Diagrams for Embeddings of Polygons

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We introduce the concept of a "convex embedding" for generalized polygons. This concept emerges from a study of convex subcomplexes of buildings. We review some results on embeddings of generalized polygons in this perspective. We also relate it to some (known) characterization theorems.

1 Introduction

The main examples of buildings, in particular of spherical buildings, are obtained from a restricted class of buildings (the so called *split buildings*) by considering fixed point sets of automorphisms. This way, one obtains *Tits diagrams* from Coxeter diagrams. Geometrically, this fixed point structure is a *convex subcomplex* of the building and one might ask whether there are such complexes not related to automorphisms. Examples are given by buildings related to groups of mixed type in small characteristics. In this paper, we describe this phenomenon in a geometric way. This enables us to recognize other examples, and to relate this to the theory of embeddings of polygons in projective spaces. For instance, the classical hexagon defined on a quadric in projective 6-space can be seen as such a convex subcomplex (called a folding below), but it does not arise from an automorphism of that quadric. Let us also mention by passing that, even if there is an automorphism around, it might still be hard to recognize the subcomplex. A striking example is given by the (relatively new) Moufang quadrangles discovered by Richard Weiss in 1997, which are convex subcomplexes of certain buildings of type F_4, and which escaped Tits' conjecture about Moufang quadrangles for about 20 years.

More exactly, we will define below *foldings* and *twistings*, which are both convex subcomplexes, but the former uses all types of the original building, while the latter does not. The results for foldings are easier and more convenient to state, but the proofs are basically the same, and therefore, we have chosen to concentrate on foldings for the general theory. We give examples, though, of both foldings and twistings. Our main result is in fact that foldings and twistings in spherical buildings are Moufang buildings. Relating this to the theory of embeddings, this firstly explains why only Moufang polygons turn up in classification results of embeddings, and, secondly, explains the conditions that are frequently used to characterize and classify certain embeddings. This will be explained in detail in Section 4.

The beauty of the theory exposed in this paper lies partly in the following observations. When twisting a building $\mathcal{B}$, the type of the new building $\widetilde{\mathcal{B}}$ can be determined by a calculation in the Weyl group, hence the standard apartment of $\mathcal{B}$ is responsible for the type. Now, if one wants to go back from $\widetilde{\mathcal{B}}$ to $\mathcal{B}$, then one needs to reconstruct the missing elements of $\mathcal{B}$. The existence of these missing elements puts some structure on $\widetilde{\mathcal{B}}$, hence some geometric conditions under which the reconstruction is possible, if starting from a general building of the same type as $\widetilde{\mathcal{B}}$. This produces a characterization of $\widetilde{\mathcal{B}}$. The main observation is now that these conditions in general can already be deduced from the (standard) apartments of $\widetilde{\mathcal{B}}$ and $\mathcal{B}$. Very good examples are provided by the symplectic quadrangles, the split Cayley hexagons and Ree-Tits octagons. We discuss these in Sections 4 and 5. The procedure to convert the standard apartment of $\widetilde{\mathcal{B}}$ into the standard apartment of $\mathcal{B}$ will be called *unfolding*.

Let us finally remark that the folded and twisted diagrams related to the folded and twisted buildings are a generalization of the above mentioned Tits diagram. In the lecture of the conference, the second author called these generalizations *Mühlherr diagrams*, which, for obvious reasons due to the first author, will not be done in the present paper.

2 Folding Buildings

Foldings of Coxeter diagrams

Let I be a set. A *Coxeter diagram over* I is a symmetric matrix $M = (m_{ij})_{i,j\in I}$ such that $m_{ij} \in \{2, 3, \ldots\} \cup \{\infty\}$ for all $i \neq j \in I$ and such that all entries on the diagonal are equal to 1. Given a Coxeter diagram over I and a subset J of I then M_J denotes the restriction of M to J.

Let M be a Coxeter diagram over I. A Coxeter system of type M is a pair consisting of a group W and a set $S = \{s_i \mid i \in I\}$ of generators of W such that $s_i s_j$ has order m_{ij} for all $i, j \in I$ and such that the set S together with these relations form a presentation for W. Given a Coxeter system (W, S) of type M and a subset J of I, then we put $S_J = \{s_j \mid j \in J\}$ and $W_J = \langle S_J \rangle$. If (W, S) is a Coxeter system, the we have a natural length function from W into the set of natural numbers, which assigns to each element of W the length of a shortest representation as a product of elements of S; the length of $w \in W$ will be denoted by $l(w)$.

For a given Coxeter diagram M, there exists up to isomorphism only one Coxeter system (W, S) of type M. The diagram is called *spherical* if the corresponding Coxeter group W is finite. In that case there exists a unique element $r \in W$ such that $l(r) > l(w)$ for all $w \in W \setminus \{r\}$; moreover r is an involution.

Let M be a Coxeter diagram over I. A subset J of I is called spherical if M_J is a spherical diagram. In that case we let r_J denote the longest element in W_J. A partition $\tilde{I}$ of I is called a *spherical partition of* M if all elements of $\tilde{I}$ are spherical subsets of I. Given a spherical partition $\tilde{I}$ of M, then we define $\tilde{W} := \langle r_{\tilde{i}} \mid \tilde{i} \in \tilde{I} \rangle$. The spherical partition $\tilde{I}$ is called a *folding of* M if for all $\tilde{w} \in \tilde{W}$ and all $\tilde{i} \in \tilde{I}$ one has $l(\tilde{w} r_{\tilde{i}}) = l(\tilde{w}) + \epsilon l(r_{\tilde{i}})$ for an $\epsilon \in \{1, -1\}$. Given a folding, then we put $R = \{r_{\tilde{i}} \mid \tilde{i} \in \tilde{I}\}$. It follows from [4] that $(\tilde{W}, R)$ is a Coxeter system of type $(o(r_{\tilde{i}} r_{\tilde{j}}))_{\tilde{i},\tilde{j}\in\tilde{I}}$. This matrix is denoted by $\tilde{M}$.

