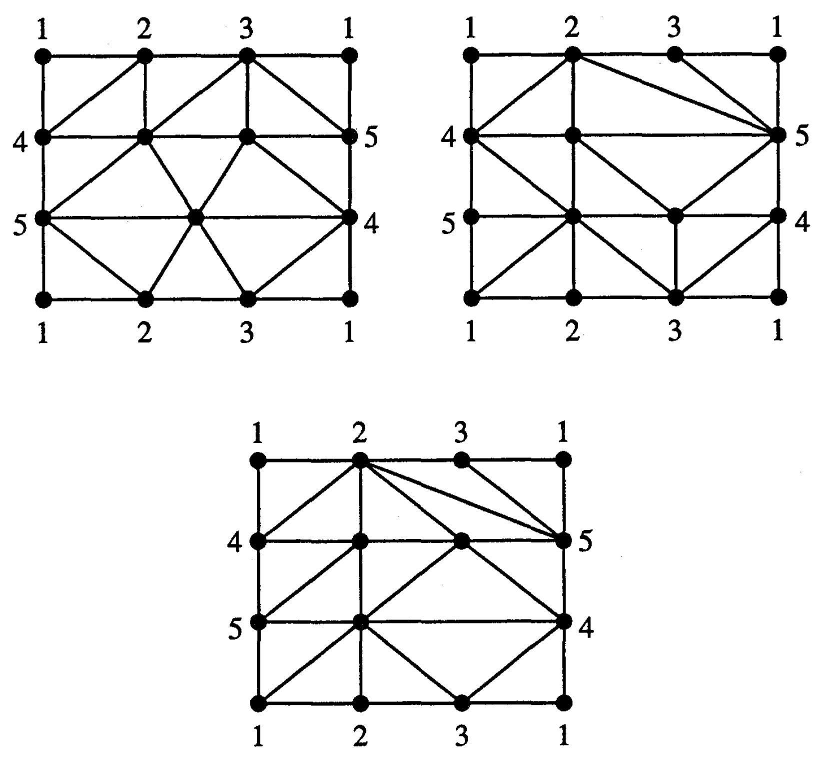


Figure 4: The minimal triangulations on the Klein bottle

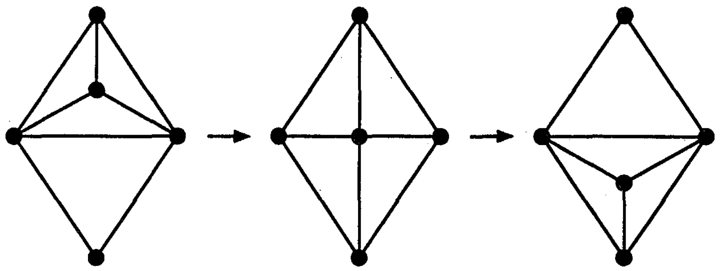


Figure 5: Moving a vertex of degree 3

two. Observe that any two of the three minimal triangulations of the Klein bottle can be transformed into each other by diagonal flips.

There is a common key fact for the proofs in [3] and in [10], which is that a vertex of degree 3 can be moved to any place in a triangulation. For example, two diagonal flips carry a vertex of degree 3 in the upper triangle to the lower in Figure 5. Since a triangulation is connected, we can move such a vertex step by step to a face arbitrarily chosen in the triangulation.

From this fact, we can establish the following logic. Let G_1 and G_2 be two triangulations with the same number of vertices on the same closed surface. Suppose that we can deform both of them into ones which have vertices of degree 3, say $v_i \in V(G_i)$, by diagonal flips. By induction hypothesis, $G_1 - v_1$ can be transformed into $G_2 - v_2$ by a sequence of diagonal flips. Translate the sequence naturally into a sequence of diagonal flips starting from G_1. When one of two triangles sharing an edge to which a diagonal flip is applied contains a vertex v of degree 3 corresponding to v_1, then we first move v to another face far from these triangles and next carry out the diagonal flip. Through this sequence, G_1 can be transformed into G_2 with v_2 moved to somewhere. Finally we get G_2, modifying the position of v_2 in G_2, and conclude that G_1 and G_2 are equivalent to each other under diagonal flips.

To complete the proof, we have to discuss about whether or not the assumption of G_1 and G_2 actually holds and about the initial stage of our induction. In fact, these two problems can be solved together as follows.

In general, a triangulation G on a closed surface F^2 is said to be *pseudo-minimal* if G is not equivalent to any triangulation which has a vertex of degree 3 under diagonal flips. (This terminology has appeared in [9].) Then the assumption of G_1 and G_2 can be rephrased into that neither G_1 nor G_2 are pseudo-minimal. So it suffices to determine what the pseudo-minimal triangulations are.

Let G be a pseudo-minimal triangulation on a closed surface F^2 and deform G by diagonal flips so as to minimize its minimum degree $\delta(G)$. Choose a

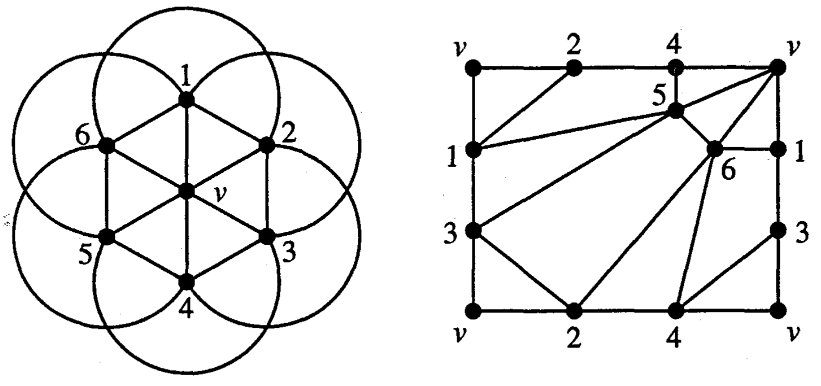


Figure 6: Finding pseudo-minimal triangulations

vertex v of G with $\deg v=\delta(G)=n$ and let $u_1, \ldots, u_n$ be the neighbors of v lying around v in this cyclic order. If u_i and u_{i+2} were not adjacent in G, then we could switch the diagonal vu_{i+1} to u_iu_{i+2} in the rectangle $vu_iu_{i+1}u_{i+2}$ and the degree of v would be reduced by one. This contradicts the assumption of G after the deformation and the choice of v. Thus, G must have edges $u_1u_3, \ldots, u_{n-2}u_n, u_{n-1}u_1, u_nu_2$. Then let F_n denote the wheel structure around v with these chords. It is clear that if n is sufficiently large, then F_n cannot be embedded on the torus, the projective plane and the Klein bottle.

For example, F_7 cannot be embedded on the torus and moreover the embedding of F_6 is strictly restricted on the torus. In fact, it is unique as shown in Figure 6. Discussing what happens in quadrilateral regions, we can conclude that the whole G including this embedding of F_6 is isomorphic to K_7. On the other hand, if v lies on either F_4 or F_5, we can find a sequence of diagonal flips to make a vertex of degree 3, but this is contrary to the assumption of G being pseudo-minimal. Thus, K_7 is the unique pseudo-minimal triangulation of the torus. This implies that the initial stage of our induction is trivial for the torus and that the required assumption holds for any toroidal triangulation with 8 or more vertices.

By similar arguments, Negami and Watanabe [10] proved Theorems 3 and 4 for the projective plane and the Klein bottle. Note that those depend on what the surfaces are and the higher the genus of a closed surafce is, the harder the arguments are. They will not work to prove Wagner's theorems for general surfaces.

2. Equivalence of graphs on surfaces

In the previous section, we did not refer to how we compare two given triangulations. What does it mean that two triangulations are the same? Here we will discuss it to understand the more delicate meaning of Wagner's theorem.

More generally, let G_1 and G_2 be two graphs embedded on closed surfaces F_1^2 and F_2^2, respectively. They are said to be *homeomorphic* to each other if there is a homeomorphism $h: F_1^2 \to F_2^2$ with $h(G_1)=G_2$ which induces an isomorphism from G_1 to G_2. In this case, we say that G_1 and G_2 are the same one, *up to homeomorphism.*

For example, consider the unique embedding of K_7 on the torus, given in Figure 3. First, identify the top and bottom edges of the rectangle to get a cylinder horizontally placed and next identify the both ends of this cylinder to get one embedding of K_7 on the torus. Now twist one end of the cylinder several times before identifying the two ends in the second step and identify them. Then we obtain another embedding of K_7 on the torus, which looks more complicated than the first one. It is clear that the natural correspondence between the two K_7's extends to a homeomorphism between the two tori. That is, these two embeddings of K_7 on the tori are homeomorphic to each other even if their appearances are completely different.

Now recall the statement of Dewdney's theorem, Theorem 2 in the previous section. This must guarantee the existence of a sequence of diagonal flips which transforms the first embedding of K_7 into the second. However, any diagonal flip cannot be applied to any edge of K_7 since any pair of vertices are already joined by an edge in K_7. That is, it is impossible to transform the first K_7 into the second only by diagonal flips. Thus, these two K_7's must be the same one to solve this dilemma. That is the reason why the phrase "up to homeomorphism" is added to the statement of Theorem 2, as well as Theorem 4.

On the other hand, such a modification is not needed for the original version of Wagner's theorem since the proof gives us an algorithm which deforms a given triangulation on the sphere directly into the standard form. Precisely speaking, we need continuous deformations, in addition to diagonal flips, to move a triangulation freely around the sphere and to adjust the curvatures of its edges. Such a deformation will be formulated as follows.

An *ambient isotopy* $H: F^2 \times I \to F^2$ is a continuous map such that the map $h_t: F^2 \to F^2$, defined by $h_t(x)=H(x, t)$, is a homeomorphism over F^2 for each $t \in I$ with h_0 the identity map of F^2, where I stands for the interval $[0,1]$. Two graphs G_1 and G_2 embedded on F^2 are said to be *ambient isotopic* or simply *isotopic* to each other if there is an ambient isotopy $H: F^2 \times I \to F^2$ such that $h_1(G_1)=G_2$. By definition, $h_0(G_1)=G_1$. Intuitively, we can say that an ambient isotopy H deforms G_1 into G_2 continuously, regarding $h_t(G_1)$ as the position of

G_1 at time t.

Note that we classified the pseudo-minimal triangulations up to homeomorphism by a tacit consent in the sketch of proofs in the previous section. Thus, our combinatorial arguments usually conclude everything only up to homeomorphism. In particular, it can be proved in a standard topological argument that any homeomorphisim $h: P^2 \to P^2$ over the projective plane P^2 extends to an ambient isotopy $H: P^2 \times I \to P^2$ so that $h_1 = h$. This implies that any fact up to homeomorphism also holds up to ambient isotopy for the projective plane, as well as Theorem 3.

Recall that we apply any diagonal flip to a triangulation only if it preserves the simpleness of the triangulation. Note that if we do not mind diagonal flips' breaking the simpleness, our problem will be easy, so that we can prove a general theorem up to ambient isotopy, as below.

Up to the end of this section, we allow graphs to have loops and multiple edges. Such a graph G embedded on a closed surface F^2 is called a *pseudo-triangulation* on F^2 if the boundary of each face of G is just a closed walk of length 3, possibly not a cycle. Thus, K_3 on the sphere is now a pseudo-triangulation. Place a vertex v_0 with one loop e_0 on the sphere so that e_0 divides the sphere into two monogonal regions R_1 and R_2, and put a vertex v_1 to R_1 and v_2 to R_2 with edges v_0v_1 and v_0v_2 added. The resulting graph is also a pseudo-triangulation on the sphere.

A diagonal flips in a pseudo-triangulation is defined as well as in a usual triangulation, but is not required to preserve the simpleness of graphs.

THEOREM 5 *Any two pseudo-triangulations on a closed surface with the same number of vertices are equivalent to each other under diagonal flips, up to ambient isotopy.*

Proof. Let G_1 and G_2 be two pseudo-triangulations on a closed surface F^2. We shall deform G_1 by diagonal flips and G_2 by an ambient isotopy of F^2, as follows. Put G_1 and G_2 together on F^2 so that $V(G_1) = V(G_2)$.

First deform G_2 by an ambient isotopy fixing $V(G_2)$ to minimize the crossings of edges of G_1 and G_2. For example, let A be a face of G_1 with boundary cycle uvw and e an edge of G_2 incident to u which goes through A. If e reaches to an inner point x of the edge uv, then we push the arc of e between u and x toward uv and push it off moreover to eliminate the crossing point x. After these deformations, we can assume either that e is placed along uv or uw in parallel, or that e crosses the opposite edge vw.

In the latter case, if vw is shared by two distinct faces A and B (Figure 7 (i)), then we replace vw with the other diagonal uz of the rectangular region

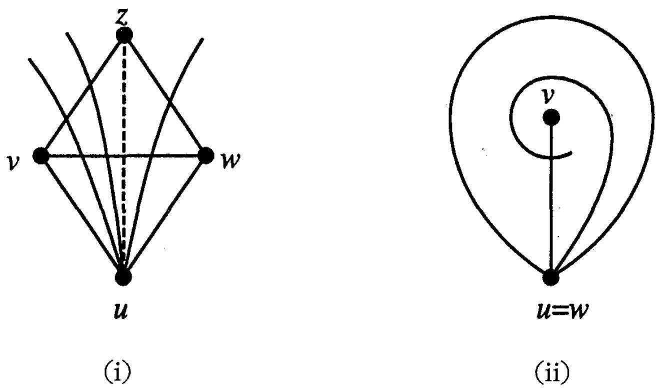


Figure 7: Eliminating crossings

$A \cup B$ with boundary *uvzw*, by the diagonal flip, so that the arc $e \cap (A \cup B)$ is contained in one of trianglar faces with boundaries *uvz* and *uwz*. If both sides of *vw* are incident to the face A (Figure 7 (ii)), then *vw* has to coincide with *vu* and is contained in the 2-cell region bounded by the self-loop *uw*. The arc of e starting from u goes around v, crossing *uv* several times, and either terminates at v or comes back. In this case, we can find an ambient isotopy which eliminates all the crossings on *uv*.

Eliminating crossings in the above ways, we can assume finally that the edges of G_1 and G_2 do not meet each other except at their ends and are placed in parallel in pairs. Now there is an ambient isotopy which carries each edge of G_2 onto the corresponding edge of G_1. ■

3. Irreducible triangulations

As is mentioned in Section 1, the final stages of their proofs of Theorems 2, 3 and 4 depend strongly on the individual surfaces although they have a large common part. This causes the difficulty in generalizing those theorems for other surfaces. To solve this problem, Negami has proposed recently a nice idea which connects the diagonal flip with the contraction of edges and proved the following for general surfaces in [9].

THEOREM 6 (Negami, 1992). *For any closed surface F^2, there exists a positive integer $N=N(F^2)$ such that if G_1 and G_2 are two triangulations on F^2 with $|V(G_1)|=|V(G_2)|\geq N$, then G_1 and G_2 are equivalent under diagonal flips, up to homeomorphism.*

For example, the integers N for the sphere, the torus, the projective plane and the Klen bottle are 4, 7, 6, 8 in order, which follows from Theorems 1, 2, 3 and 4. Each of these values is equal to the minimum number of vertices that the triangulations in each surface have, here denoted by $\Delta(F^2)$. Of cource, if any two triangulations on F^2 with the same number of vertices are equivalent to each other under diagonal flips, the value of $N(F^2)$ will coincide with $\Delta(F^2)$.