Coxeter diagrams will also be considered as graphs in the usual way; in particular we will talk about the *nodes* of a diagram. An *irreducible* diagram is then a connected one.

Foldings of rank 2

Let M be a Coxeter diagram over I and let $\tilde{I}$ be folding of M. The rank of the folding is defined to be the cardinality of $\tilde{I}$. Later on we will be interested in rank 2 foldings of irreducible spherical diagrams. These can be described as follows

Proposition 2.1 *Let M be an irreducible spherical Coxeter diagram over I and let $\tilde{I} = \{\alpha, \beta\}$ be a rank 2 folding of M. Then either $\tilde{I}$ corresponds to the bipartite partition of the diagram and $\tilde{m}_{\alpha\beta}$ is equal to the Coxeter number of M or one of the following holds (notation from* [1]*):*

Type I $M = A_{2n}, \alpha = \{\dots n-4, n-2, n, n+1, n+3, n+5, \dots\}, \beta = \{\dots, n-5, n-3, n-1, n+2, n+4, n+6, \dots\}$ *and* $\tilde{m}_{\alpha\beta} = 2n$.

Type II $M = E_6, \alpha = \{1, 2, 6\}, \beta = \{3, 4, 5\}$ *and* $\tilde{m}_{\alpha\beta} = 8$.

Type III $M = F_4, \alpha = \{1, 4\}, \beta = \{2, 3\}$ *and* $\tilde{m}_{\alpha\beta} = 8$.

The previous proposition is proved in [5] for the diagrams A_n, C_n and D_n; there are only finitely many cases remaining, which can easily be handled.

Buildings

Let I be a set, let M be a Coxeter matrix over I and let (W, S) be the Coxeter system of type M. A *building* of type M is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and where $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a *distance function* satisfying the following axioms where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

Given a building $\mathcal{B} = (\mathcal{C}, \delta)$, then the elements of $\mathcal{C}$ are called *chambers*. Given a set $J \subseteq I$ and $x \in \mathcal{C}$, the *J-residue* of x is the set $R_J(x) = \{y \in \mathcal{C} | \delta(x, y) \in W_J\}$. Each J-residue is a building of type M_J with the distance function induced by δ (cf [8];p.30). Note that a set consisting of one element $c \in \mathcal{C}$ is the $\emptyset$-residue of c. The rank of a J-residue is the cardinality of J. The building $\mathcal{B}$ is called *thick* if each rank 1 residue of $\mathcal{B}$ contains at least 3 chambers. The numerical distance between two chambers $c, d \in \mathcal{C}$ is defined by $l(c, d) := l(\delta(c, d))$.

Let $\tilde{I}$ be a folding of M. A subset $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is called folding of type $\tilde{I}$ if $\delta(\tilde{c}, \tilde{d}) \in \tilde{W}$ for all $\tilde{c}, \tilde{d} \in \tilde{\mathcal{C}}$ and if $\tilde{\mathcal{B}} := (\tilde{\mathcal{C}}, \tilde{\delta})$ is a building of type $\tilde{M}$, where $\tilde{\delta}$ denotes the restriction of δ to $\tilde{\mathcal{C}}$. The folding $\tilde{\mathcal{C}}$ is called thick if the building $\tilde{\mathcal{B}}$ is a thick building.

Projections and Opposition

Throughout this subsection let M be a Coxeter diagram over a set I and let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type M. Let $R \subseteq \mathcal{C}$ be a residue in $\mathcal{B}$ and let c be a chamber of $\mathcal{B}$. Then there exists a unique chamber $d \in R$ such that $l(c, x) = l(c, d) + l(d, x)$ for all $x \in R$. This chamber is called the *projection of* c *onto* R and it will be denoted by $\mathrm{proj}_R c$.

Suppose now that M is spherical and let r denote the longest element in the corresponding Coxeter group. Two chambers in a building of type M are called *opposite* if their distance is equal to r.

The following observations follow easily from the definitions:

Lemma 2.2 *Let $\tilde{\mathcal{C}}$ be a folding of the building $\mathcal{B} = (\mathcal{C}, \delta)$ of type $\tilde{I}$. Let $\tilde{J}$ be a subset of $\tilde{I}$ and put $J = \cup_{\tilde{j} \in \tilde{J}} \tilde{j}$. Let $\tilde{c}, \tilde{d} \in \tilde{\mathcal{C}}$ and let R (respectively $\tilde{R}$) denote the J-residue (respectively the $\tilde{J}$-residue) of $\tilde{d}$. Then the projection of $\tilde{c}$ on R in $\mathcal{B}$ and the projection of $\tilde{c}$ on $\tilde{R}$ in $\tilde{\mathcal{B}}$ coincide. In particular,* $\mathrm{proj}_R \tilde{c} \in \tilde{\mathcal{C}}$.