In general, Ringel determined in [12] and [14] the precise values of $\Delta(F^2)$ for all the closed surfaces and showed that the complete graphs attain those values in most of cases. Since the complete graph cannot be deformed into any other graph by diagonal flips, if $K_{\Delta(F^2)}$ triangulates F^2 and has two or more embeddings of different homeomorphism types, then $N(F^2)$ must be greater than $\Delta(F^2)$.

Theorem 6 says nothing about the value of $N(F^2)$ but only its existence, while Negami's proof in [9] suggests how to determine $N(F^2)$, related to the notion of irreducible triangulations, defined below.

Let G be a triangulation on a closed surface F^2 and ac an edge of G. Then there are two triangles abc and acd sharing this edge ac in G. The *contraction* of the edge ac is to shrink ac to a point $a=c$ until the two faces abc and acd degenerate to edges $ab=cb$ and $ad=cd$, as shown in Figure 8, so that we obtain another triangular embedding of a graph G'. However we forbit ourselves to carry out a contraction of an edge, as well as a diagonal flip, if it deforms a given triangulation into a non-simple graph, that is, if a and c have another common neighbor other than b and d. If a triangulation G can be transformed into another G' by contracting edges under this rule, then G is said to be *contractible* to G'.

A triangulation G on a closed surface F^2 is said to be *irreducible* if G is not contractible to any other triangulation on F^2. For example, the complete graph K_4 on the sphere is irreducible since the contraction of each edge yields K_3 on the sphere, which is not a triangulation. However, this example is so special.

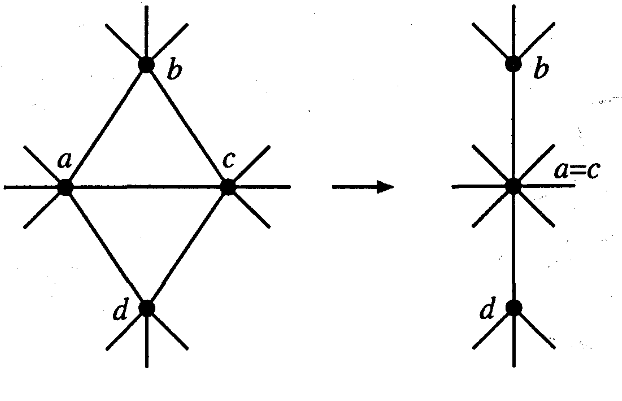


Figure 8: Contraction of an edge

In fact, an irreducible triangulation can be defined, with the unique exception K_4, as one each of whose edge is contained in at least three triangles, two of which bound faces but the other do not. Obviously by defintion, any triangulation on F^2 is contractible to one of irreducible triangulations of F^2.

One of the key facts in Negami's proof is that if a triangulation G is contractible to another triangulation T, then G is equivalent to $T+\Delta_m$ under diagonal flips with $|V(G)|=|V(T)|+m$, where $T+\Delta_m$ denotes any triangulation obtained from T by adding the standard form Δ_m of the spherical triangulation with $m+3$ vertices to any face of T. Note that this notation $T+\Delta_m$ is so ambiguous since it depends on the choice of a face to which Δ_m is added, but clearly any two triangulations denoted by $T+\Delta_m$ are equivalent to each other under diagonal flips.

Figure 9 illustrates how a contraction of an edge is related to diagonal flips, which is the essence of his proof. Roughly speaking, the contraction of an edge can be realized by deleting a vertex of degree 3 following diagonal flips. This implies that if G is not irreducible, then G is not pseudo-minimal.

Another key fact is that for any two triangulations G_1 and G_2, there is a non-negative integer m_1 and m_2 such that $G_1+\Delta_{m_1}$ is equivalent to $G_2+\Delta_{m_2}$ under diagonal flips, where G_1 and G_2 are not assumed to have the same number of vertices. In this case, G_1 and G_2 are said to be *stably equivalent* to each other under diagonal flips. To prove this fact, consider a common refinement G of G_1 and G_2, and observe that G is contracitible to both G_1 and G_2. By the first fact, G is equivalent to both $G_1+\Delta_{m_1}$ and $G_2+\Delta_{m_2}$ and hence so they are. Note that if $G_1+\Delta_{m_1}$ is equivalent to $G_2+\Delta_{m_2}$, then $G_1+\Delta_{m_1+k}$ is equivalent to $G_2+\Delta_{m_2+k}$ for any positive integer $k>0$.

To complete the proof of Theorem 6, we need one more fact that there

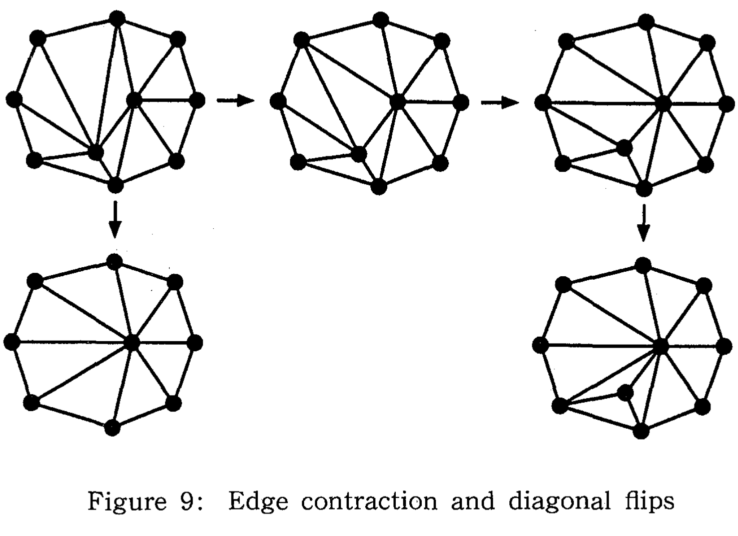


Figure 9: Edge contraction and diagonal flips

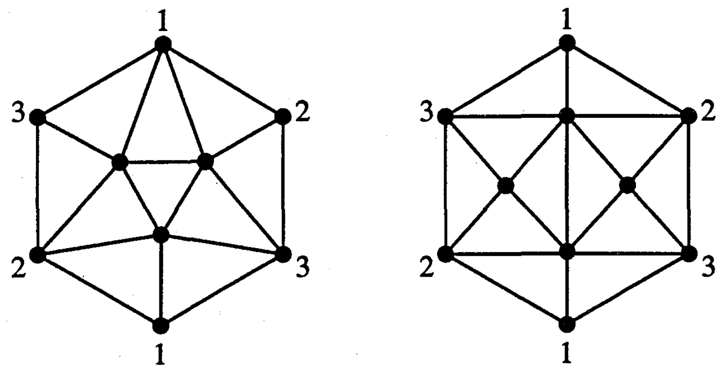


Figure 10: Irreducible triangulations of the projective plane

exist only finitely many irreducible triangulations of a given closed surface. To show this, Negami used Wagner's conjecture, whose affirmative solution is given in [15]. In general, a graph H is called a *minor* of a graph G if H can be obtained from G by deleting and contracting edges. Wagner's conjecture states that every infinite series of graphs includes a pair of graphs one of which is a minor of the other and this fact is known to be independent of Peano's axioms. It will be interesting to try an elementary proof of the finiteness of the irreducible triangulations.

For example, it is easy to show in an elementary way that the sphere has unique irreducible triangulation as K_4. Barnette also showed in [2] that there are precisely two irreducible triangulations of the projective plane, given in Figure 10, one of which is the unique embedding of K_6 on the projective plane. It is easy to see that the second one with 7 vertices can be transformed into $K_6+\Delta_1$ by three diagonal flips. This implies that any triangulation on the projective plane is contractible to one of these and hence is equivalent to $K_6+\Delta_m$ under diagonal flips in either case, which follows from the arguments in Section 1.

On the other hand, there exist precisely 21 irreducible triangulations of the torus. Lavrenchenko determined their complete list in [5]. The only pseudo-minimal one among those is the unique embedding of K_7 as is mentioned at the end of Section 1 and hence any triangulation is equivalent to $K_7+\Delta_m$.

The irreducible triangulations of the Klein bottle have never been classified yet. For example, prepare two copies of irreducible triangulations in Figure 10 and glue them together along a pair of faces. Then we obtain an irreducible triangulation with 9, 10 or 11 vertices on the connected sum of two projective plane, which is homeomorphic to the Klein bottle. There will be another type of irreducible triangulations on the Klein bottle which do not split in this way.

In general, let $T_1, \dots, T_n$ be all the irreducible triangulations of a given closed surface F^2. Then there is a posivite interger $N=N(F^2)$ such that if $|V(T_i+\Delta_{m_i})|=|V(T_j+\Delta_{m_j})|\geq N$, then $T_i+\Delta_{m_i}$ is equivalent to $T_j+\Delta_{m_j}$ under diagonal flips, since they are stably equivalent to each other and finite in number. Choose any triangulation G on F^2 so that G has at least N vertices and suppose that G is contractible to T_i. Then G is equivalent to $T_i+\Delta_{m_i}$ with a suitable m_i and hence to $T_1+\Delta_{m_1}$ with $|V(G)|=|V(T_1)|+m_1$ by the first key fact and the property of N. Therefore, any two triangulations on F^2 with at least N and the same number of vertices can be transformed into each other by diagonal flips via the standard form $T_1+\Delta_{m_1}$ chosen here.

These arguments suggest that if we have the complete list of irreducible triangulations of a closed surface F^2, then the value of $N(F^2)$ in Theorem 6 will be determined. More precisely, the number $N(F^2)$ is equal to the minimum number N such that for any two of pseudo-minimal triangulations T_i and T_j, if $|V(T_i+\Delta_{m_i})|=|V(T_j+\Delta_{m_j})|\geq N$, then $T_i+\Delta_{m_i}$ is equivalent to $T_j+\Delta_{m_j}$, under diagonal flips. Note that the complete list of pseudo-minimal triangulations is a subset of that of irreducible triangulations and that any triangulation obtaited from a pseudo-minimal one by diagonal flips is also pseudo-minimal. Thus, if $N(F^2)>\Delta(F^2)$, then there will be a class of pseudo-minimal triangulations, but not minimal, which is closed under diagonal flips. However, we conjecture that any pseudo-minimal triangulation is minimal. See [9] for the details.

4. Bipartite quadrangulations

Now we will deal with a new class of graphs on closed surfaces, called quadrangulations and describe Nakamoto's results on them in [6] with additional observations. Some part goes in parallel to the previous arguments on triangulations, but there are several phenomena peculiar to quadrangulations.

A graph G 2-cell embedded on a closed surface F^2 is called a *quadrangulation* on F^2 if G is simple and if each face of G is bounded by a cycle of length 4. Thus, the simplest example of quadrangulations is the cycle C_4 of lenght 4 on the sphere. Recall that we exclude $K_3=C_3$ on the sphere in case of triangulations. If we respected this convention, then we should exclude C_4 and should take the complete bipartite graph $K_{2,3}$ on the sphere with three faces as the smallest quadrangulation on the sphere. However, any choice from these does not play an essential role in our arguments below.

To construct another example of a typical quadrangulation, consider a rectangle subdivided by an $(n+1)\times(m+1)$ grid so that its vertical and horizontal edges have length n and m respectively and identify each opposite pair of edges. Then we obtain a quadrangulation on the torus which is isomorphic to $C_n \times C_m$ as a graph. If both n and m are even, then $C_n \times C_m$ is *bipartite*, that is, we can assign black and white to its vertices so that any pair of adjacent vertices get different colors. Otherwise, $C_n \times C_m$ contains an odd cycle. This example guarantees the existence of non-bipartite quadrangulations although all the quadrangulations on the sphere are bipartite.

Here we define two diagonal transformations for the quadrangulations as follows. Let $v_1v_2v_3v_4$ and $v_1v_4v_5v_6$ be the boundary cycles of two faces sharing an edge v_1v_4. Then the *diagonal slide* is to replace v_1v_4 with another diagonal v_2v_5 (or v_6v_3) of the hexagonal region bounded by the cycle $v_1v_2v_3v_4v_5v_6$, as shown in Figure 11. We do not carry out the diagonal slide, as well as a diagonal flips, if it yields a non-simple graph.

Another transformation is called the *diagonal rotation*, which generalizes the notion defined in [6]. Let v be a vertex of degree n in a quadrangulation G and let $vw_1b_1w_2, vw_2b_2w_3, \dots, vw_nb_nw_1$ be the boundary cycles of the n faces incident to v in G. The diagonal rotaiton around v is to rotate the n edges $vw_1, \dots, vw_n$ around v to be $vb_1, \dots, vb_n$ in the $2n$-gonal region bounded by the cycle $w_1b_1 \cdots w_nb_n$. Clearly, this does not destroy the simpleness of a quadrangulation. Nakamoto has introduced only a diagonal rotation around a vertex of degree 2 and we also will use only it in this section.

Notice that both a diagonal slide and a diagonal rotation preserve the bipartiteness of quadrangulations, as the coloring of vertices with black and white suggests in both Figures 11 and 12. That is, a quadrangulation G is bipartite if and only if so is any quadrangulation G' obtained from G by one of these deformations. Thus, it is impossible to transform a bipartite quadrangulation into a non-bipartite one. It is another important fact that any diagonal slide does not change the number of black vertices and of white

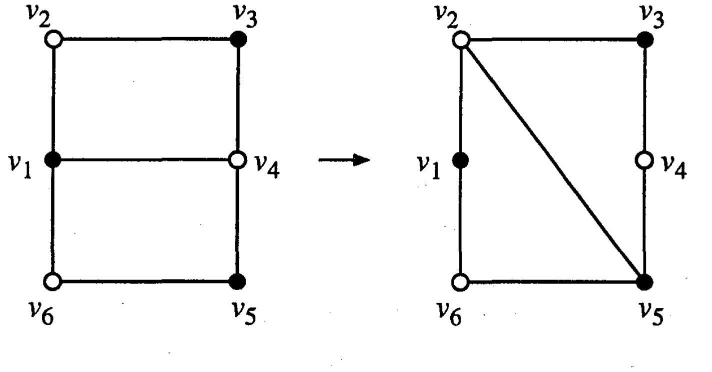


Figure 11: Diagonal slide

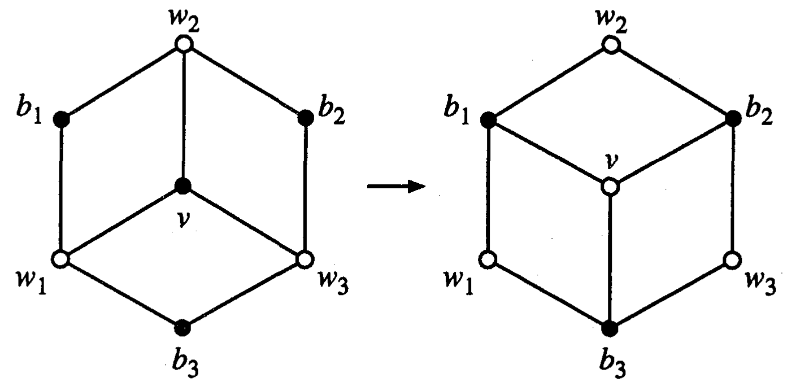


Figure 12: Diagonal rotation

vertices while a diagonal rotation around v changes the color of v in a bipartite quadrangulation with black and white vertices. This implies that two bipartite quadrangulations with partite sets of different sizes cannot be transformed into each other by only diagonal slides. Therefore, we need also diagonal rotations to establish the following theorem.