Lemma 2.3 *Let $\tilde{I}$ be a folding of the Coxeter diagram M over I and let $\tilde{J}$ be a spherical subset of $\tilde{I}$ with respect to the Coxeter diagram $\tilde{M}$. Put $J = \cup_{\tilde{j} \in \tilde{J}} \tilde{j}$. Then M_J is spherical and the longest element of $\tilde{W}_{\tilde{J}}$ coincides with the longest element in W_J. In particular, if $\tilde{\mathcal{C}}$ is a folding of a building $\mathcal{B}$ of type M and if $\tilde{c}, \tilde{d}$ are* *contained in a $\tilde{J}$-residue of $\tilde{\mathcal{B}}$. Then they are opposite in that residue if and only if they are opposite in the corresponding J-residue of $\mathcal{B}$.*

Using the previous two lemmas and similar ideas as in the proof of Tits' Rigidity-Theorem for spherical buildings (Theorem 4.1.1 in [18]) one can easily prove the following proposition:

Proposition 2.4 *Let $\tilde{\mathcal{C}}$ be a thick folding of the spherical building $\mathcal{B}$ of type $\tilde{I}$, let $\tilde{c}, \tilde{d}$ be two opposite chambers in $\tilde{\mathcal{B}}$. Given a folding $\tilde{\mathcal{C}}'$ of $\mathcal{B}$ of type $\tilde{I}$ such that $\tilde{c}, \tilde{d} \in \tilde{\mathcal{C}}'$ and such that $R_{\tilde{i}}(\tilde{c}) \cap \tilde{\mathcal{C}} = R_{\tilde{i}}(\tilde{c}) \cap \tilde{\mathcal{C}}$ for all $\tilde{i} \in \tilde{I}$, then $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}'$.*

The Moufang property of spherical buildings

Let $\mathcal{B}$ be a building. An automorphism of $\mathcal{B}$ is called *unipotent* if it fixes at least one chamber and if it fixes all chambers in each panel, in which it fixes at least 2 chambers. The following lemma is immediate.

Lemma 2.5 *Suppose that a unipotent automorphism of the building $\mathcal{B}$ fixes two opposite chambers in a spherical residue R of $\mathcal{B}$. Then it fixes each chamber in R.*

Suppose now that $\mathcal{B} = (\mathcal{C}, \delta)$ is a spherical building. Given a chamber $c \in \mathcal{C}$ then c^{op} denotes the set of all chambers of $\mathcal{B}$ which are opposite to c. Moreover, given a chamber $d \in c^{op}$, then c_d^{op} denotes the set of all chambers x in c^{op} such that $\mathrm{proj}_P x = \mathrm{proj}_P d$ for all panels P containing c.

The equivalence of (i) and (ii) of the following condition is proved in [5]; the equivalence of (i) and (iii) can be established in a similar way.

Proposition 2.6 *Let $\mathcal{B}$ be a thick and irreducible building of spherical type. Then the following conditions are equivalent:*

(i) *$\mathcal{B}$ is Moufang in the sense of* [18].

(ii) *For each chamber c of $\mathcal{B}$ there is a transitive group of unipotent automorphisms of $\mathcal{B}$ fixing c and acting transitively on c^{op}.*

(iii) *For any pair c, d of opposite chambers in $\mathcal{B}$ there is a group of unipotent automorphisms of $\mathcal{B}$ fixing c and acting transitively on c_d^{op}.*

Foldings of spherical Moufang buildings

Throughout this subsection we assume that M is an irreducible spherical Coxeter diagram over a set I, that $\tilde{I}$ is a folding and that $\tilde{M}$ is the Coxeter diagram over $\tilde{I}$ which is obtained by the folding $\tilde{I}$. Note that $\tilde{M}$ is also an irreducible spherical Coxeter diagram. We assume furthermore that $\mathcal{B} = (\mathcal{C}, \delta)$ is a Moufang building of type M and that $\tilde{\mathcal{C}}$ is a thick folding of $\mathcal{B}$ of type $\tilde{I}$. We obtain hence in this way a building $\tilde{\mathcal{B}} = (\tilde{\mathcal{C}}, \tilde{\delta})$.

The following observation is an easy consequence of Lemmas 2.3 and 2.5

Lemma 2.7 *Let u be a unipotent automorphism of $\mathcal{B}$ which stabilizes an element of $\tilde{\mathcal{C}}$ and which normalizes $\tilde{\mathcal{C}}$. Then the restriction of u onto $\tilde{\mathcal{C}}$ is a unipotent automorphism of $\tilde{\mathcal{B}}$.*