THEOREM 7 (Nakamoto, 1993). *For any closed surface F^2, there exists a positive integer $M=M(F^2)$ such that if G_1 and G_2 are two bipartite quadrangulations on F^2 with $|V(G_1)|=|V(G_2)|\geq M$, then G_1 and G_2 are equivalent under diagonal slides and diagonal rotations around vertices of degree 2, up to homeomorphism.*

The diagonal rotation plays the role to change the sizes of partite sets of a quadrangulation, so we can expect that only diagonal slides are needed to transform bipartite quadrangulations with partite sets of the same size. The following theorem can be regarded as a refinement of Theorem 7 and is the one as we expect. Here we assume that any bipartite quadrangulation G has a fixed coloring of vertices with black and white and denote the sets of black and white vertices by $V_B(G)$ and $V_W(G)$, respectively.

THEOREM 8 (Nakamoto, 1993). *For any closed surface F^2, there exist a pair of positive integers $B=B(F^2)$ and $W=W(F^2)$ such that if G_1 and G_2 are two bipartite quadrangulations on F^2 with $|V_B(G_1)|=|V_B(G_2)|\geq B$ and $|V_W(G_1)|=|V_W(G_2)|\geq W$, then G_1 and G_2 are equivalent under diagonal slides, up to homeomorphism.*

Let G be a quadrangulation on a closed surface F^2 and $abcd$ the boundary cycle of a face A of G. The *contraction* of the face A at a and c is to shrink the region of A until the two paths bad and bcd coincide, as shown in Figure 13. There are two ways to contract one face at two diagonal pairs of vertices on its boundary. A face A with boundary cycle $abcd$ as above is said to be *contractible* (at a and c) if the contraction of A (at a and c) yields another

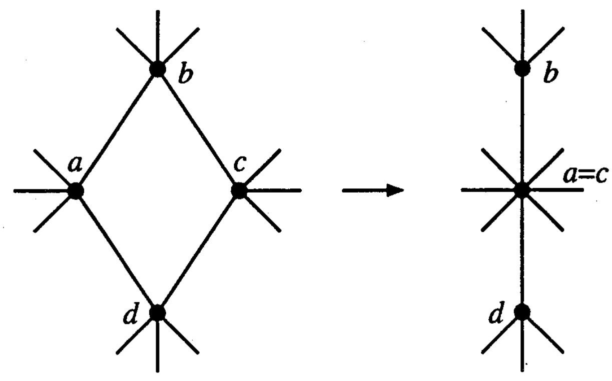


Figure 13: Contraction of a face

quadrangulation G', that is, if G' is simple. A quadrangulation G is said to be *contractible* to another quadrangulation G' if we can transform G into G' by contractions of faces, preserving the simpleness of quadrangulations.

As the reader might expect, the contraction of a face corresponds to the contraction of an edge in a triangulation and plays the same role. Thus, an *irreducible* quadrangulation on a closed surface is defined as one that is not contractible to any other quadrangulation on the same surface, and a quadrangulation G is irreducible if and only either if any diagonal pair of vertices a and c on the boundary cycle $abcd$ of each face has the third common neighbor other than b and d or if they are joined by an edge ac in G. The second condition however implies that G is not bipartite, and hence any irreducible bipartite quadrangulation, we deal with here, satisfies only the first one.

Figure 14 suggests how the face contraction is related to the diagonal slides, which is very similar to the relationship between the edge contraction and the diagonal flips for triangulations. Roughly speaking, any face contraction can be realized by deletion of a vertex of degree 2 following several diagonal slides. From this, we can conclude that if a quadrangulation G is contractible to an irreducible quadrangulation T, then G is equivalent to T with a suitable number of vertices of degree 2 added in order, denoted by $T+\Gamma_m$, under diagonal slides and diagonal rotations. Note that we need here the diagonal rotations to modify the sizes of partite sets.

For example, every quadrangulation G on the sphere is contractible to the smallest quadrangulation C_4 and is equivalent to $C_4+\Gamma_m$ with $|V(G)|=m+4$. Thus, any two quadrangulations on the sphere are equivalent to each other under diagonal slides and diagonal rotations around vertices of degree 2. We shall show a similar observation for the projective plane at the end of this

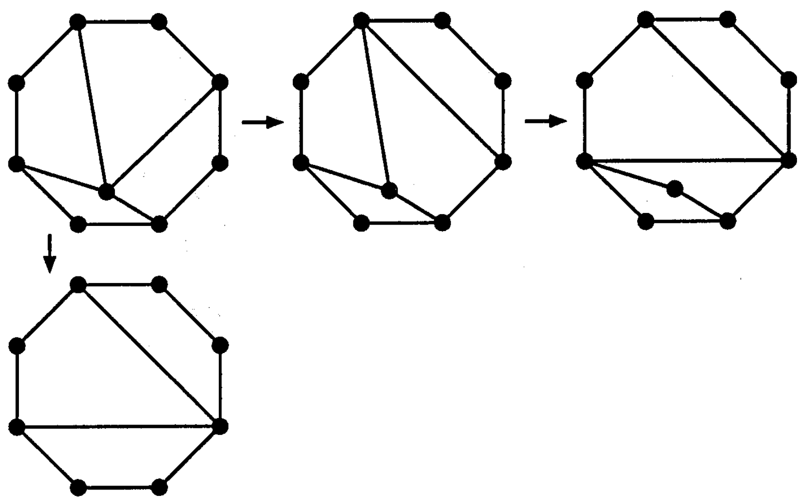


Figure 14: Face contraction and diagonal slides

paper.

To carry out the same arguments as for the triangulations in the previous section, the finiteness of irreducible quadrangulations on a closed surface plays the important role as well as that of irreducible triangulations and can be proved easily if we restrict those to only bipartite ones, assuming Wagner's conjecture. For we can translate the problem into that within the graph-minor arguments in this case, as follows.

Let G be a bipartite quadrangulation on a closed surface F^2 with partite sets $V_B(G)$ and $V_W(G)$ and let G_B be the graph with vertex set $V(G_B)=V_B(G)$ and with edges each of which joins two black vertices in the boundary cycle of each face of G. We can also define G_W similarly as one with $V(G_W)=V_W(G)$. Then G_B and G_W are the duals of each other, that is, $(G_B)^*=G_W$ and $G_B=(G_W)^*$.

Conversely, given a 2-cell embedding of a graph H on a closed surface F^2 with dual H^*, we can construct a quadrilateral embedding of a bipartite graph, denoted by $R(H)$, with partite sets $V(H)$ and $V(H^*)$ by drawing edges radially from each vertex $v^*\in V(H^*)$ to all the vertices $v\in V(H)$ on the boundary of the face which contains v^*. This bipartite graph $R(H)$ is called the *radial graph* of H on F^2. The radial graph $R(H^*)$ of the dual H^* of H clearly coincides with $R(H)$. Note that the quadrilateral embedding $R(H)$ is a quadrangulation of F^2 if and only if the boundary of any face of H is a cycle; otherwise, $R(H)$ would not be simple.

In the previous notation and this terminology, we can say that G is the radial graph of both G_B and G_W, that is, $G=R(G_B)=R(G_W)$. Observe that the contraction of a face *abcd* of G at a pair of black vertices a and c induces the contraction of the edge ac in G_B while that at a pair of white vertices b and d induces the deletion of the edge ac in G_B. This implies that the graph G_B' whose radial graph is the result G' of any face contraction is a minor of G_B. Thus, the finiteness of irreducible quadrangulations on F^2 can be translated into the finiteness of the minimal elements, with respect to the minor relation, among those graphs on F^2 each of whose faces has a cycle boundary. Wagner's conjecture guarantees the latter.

This argument does not work for the non-bipartite quadrangulation even if the same fact holds for them. However, the essential difficulty in our proof for the non-bipartite ones arises in the arguments of the *stable equivalence*. Two quadrangulations G_1 and G_2 are said to be *stably equivalent* to each other if there exist a pair of non-negative integers m_1 and m_2 such that $G_1+\Gamma_{m_1}$ and $G_2+\Gamma_{m_2}$ are equivalent to each other under diagonal slides and rotations, where $G+\Gamma_m$ denotes a quadrangulation obtained from G by adding m vertices of degree 2 to faces in order.

Recall the argument on the stable equivalence over the triangulations, which is quite simple. Negami [9] found a common refinement of any two triangulations and showed that this is contractible to both of them. In our case, we can also find easily a common refinement of any two quadrangulations on the same closed surface, but it is hardly possible to find their common refinement contractible to both of them. In fact, we can construct a pair of non-bipartite quadrangulations which makes it impossible, accroding to the theory of cycle parities in the next section.

Under this situation, Nakamoto [6] found two bipartite quadrangulations $\tilde{G}_1$ and $\tilde{G}_2$, with the same number of vertices, other than a common refinement G for any pair of quadrangulations G_1 and G_2 so that each of $\tilde{G}_i$ is contractible to G_i and both of them are contractible to G, not requiring G to be contractible to both G_1 and G_2. Then $\tilde{G}_i$ is equivalent to both $G_i+\Gamma_{m_i}$ and $G+\Gamma_{n_i}$ for each i, and hence $G_1+\Gamma_{m_1}$ and $G_2+\Gamma_{m_2}$ can be transformed into each other via $G+\Gamma_{n_1}=G+\Gamma_{n_2}$ by diagonal slides and rotations.

Now we have prepared the same facts as what we need to prove Negami's theorem for the triangulations and hence the same logic works for the proof of Theorem 7. However, we have to discuss more carefully to prove Theorem 8. See [6] for the details.

Nakamoto also has shown in [6] examples of non-bipartite quadrangulations on the Klein bottle which have the same and arbitrarily large number of vertices but which are not equivalent under diagonal slides. We shall discuss these with more general arguments in the next section. Here we

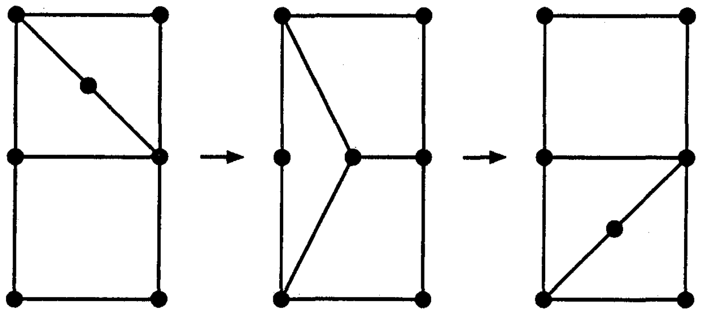


Figure 15: Moving a vertex of degree 2

should note that the diagonal rotation around a vertex of degree 2 is not needed for non-bipartite quadrangulations. For it can be realized as a sequence of diagonal slides.

In his theory, the vertices of degree 2 in quadrangulations play the same role as ones of degree 3 in triangulations. For example, a vertex v of degree 2 can be moved to any place in a quadrangulation only by diagonal slides, as shown in Figure 15, but we cannot control freely the possition of the two edges around v without diagonal rotations. However, moving v along an odd cycle takes the same effect as the diagonal rotation around v. If the vertex v is not incident to such a cycle C, then we add diagonal slides before and after this so that they carry v to a face incident to C and back to the quadrilateral region where v was placed. This operation is always possible when the quadrangulation is not bipartite, since it contains an odd cycle.

Finally, we shall show another observation on bipartite quadrangulations in [6]. A quadrangulation G on a closed surface F^2 is said to be *pseudo-minimal* if G cannot be transformed into one which has a vertex of degree 2 by diagonal slides, similarly to the definition of pseudo-minimal triangulations. For example, the complete bipartite graph $K_{m,n}$ is clearly pseudo-minimal on a closed surface F^2 if it can quadrangulate F^2 since no diagonal slide can be applied to it.

In case of triangulations, we have never found inequivalent pseudo-minimal ones on the same closed surface. However, it is easy to find such examples for quadrangulations. To do this, consider the following formula of the *genus* of $K_{m,n}$, given by Ringel in [13].

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil \qquad (m,\ n \geq 2)$$

Moreover, the complete bipartite graph $K_{m,n}$ quadrangulates the orientable closed surface S_g of genus g if

$$g=\frac{(m-2)(n-2)}{4}$$

is an integer. Thus, for all the pairs (m, n) with this condition, $K_{m,n}$'s are pseudo-minimal on F^2 and are not equivalent to one another under diagonal slides and diagonal rotations around vertices of degree 2.

For example, $K_{3,6}$ and $K_{4,4}$ are inequivalent pseudo-minimal quadrangulations on the torus S_1. Of cource, they are stably equivalent and indeed $K_{3,6}+\Gamma_n$ is equivalent to $K_{4,4}+\Gamma_m$ if and only if $3+6+n=4+4+m$ and if $n, m > 0$. This implies that the number $M(S_1)$ for the torus in Theorem 7 must be equal to or greater than 10. Thus, $M(F^2)$ does not coincide with the number of vertices of the minimal quadrangulations on F^2 in general, in contrast to our conjecture on the triangulations.

5. Cycle parity of even embeddings

In this section, we shall work in a more geneal situation. Our theory below will enable us to discuss more accurately on non-bipartite quadrangulations in terms of algebraic topology.

A face A of a graph G on a closed surface F^2 is said to be *even* (or *odd*) if its boundary walk has an even (or odd) length and an embedding of G on F^2 is said to be *even* if each face of G is a 2-cell and even. For example, any 2-cell embedding of a bipartite graph on a closed surface is even since it contains no odd cycle. In particular, a connected graph has an even embedding on the sphere if and only if it is planar and bipartite. Negami has shown in [7] and [8] a characterization of those graphs that have even embeddings on the projective plane, using the notion of "coverings" of graphs. It is so difficult to prove its analogy for other surfaces.

An even embedding of a graph on a closed surface has a nice property from a point of view of algebraic topology, as shown below, which is based on the next lemma. In general, a closed curve on a surface F^2 can be represented as the image of the unit circle $S^1=\{(x, y)\in \mathbf{R}^2: x^2+y^2=1\}$ by a continuous map $\gamma: S^1\to F^2$. Two closed curves represented with γ_0, γ_1 are said to be *homotopic* to each other if there is a continuous map $\Phi: S^1\times I\to F^2$ such that $\Phi_0=\gamma_0$ and $\Phi_1=\gamma_1$, where $\Phi_t: S^2\to F^2$ is the continuous map defined by $\Phi_t(x)=\Phi(x, t)$ for each $t\in I$. Since any cycle in a graph G embedded on F^2 can be regarded as a closed curve on F^2, we can say whether or not two given cycles in G are homotopic to each other on F^2.