Let $\tilde{c}$ be a chamber in $\tilde{\mathcal{C}}$, let $\tilde{d}, \tilde{d}' \in \tilde{\mathcal{C}}$ be two chambers opposite to $\tilde{c}$ and suppose that $\mathrm{proj}_{R_{\tilde{i}}(\tilde{c})}\tilde{d} = \mathrm{proj}_{R_{\tilde{i}}(\tilde{c})}\tilde{d}'$ for each $\tilde{i} \in \tilde{I}$. The unique unipotent automorphism u of $\mathcal{B}$ which sends $\tilde{d}$ to $\tilde{d}'$ stabilizes $x_{\tilde{i}} := \mathrm{proj}_{R_{\tilde{i}}(\tilde{c})}\tilde{d}$ for each $\tilde{i}$ and as $x_{\tilde{i}}$ is opposite to $\tilde{c}$ in $R_{\tilde{i}}(\tilde{c})$ it follows from lemma 2.5 that u induces the identity on this residue. Let $\mathcal{C}'$ denote the image of $\mathcal{C}$ under u. It follows that $\mathcal{C}'$ is also a folding of $\mathcal{B}$ of type $\tilde{I}$. Moreover, $\tilde{c}, \tilde{d}'$ are opposite chambers contained in both foldings and by the considerations above we have $R_{\tilde{i}}(\tilde{c}) \cap \tilde{\mathcal{C}} = R_{\tilde{i}}(\tilde{c}) \cap \tilde{\mathcal{C}}'$. We conclude by Proposition 2.4 that the foldings $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$ coincide and hence that u stabilizes $\tilde{\mathcal{C}}$. Now, by Lemma 2.7, the restriction of u to $\tilde{\mathcal{C}}$ is a unipotent automorphism of $\tilde{\mathcal{B}}$, which sends $\tilde{d}$ onto $\tilde{d}'$. Finally we obtain by the third characterization of irreducible spherical Moufang buildings in Proposition 2.6 the following theorem:

Theorem 2.8 *A thick folding of rank at least 2 of an irreducible spherical building is Moufang.*

3 Twisting Buildings

Twisting Coxeter diagrams

Let M be a Coxeter diagram over I. A *generalized spherical partition of M* is a pair $(K, \tilde{J})$ consisting of a subset K of I and a partition $\tilde{J}$ of $J := I \setminus K$ such that $K \cup \tilde{j}$ is a spherical subset of I and such that $r_{K\cup\tilde{j}}$ centralizes $r_{\tilde{j}}$ for each $\tilde{j} \in \tilde{J}$. Given a generalized spherical partition of M, then we put $\tilde{W} = \langle r_{K\cup\tilde{j}} \mid \tilde{j} \in \tilde{J}\rangle$. We call the generalized partition $(K, \tilde{J})$ a twisting of M if for each $\tilde{j} \in \tilde{J}$ and each $\tilde{w} \in \tilde{W}$ the following holds: $l(\tilde{w} r_{K\cup\tilde{j}}) = l(\tilde{w}) + \epsilon l(r_{K\cup\tilde{j}})$ for an $\epsilon \in \{1, -1\}$. Given a twisting $(K, \tilde{J})$ then $(\tilde{W}, R)$ is a Coxeter system of a certain type $\tilde{M}$ over $\tilde{J}$, where $R = \{r_{K\cup\tilde{j}} \mid \tilde{j} \in \tilde{J}\}$.

Twisting spherical buildings

In this section we consider buildings as simplicial complexes and we freely make use of the definitions of [18]. Let M be a Coxeter diagram over I. A building Δ of type M is hence a simplicial complex $(\mathbf{S}, \subseteq)$ which is endowed with an apartment system. The elements of $\mathbf{S}$ are the simplices of Δ and we have natural type function $\tau : \mathbf{S} \to 2^I$. The buildings considered in this section will always be assumed to be of spherical type. A twisting of a spherical building $\Delta = (\mathbf{S}, \subseteq)$ is a subset $\tilde{\mathbf{S}}$ such that $\mathrm{proj}_{\tilde{A}}\tilde{B}$ is contained in $\tilde{\mathbf{S}}$ for all $\tilde{A}, \tilde{B} \in \mathbf{S}$, such that for each $\tilde{A} \in \tilde{\mathbf{S}}$ there exists $\tilde{B} \in \tilde{\mathbf{S}}$ which is opposite to $\tilde{A}$ in Δ and such that each comaximal element in $\tilde{\mathbf{S}}$ is contained in at least 3 maximal ones.

Given a twisting $\tilde{\mathbf{S}}$ of a spherical building $\Delta = (\mathbf{S}, \subseteq)$ then the partial ordered set $\tilde{\Delta} = (\tilde{\mathbf{S}}, \subseteq)$ is a thick spherical building and the restriction of the type function τ onto $\tilde{\mathbf{S}}$ is a type function on $\tilde{\Delta}$. In particular, all chambers of $\tilde{\Delta}$ have the same type J, which is a subset of I and the types of the vertices in $\tilde{\Delta}$ yield a partition $\tilde{J}$ of J. The data consisting of $I \setminus J$ and the partition $\tilde{J}$ of J is called the type of the twisting $\tilde{\mathbf{S}}$. It is a twisting of the Coxeter diagram M in the sense of the previous subsection.

Just as in the case of a folding, one can show in a similar fashion, the following result.

Theorem 3.1 *A thick twisting of rank at least* 2 *of an irreducible spherical building is Moufang.*

4 Embeddings of polygons

Convex embeddings

In the rank 2 case, twisted and folded buildings very often produce embeddings of generalized polygons in projective spaces in the classical sense. Let us first define what we mean with an *embedding*.

Let $\mathcal{P}$ be a generalized polygon (a spherical rank 2 building; we view $\mathcal{P}$ as a simplicial complex of dimension 2, where we call the vertices of one type "points" and the ones of the other type "lines", furthermore a simplex of dimension 1 is called a "flag" and the vertices of such a simplex are called "incident"), and let $\mathrm{PG}(d,k)$ be the d-dimensional projective space defined over the skew field k (for the definitions below this projective space may be infinite dimensional, but for the purpose of the present paper, we may assume that d is a natural number). An *embedding* of $\mathcal{P}$ in $\mathrm{PG}(d,k)$ is an injective identification of the point set and the line set of $\mathcal{P}$ with a point set and a line set, respectively, of $\mathrm{PG}(d,k)$ in such a way that incident elements in $\mathcal{P}$ are also incident in $\mathrm{PG}(d,k)$ (but the converse is not required), and with the additional condition that the point set of $\mathcal{P}$ generates $\mathrm{PG}(d,k)$.