LEMMA 9. *Let G be a graph 2-cell embedded on a closed surface F^2 with only even faces. Then the lengths of two cycles in G have the same parity if they are homotopic to each other on F^2.* ■

We need some technical arguments if we want to prove this lemma in a strict way. The phenomenon is however quite simple. We can understand that any two homotopic cycles in G can be transformed into each other on F^2 by a sequence of the following elementary deformation, which preserves the parity of their lengths; Let C be a closed walk and W the boundary walk of a face A, and suppose that a segment of W is contained in C. Then we can carry this segment continuously across A to the opposite segment of W, fixing its ends. It is obvious that if W has an even length, then the length of the closed walk obtained from C by this deformation has the same parity as that of C.

Let F^2 be a closed surface and G a graph 2-cell embedded on F^2. Roughly speaking, the fundamental group $\pi_1(F^2)$ of F^2 (with a suitable base point) is a group consisting of all the closed curves on F^2, classified up to homotopy, with natural compositions as its multiplication. Since each face of G is simply connected, any closed curve on F^2 can be carried homotopically into G. That is, it is homotopic to a closed walk in G. This implies that some cycles in G generate $\pi_1(F^2)$.

By the above lemma, any two cycles of G have the same parity if they represent the same element of $\pi_1(F^2)$. Thus, we can assign "0" or "1" to each element of $\pi_1(F^2)$, according to whether the element can be represented as an even cycle or an odd cycle. Denote this assignment by $\tilde{\rho}_G : \pi_1(F^2) \to \boldsymbol{Z}_2$. This map $\tilde{\rho}_G$ is a group homomorphism and depends only on the embedding of G on F^2. (In other words, if we re-embed G on F^2, then $\tilde{\rho}_G$ also changes in general.)

By well-known facts on homotopy and homology groups, $\tilde{\rho}_G$ factors through $H_1(F^2; \boldsymbol{Z}_2)$ uniquely, that is, there is a unique homomorphism ρ_G: $H_1(F^2; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ such that $\tilde{\rho}_G = \rho_G \cdot q$, where $q : \pi_1(F^2) \to H_1(F^2; \boldsymbol{Z}_2)$ is the composition of the abelianization and reduction modulo 2. This homomorphism ρ_G can be regarded as an element of the cohomology group $H^1(F^2; \boldsymbol{Z}_2)$. We shall call this the *cycle parity* of G on F^2.

It is easy to see that both the diagonal slide and the diagonal rotation aroud a vertex of any degree preserve the parity of the length of cycles with the same homotopy type on F^2. Thus, we have the following theorem.

THEOREM 10. *Let G_1 and G_2 be two quadrangulations on a closed surface F^2. If G_1 is equivalent to G_2 under diagonal slides and rotations up to ambient isotopy, then $\rho_{G_1} = \rho_{G_2}$ in $H^1(F^2; \boldsymbol{Z}_2)$.* ■

For example, the sphere S^2 has the trivial cohomology $H^1(S^2; \boldsymbol{Z}_2)=0$, so all the quadrangulations on the sphere have the same cycle parity, which assigns 0 to all of their cycles. This corresponds to the fact that every quadrangulation on the sphere is bipartite. On the other hand, $H^1(P^2; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ for the projective plane P^2 and this consists of two elements, one of which assigns 0 to all the cycles and the other assigns 1 to each essential cycle in P^2. They are the cycle parities of bipartite and non-bipartite quadrangulations on the projective plane, respectively, which are not equivalent, as we have seen in the previous section. (A cycle is said to be *essential* on a closed surface F^2 if it does not bound any 2-cell region on F^2. In this case, the cycle represents a non-trivial element in $\pi_1(F^2)$.)

It should be noticed that two quadrangulations equivalent up to homeomorphism might have distinct cycle parities. In general, two elements ρ_1 and ρ_2 in $H^1(F^2; \boldsymbol{Z}_2)$ are said here to be *congruent* if there is a homeomorphism $h: F^2 \to F^2$ which induces a homomorphism $h^*: H^1(F^2; \boldsymbol{Z}_2) \to H^1(F^2; \boldsymbol{Z}_2)$ with $h^*(\rho_2)=\rho_1$. In this case, for each closed curve γ on F^2, ρ_2 assigns the same parity to $h(\gamma)$ as ρ_1 assigns to γ. Thus, the cycle parities ρ_G and $\rho_{h(G)}$ of quadrangulations G and h(G) are congruent for any homeomorphism $h: F^2 \to F^2$, and we have:

THEOREM 11. *Let G_1 and G_2 be two quadrangulations on a closed surface F^2. If G_1 is equivalent to G_2 under diagonal slides and rotations up to homeomorphism, then the cycle parity ρ_{G_1} is congruent to ρ_{G_2} in $H^1(F^2; \boldsymbol{Z}_2)$.* ■

Note that the number of congruence classes of $H^1(F^2; \boldsymbol{Z}_2)$ depends on the homeomorphism type of the surface F^2 rather than the algebraic struture of $H^1(F^2; \boldsymbol{Z}_2)$. For example, the torus T^2 and the Klein bottle K^2 have the same cohomology

$$H^1(T^2; \boldsymbol{Z}_2) \cong H^1(K^2; \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$$

but the numbers of their congruence classes are different, as shown below.

Choose any two simple closed curves on the torus which cross each other at only one point and call them the *meridian* and the *longitude*, respectively. Then $H^1(T^2; \boldsymbol{Z}_2)$ consists of four elements (0, 0), (1, 0), (0, 1), (1, 1) and can be generated by (1, 0) and (0, 1), one of which assigns 1 to the meridian and 0 to the longitude and the other assigns 0 to the meridain and 1 to the longitude. It is easy to see that all the three non-trivial elements are congruent to each other. Thus, the torus has precisely two congruence classes of cycle parities.

The above is based on the fact that any system of two simple closed

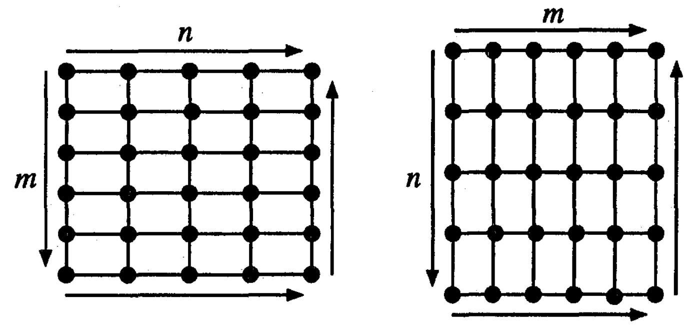


Figure 16: Inequivalent quadrangulations on the Klein bottle

curves on the torus crossing each other at a point can be chosen as a "meridian-longitude system". On the other hand, a simple closed curve on the Klein bottle which cuts open the Klein bottle into an annulus is unique up to ambient isotopy and is called the *meridian* of the Klein bottle. This implies that (1, 0) is congruent to (1, 1) but is not to (0, 1). Thus, the Klein bottle has three congruence classes of cycle parities. The trivial one (0, 0) corresponds to the class of bipartite quadrangulations and the other two correspond to those of non-bipartite ones, which are not equivalent up to homeomorphism.

For example, Figure 16 gives us two inequivalent non-bipartite quadrangulations on the Klein bottle, obtained from rectangles of size $m \times n$ and of size $n \times m$ with m odd and n even, which are the same ones given in [6]. To get the whole Klein bottle from each rectangle, identify each horizontal pair of paths in parallel and each vertical pair of paths in anti-parallel to be the meridian. They have the same number of vertices, namely $m \times n$ vertices but cannot be transformed into each other by diagonal slides since their cycle parities (1, 0) and (0, 1) are not congruent.