Suppose that we are given a generalized polygon $\mathcal{P}$ as a folding of a certain spherical building $\mathcal{B}$ of type M, where M is a Coxeter diagram over I, and let $\tilde{I}$ be its type. Suppose also that the building $\mathcal{B}$ is a folding of an A_n-building (i.e. a projective space) $\mathcal{S}$ and let (putting $J = \{1,2,\ldots,n\}$) $\widetilde{J}$ be its type. Suppose these foldings satisfy the following two properties:

1. Denote by a the node in J corresponding to the point set of $\mathcal{S}$. Let $\widetilde{a}$ be the class of $\widetilde{J}$ containing a. Let b be the node of I corresponding to the class $\widetilde{a}$ of $\widetilde{J}$, and let $\widetilde{b}$ be the class of $\tilde{I}$ containing b. Then $\widetilde{b}$ corresponds to the point set of $\mathcal{P}$.

2. Denote by a' the node in J corresponding to the line set of $\mathcal{S}$. Let $\widetilde{a}'$ be the class of $\widetilde{J}$ containing a'. Let b' be the node of I corresponding to the class $\widetilde{a}'$ of $\widetilde{J}$, and let $\widetilde{b}'$ be the class of $\tilde{I}$ containing b'. Then $\widetilde{b}'$ corresponds to the line set of $\mathcal{P}$.

We then obtain a folding of $\mathcal{S}$ into $\mathcal{P}$ such that the simplices of $\mathcal{S}$ that correspond to the points of $\mathcal{P}$ contain points of $\mathcal{S}$, and the simplices of $\mathcal{S}$ that correspond to the lines of $\mathcal{P}$ contain lines of $\mathcal{S}$. If we now consider the points and lines of $\mathcal{S}$ that are precisely contained in a simplex belonging to the folding, then we obviously obtain an embedding of $\mathcal{P}$ into the projective subspace of $\mathcal{S}$ generated by the points and lines of the embedded polygon.

Obviously, one can generalize the previous observations to twistings.

Let us call an embedding obtained from a folding (or, more generally, a twisting) as just described a *convex embedding*. Two main questions immediately arise.

1. Can one classify all convex embeddings of generalized polygons?

2. Under which conditions is a particular class of embeddings convex?

Both questions have indirectly received much attention in the literature. Indeed, these questions are very closely related to the classification and characterization of (lax) embeddings of polygons in the sense of [14]. It is well known that it is for the moment an impossible task to classify all embeddings of polygons, even in the finite case. Hence, one considers extra conditions. These additional conditions are very often just part of the geometric interpretation of the appropriate twisted A_n-diagram (see the examples below). Also, these conditions are chosen in the best economical way, and very often the first steps in the classification is to prove the remaining part of the geometrical interpretation. In this way, one singles out the convex embeddings, which are usually called the "standard embeddings" or the "natural embeddings", obtaining characterizations for these. This is a job worthwhile doing because there are a lot of non-convex embeddings, and even some classes of those can be classified under some mild conditions (see examples below).

Examples

In this subsection, we review some results on embeddings and put these in the perspective of the present paper.

Generalized quadrangles

In all examples of convex embeddings of generalized quadrangles (think of the "natural embeddings" of classical quadrangles as quadrics or Hermitian varieties in projective space), the corresponding twisted diagram has the property that the nodes belonging to points and hyperplanes are in the same class. Hence, for each such embedding of a quadrangle $\mathcal{Q}$, to every point p of $\mathcal{Q}$ corresponds a unique hyperplane H of the projective space. This is precisely the hyperplane spanned by the points of $\mathcal{Q}$ collinear with p (i.e., incident with a common line with p). This motivates the following definition (and we phrase it in general).

Let the generalized polygon $\mathcal{P}$ be embedded in the projective space $\mathrm{PG}(d,k)$, for some field k, then we call the embedding *polarized* if for each point p of $\mathcal{P}$ the set of points of $\mathcal{P}$ not opposite p does not span $\mathrm{PG}(d,k)$.

Note that the condition involved in a polarized embedding is a little weaker than just hypothesizing a hyperplane for each point. But in most cases, one shows that for a polarized embedding, the set of points not opposite a given point p spans a hyperplane that does not contain any point opposite p. Note also that the definition of polarized embedding implies that $\mathcal{P}$ has even diameter.

Now we introduce another condition for embeddings.

Let the generalized polygon $\mathcal{P}$ be embedded in the projective space $\mathrm{PG}(d,k)$, for some field k, then we call the embedding *full* if for each line L of $\mathcal{P}$ the set of points of $\mathcal{P}$ incident with L in $\mathcal{P}$ coincides with the set of points of $\mathrm{PG}(d,k)$ incident with L in $\mathrm{PG}(d,k)$.

Applied to the case of generalized quadrangles, we see that the notion of polarized embedding is a geometric translation of one single feature of the twisted diagram behind the convex embedding. But it appears to be enough to be able to classify, and by inspection of the list obtained in [10, 11], together with the results in [3] and [13], we may state:

Theorem 4.1 *Every full embedding of a generalized quadrangle into projective space is a polarized embedding. Also, all polarized embeddings can be explicitly enumerated. They are all convex, except if the quadrangle is isomorphic to the unique quadrangle* $W(2)$

with three points per line and three lines per point, and the projective space has characteristic different from 2 ($W(2)$ has a unique exceptional non convex embedding in $\mathrm{PG}(4,k)$ *for every field k of characteristic $\neq 2$). In the finite case, every convex embedding is obtained from a full embedding by just extending the field of the ambient projective space.*

The results in [15] show that in may cases the geometric conditions can be weakened (sometimes just a bound on the dimension of the ambient projective space is enough). Also, all projections of the convex embeddings in lower-dimensional spaces are characterized in [15]: these are examples of classifications of non convex embeddings under very mild conditions.

Generalized hexagons

Here are some remarkable cases. Let us first determine the twisted diagrams of some Moufang hexagons. We consider the hexagons obtained from a triality, as discovered and explained by Tits in [16]. Let first $\mathcal{H}$ be the (so called split Cayley) generalized hexagon corresponding to the group $G_2(k)$, for some field k, where we choose the line set in such a way that it corresponds to a line set of the triality quadric. It is well known, see [16] again, that $\mathcal{H}$ lives on a non-degenerate quadric $Q(6,k)$ of maximal Witt index in projective 6-space, and that the points of $\mathcal{H}$ collinear to any point p of $\mathcal{H}$ are exactly the points of a plane $p^\perp$ of $Q(6,k)$ (the lines of $\mathcal{H}$ are some lines of $Q(6,k)$). Hence we may regard the points of $\mathcal{H}$ really as point-plane flags and we obtain a convex subcomplex of $Q(6,k)$. Hence the embedding is convex and the folded diagram is the bipartite C_3-diagram. Combined with the folded diagram of $Q(6,k)$, we obtain a folded A_6-diagram of type I for $\mathcal{H}$ (see Proposition 2.1). Geometrically, we consider the flags of type $\{p, p^\perp, (p^\perp)^\theta, p^\theta\}$, where p is any point of $\mathcal{H}$, and where θ is the (possibly degenerate) polarity associated with $Q(6,k)$. These flags form the vertices of one type. The vertices of the other type are obtained by considering a line L of $\mathcal{H}$ and defining the flag $\{L, L^\theta\}$. These two sets of flags form a folding, as defined in Section 2.

If the characteristic of k is equal to 2, then we can project the whole situation from the kernel of $Q(6,k)$ into a hyperplane of the ambient projective space and we obtain as folded diagram the bipartite A_5-diagram. For later reference, we call this full embedding a *symplectic* embedding (it lives inside a symplectic space, see 2.4.14 of [23] for more details).

Now consider any Moufang hexagon $\mathcal{T}$ obtained from a triality τ, and choose as line set the elements of $\mathcal{T}$ that are lines of the corresponding triality quadric $Q^+(7,k)$ (a quadric of maximal Witt index in $\mathrm{PG}(7,k)$). Let θ be the polarity (possibly degenerate) associated with $Q^+(7,k)$. To each point p of $\mathcal{T}$, we associate the flag $F_p = \{p, p^\tau \cap p^{\tau^2}, (p^\tau \cap p^{\tau^2})^\theta, p^\theta\}$. To each line L of $\mathcal{T}$ we associate the flag $F_L = \{L, L^\theta\}$. Then the set of all flags F_p, F_L, with p a point of $\mathcal{T}$ and L a line of $\mathcal{T}$, forms a convex subcomplex. The diagram is not a folded one, but a twisted A_7, where one partitions the subset of types $\{1,2,3,5,6,7\}$ into $\{1,3,5,7\}$ and $\{2,6\}$ (type i means as usual "$(i-1)$-dimensional subspaces"). The corresponding embeddings (which are well known) are hence convex. If the field automorphism involved in τ is trivial, then the embedding actually is contained in a hyperplane, and it is the previous example: this is an example for which the embedding derived from the twisting does not generate the whole projective space in which the twisting occurs.

Now, in all these examples of twisted diagrams, the class of types containing the node corresponding to the points of the projective space also contains the nodes corresponding to planes and hyperplanes. So a partial geometric interpretation of these diagrams is to say that the corresponding embedding is polarized and also *flat*, if flat means the following:

Let the generalized polygon $\mathcal{P}$ be embedded in the projective space $\mathrm{PG}(d,k)$, for some field k, then we call the embedding *flat* if for each point p of $\mathcal{P}$ the set of points of $\mathcal{P}$ collinear to p is contained in a plane of $\mathrm{PG}(d,k)$.

It turns out that, at least in the finite case, the flat polarized embeddings of hexagons can be classified, and they are all convex. In fact, we have the following theorem, see [14]:

Theorem 4.2 *Let $\mathcal{H}$ be a generalized hexagon embedded in the projective space* $\mathrm{PG}(d,k)$*. If the embedding is flat and polarized, then $\mathcal{H}$ is a Moufang hexagon and every unipotent automorphism of $\mathcal{H}$*

is induced by a linear transformation of PG(d,k)*. If moreover* k *is finite, then the embedding is convex,* $\mathcal{H}$ *arises from a triality and the embedding arises by field extension from either the full embedding corresponding to that triality, or from a symplectic embedding (and in this case* $\mathcal{H}$ *is the split Cayley hexagon over a field of characteristic* 2*).*

The classification in the infinite case is still open, but this should not be too hard a problem (in view of the explicit list of Moufang hexagons obtained by Tits and Weiss [20]).