It is easy to see that any element in $H^1(F^2; \boldsymbol{Z}_2)$ can be the cycle parity ρ_G of a certain quadrangulation G on the closed surface F^2. Prepare a $4g$-gonal disk Ω_{4g} for the orientable closed surface S_g of genus g (or a $2g$-gonal one for the non-orientale closed surface N_g of genus g) and label its edges with a_1, b_1, a_1^{-1}, b_1^{-1}, ..., a_g, b_g, a_g^{-1}, b_g^{-1} (or c_1, c_1, ..., c_g, c_g) in this cyclic order. Identify each pair of edges with the same lables, taking their directions into account as we usually do. Then we obtain the surface S_g and the cycles a_1, b_1, ..., a_g, b_g generate $H_1(S_g; \boldsymbol{Z}_2)$. Given an element $\rho \in H^1(F^2; \boldsymbol{Z}_2)$, subdivide a_i and b_i to be paths whose lengths have the same parities as $\rho(a_i)$ and $\rho(b_i)$ have, and subdivide the whole Ω_{4g} into quadrilaterals. It is clear that $\rho_G = \rho$ for the resulting quadrangulation G on S_g. The same argument works for the non-orientable closed surface N_g with $c_1, \ldots, c_g$.

From these observations, a natural question arises; Does the number of equivalence classes of quadrangulations on a closed surface F^2 with the same number of vertices up to homeomorphism coincide with the number of congruence classes in $H^1(F^2; \boldsymbol{Z}_2)$? Equivalently, are two non-bipartite quadrangulations on F^2 equivalent under diagonal slides up to homeomorphism if their cycle parities are congruent? Following Negami's and Nakamoto's style, we conjecture that this is the case for quadrangulations with sufficiently many vertices.

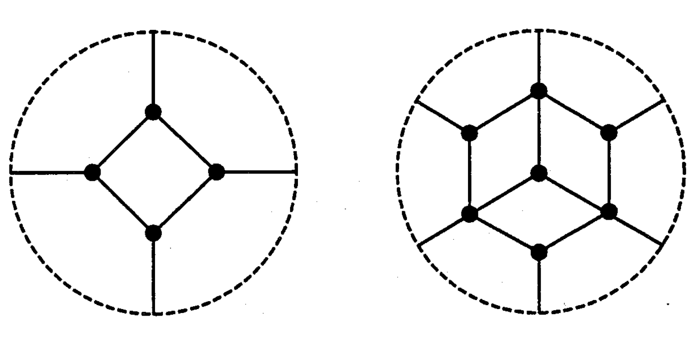
It also is easy to see that the contraction of any face of a quadrangulation G does not change the cycle parity ρ_G. Thus, if the cycle parities of two quadrangulations G_1 and G_2 on a closed surface F^2 are not congruent in $H^1(F^2; \boldsymbol{Z}_2)$, then neither G_1 nor G_2 is contractible to the other. This implies that the number of congruence classes in $H^1(F^2; \boldsymbol{Z}_2)$ gives us a lower bound for the number of the pseudo-minimal quadrangulations on F^2.

As an application of the cycle parity, we shall classify the irreducible quadrangulations on the projective plane.

THEOREM 12. *There are precisely two irreducible quadrangulations on the projective plane, as shown in Figure 17, one of which is not bipartite and is isomorphic to K_4 and the other is bipartite and is isomorphic to $K_{3,4}$.*

Proof. Let G be an irreducible quadrangulation on the projective plane P^2 and A a face of G with boundary cycle *abcd*. Then there is either (i) an edge *ac* or (ii) a path *axc* of length 2 in G since G is irreducible.

In Case (i), G contains an odd cycle *abc*, which has to be essential on P^2. This implies that the cycle parity ρ_G of G is the non-trivial one and hence every essential cycle has an odd length in this case. The diagonal pair of b and d also are joined by an edge *bd* or a path *byd* of length 2. The latter is not the case however; otherwise, the even cycle *bcdy* would be essential. Thus, we have found K_4 with vertices a, b, c, d embedded on P^2 so that *abcd* bounds



K_4 $K_{3,4}$

Figure 17: Irreducible quadrangulations on the projective plane

a face. This is the same one as the left in Figure 17 and has three quadrilateral faces, one of which is a face of G. The other two also are faces of G by Lemma 3 in [6]. This lemma implies in general that a cycle of length 4 bounds a face in an irreducible quadrangulation if it bounds a 2-cell region. Therefore, this embedding of K_4 coincides with the whole G.

In Case (ii), G contains an even cycle $abcx$ which is essential and hence ρ_G has to be trivial. This implies that G is a bipartite quadrangulation on P^2. By Euler's formula, we can conclude that G has a vertex v of degree 3. Then the union of the three faces incident to v forms a hexagonal region containing v at the center. Let $v_1v_2v_3v_4v_5v_6$ be the boundary cycle of this region with v_1, v_3 and v_5 adjacent to v. Since these three faces are not contractible, there are edges v_1v_4, v_2v_5, v_3v_6 in G. Now we have found $K_{3,4}$ with partite sets $\{v_2, v_4, v_6\}$ and $\{v, v_1, v_3, v_5\}$. This $K_{3,4}$ is embedded on P^2 with six faces in the same way as given in the right of Figure 17. By Lemma 3 in [6] again, we conclude that this embedding coincides with the whole G. ■

Combining the above theorem with Nakamoto's arguments introduced in Section 4, we can prove the following concrete theorem for the projective plane.

COROLLARY 13. *Two quadrangulations on the projective plane are equivalent to each other under diagonal slides and diagonal rotations around vertices of degree 2, up to ambient isotopy, if and only if both or neither of them are bipartite.*

Proof. Any quadrangulation G on the projective plane is contractible to one of two irreducible quadrangulations classified above, denoted simply by K_4 and $K_{3,4}$ respectively. As is shown in Section 4, this implies that G is equivalent to either $K_4+\Gamma_m$ or $K_{3,4}+\Gamma_n$, where $|V(G)|=m+4=n+7$, and G is not bipartite in the former case while G is bipartite in the latter case. Thus, the theorem follows. ■

References

[1] D. Archdeacon and B. Richter, Construction and classification of self-dual polyhedra, *J. Combin. Theory, Ser. B* **54** (1992), 37–63.
[2] D. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory, Ser. B* **33** (1982), 222–230.
[3] A. K. Dewdney, Wagner's theorem for torus graphs, *Discrete Math.* **4** (1973), 139–149.
[4] J. L. Gross and T.W. Tucker, "Topological Graph Theory", John Wiley & Sons, New York, 1987.
[5] S. Lawrenchenko, The irreducible triangulations of the torus, *Ukrain. Geom. Sb.* **30**

(1987), 52–62, in Russian.

[6] A. Nakamoto, "Diagonal transformations in quadrangulations of surfaces", Master Thesis, Yokohama National University, 1993.

[7] S. Negami, Projective-planar graphs with even duals, *J. Graph Theory* **16** (1992), 287–295.

[8] S. Negami, Projective-planar graphs with even duals II, "Graph Structure Theory", edited by N. Robertson and P. Seymour, *Contemporary Math.* **147**(1933), 363–379.

[9] S. Negami, Diagonal flips in triangulations of surfaces, to appear in *Discrete Math.*

[10] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math.* **14** (1990), 155–166.

[11] O. Ore, "The four-color problem", Academic Press, New York, 1967.

[12] G. Ringel, Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann, *Math. Annalen* **130** (1955), 317–326.

[13] G. Ringel, Das Geschlecht des vollständigen paaren Graphen, *Abh. Math. Sem. Univ. Hamburg* **28** (1965), 139–150.

[14] G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math.* **145** (1980), 121–154.

[15] N. Robertson and P. Seymour, Graph Minors XVI, Wagner's conjecture, preprint.

[16] K. Wagner, Bemekungen zum Vierfarbenproblem, *J. der Deut. Math.* Ver. 46, Abt. 1, (1963), 26–32.