Also, it is an open question whether the dual of the triality hexagons admit convex embeddings.

Again, [14] shows that weaker conditions can be used to characterize the convex embeddings. Let us remark that there are no characterizations known of non convex embeddings.

Generalized octagons
There is not much literature about embeddings of generalized octagons. In fact, the Main Result in [12] says that there can be no flat embedding of any (thick) octagon, hence, in view of the classification of the rank 2 foldings of A_n-diagrams (see Proposition 2.1), we deduce that there can be no convex embedding arising from a folding, for any generalized octagon (finite or infinite).

The Moufang octagons arise naturally from a polarity in a building of type F_4. Hence these octagons admit convex embeddings in these rank 4 buildings, and the folded diagram is of type III (notation of Proposition 2.1). One could ask for a characterization of this embedding in terms of geometric conditions. Our conjecture is that these conditions are again the translation of the folded diagram. Work in progress shows that this should indeed be the case. We leave it to the reader to formulate the geometric properties corresponding to the folded diagram of type III.

5 Unfolding Apartments

In this section, we look at a problem at the same time related and opposite to the the main ideas of the previous sections. Indeed, we constructed buildings *inside* other buildings, and used twisted diagrams to determine the type of the smaller building. Now we are interested in the following question: given that smaller building, can we reconstruct the bigger ambient building? The fact that this is possible should induce some specific properties in the smaller building that distinguishes it from other buildings of the same type. In this paragraph, we want to show that in many cases the apartments already tell us the relevant conditions (in contrast to the twisting, where the diagrams hold the geometric information; hence we talk about unfolding the apartments). We do so with three very characteristic examples.

The symplectic quadrangle

Consider the symplectic quadrangle $W(k)$, naturally embedded in the projective 3-space $\mathrm{PG}(3,k)$ (and related to a symplectic polarity). The embedding is convex and has a bipartite A_3-diagram. The apartment of $W(k)$ is an ordinary quadrangle Q. The apartment of $\mathrm{PG}(3,k)$ is a tetrahedron T, i.e. a complete graph on 4 points. Adding the diagonals of Q to Q, one obtains T. This observation suggests that, in order to reconstruct $\mathrm{PG}(3,k)$ from $W(k)$, we should take all points and lines of $W(k)$, and define some new lines by joining opposite points. In the apartment Q, this addition creates triangles. Hence in the thick case, we should create generalized triangles, i.e., projective planes. We translate the situation in the apartment in a straightforward way as follows: the points of the new lines are collinear with two opposite points, and the new lines that are "collinear" with a certain point p exactly form, together with the old lines through p, a generalized triangle. Indeed, we have the following theorem, due to Schroth [9] (we use the notation $p^\perp$ to denote the set of points collinear with p).

Theorem 5.1 *Let $\mathcal{Q}$ be any generalized quadrangle. Let p be any point of $\mathcal{Q}$ and define the geometry $\mathcal{Q}_p$ as follows. The points of $\mathcal{Q}_p$ are the points in $\mathcal{Q}$ collinear with p; the lines are the lines of $\mathcal{Q}$ through p and the intersections $p^\perp \cap p'^\perp$ for p' opposite p; incidence is the natural one. If $\mathcal{Q}_p$ is a projective plane for all points p of $\mathcal{Q}$, then $\mathcal{Q}$ is isomorphic to $W(k)$ for some field k.*

In view of the remarks preceding the theorem, the proof is straightforward: one defines $\mathrm{PG}(3,k)$ is the obvious way; the symplectic polarity follows readily. As was the case with the diagrams, we can sometimes consider weaker conditions, or alternative ones. For instance, in the finite case one just requires that $\mathcal{Q}_p$ is a linear space and that the number of points on every line is equal to the number of lines through every point, see [6]. In [23], the following general weaker theorem is proved:

Theorem 5.2 *Let $\mathcal{Q}$ be any generalized quadrangle. If for every point p of $\mathcal{Q}$ the geometry $\mathcal{Q}_p$ (defined in the previous theorem) is a partial linear space, and if for some point x this geometry $\mathcal{Q}_x$ is a projective plane, then $\mathcal{Q}$ is isomorphic to $W(k)$ for some field k.*

The split Cayley hexagon

We can try to do a similar thing with other generalized n-gons. Let us first try to find the geometric property that is symbolized by joining two points p, p' of an apartment A (an ordinary n-gon) which are joined to a common point x. Let v be the element of A opposite x (v can be a line or a point). The points p, p' are characterized by: they are all points of A at distance 2 from x and $n-2$ from v (taking distances in the incidence graph of A). So in an arbitrary generalized n-gon $\mathcal{P}$ containing A, we can define the geometry $\mathcal{P}_x$ as follows. The points of $\mathcal{P}_x$ are the points of $\mathcal{P}$ collinear with x; the lines are the lines of $\mathcal{P}$ through x, together with the sets of points collinear with x and at distance $n-2$ from some element v opposite x (and we denote such a set by x^v). Incidence is the natural one. In the thin case (the case of A), this geometry is a triangle, so $\mathcal{P}_x$ should be a projective plane. If this is the case, then we call x a *projective point*. The interesting thing is now, that we can formulate an equivalent condition which will enable us to generalize this approach in a more elegant way to points p, p' at distance $2j > 4$.

Let P be a set of points of the generalized polygon Γ. Let us denote by $P^{\perp\!\!\perp}$ the set of points in Γ not opposite any element of P. For two non-opposite elements v, w, let us call the *projection of v onto v'* the element incident with v' closest to v. Then we have by Proposition 2.8. of [2] (where the *codistance* in a generalized n-gon equals n minus the distance in the incidence graph):

Proposition 5.3 *A point p of a generalized polygon is projective if and only if for every two non-collinear points x, y in $p^\perp$, the projection of the set $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ onto any element at codistance 1 of every point in $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ is either constant or surjective.*

Applied to a generalized pentagon, we obtain a condition under which such a geometry would admit an embedding in a projective 4-space such that all points of the space are covered and such that the points collinear with a fixed point all lie in a plane. Unfortunately, no generalized pentagon can have two collinear projective points by [21]. This explains partly why there are no natural "beautiful" generalized pentagons.

Let us have a look at the generalized hexagons. Joining the points at distance 4 of an ordinary hexagon gives us the apartment of a building of type C_3. And indeed, one can easily deduce from [7] the following theorem (see also 6.3.1 of [23]):

Theorem 5.4 *If every point of a generalized hexagon $\mathcal{H}$ is projective, then $\mathcal{H}$ is isomorphic to the split Cayley hexagon (associated to Dickson's group of type G_2) naturally embedded in the quadric $Q(6, k)$ of maximal Witt index in projective 6-space. The points and lines of $Q(6, k)$ are the points and the lines of $\mathcal{H}$ together with the sets $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$, for all non-collinear non-opposite points x, y.*

In fact it again suffices that every $\mathcal{H}_p$ is a linear space and at least one $\mathcal{H}_p$ is a projective plane.

Now if we moreover join opposite points of an ordinary hexagon, then we obtain a complete graph on 6 vertices, hence the apartment of projective 5-space. Translating this to the thick case, and in view of Proposition 5.3, we require, for a generalized hexagon $\mathcal{H}$, that for every two opposite points x, y, the projection of the set $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ onto any element at codistance 1 of every point in $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ is either constant or surjective. This explains Theorem 5.1. of [2]. We rephrase that result as follows.

Theorem 5.5 *If in a generalized hexagon $\mathcal{H}$, for every two non-collinear points x, y, the projection of the set $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ onto any element at codistance 1 of every point in $\{x, y\}^{\perp\!\!\perp\perp\!\!\perp}$ is either constant or surjective, then the points of $\mathcal{H}$ and the lines of $\mathcal{H}$ together* *with all sets* $\{x,y\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$*, for* x,y *non-collinear, are the points and the lines, respectively, of a projective* 5*-space. In this case,* $\mathcal{H}$ *is isomorphic to the split Cayley hexagon over a perfect field of characteristic* 2*.*

In fact, it turns out that we only need to require the given condition for *opposite* points x, y, see again [2].

Since not all points of a generalized 7-, 8- or 9-gons can be projective, no unfolding of the apartments of these polygons will extend to the thick case. And it readily follows from [21] that the same is true for all generalized n-gons, $n \geq 10$. This is in conformity with the fact that Moufang n-gons exist only for $n = 3, 4, 6, 8$ (although also the Moufang octagons do not behave nicely in that respect).

Ree-Tits octagons

All Moufang octagons are isomorphic to Ree-Tits octagons, arising from the Ree groups of type ${}^2F_4$ (generalized by Tits [19] to non perfect fields). A geometric characterization of the *perfect* Ree-Tits octagons (i.e., where the underlying field is perfect) is presented in [22]. There, the method is precisely to reconstruct the ambient building of type F_4 (from which the Ree-Tits octagon arises as a folding). Also in this case, restriction to the apartment shows how to obtain the 24-cell (the apartment of a building of type F_4) from an ordinary octagon. In fact, the position of the missing points in the apartment helped to prove the thick case. So in this case it is not just an *a posterior* explanation; the role of the apartments were crucial for the result.

It would take too much space to reproduce and explain here the conditions of the characterization in [22]. However, let us simply write down how one constructs the 24-cell from an ordinary octagon, and then comment shortly on the necessary geometric interpretations.

Let b_i, $i \in \mathbf{Z}$ mod 8, be a natural cyclic ordering of the vertices of an ordinary octagon. Then we define new vertices a_j, $j \in \mathbf{Z} + \frac{1}{2}$ mod 8, and join a_j to b_i with $i \in \{j - \frac{3}{2}, j - \frac{1}{2}, j + \frac{1}{2}, j + \frac{3}{2}\}$. Also, we define new vertices c_ℓ, $\ell \in \mathbf{Z}$ mod 8 and join c_ℓ to $b_{\ell-1}$, $b_{\ell+1}$, $c_{\ell-3}$ and $c_{\ell+3}$, and also to a_j for $j \in \{\ell - \frac{5}{2}, \ell - \frac{1}{2}, \ell + \frac{1}{2}, \ell + \frac{5}{2}\}$.

In the thick case, the vertices b_i are the points of the Ree-Tits octagon $\mathcal{O}$, the vertices a_j are the points of the building $\mathcal{B}$ of type F_4

collinear to collinear points of the octagon (and they are recognized inside $\mathcal{O}$ as sets of lines satisfying certain geometric conditions), and the vertices c_ℓ are the vertices of $\mathcal{B}$ collinear to all points of a trace in $\mathcal{O}$, and they are recognized inside $\mathcal{O}$ as certain classes of points related to the corresponding trace (but one trace gives rise to different vertices in $\mathcal{B}$; this is an extra complication). For more details, see [22].

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