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DIALOGUES AS A FOUNDATION FOR INTUITIONISTIC LOGIC

SUMMARY OF CONTENTS

The principal content of this article is a (new) foundation for intuitionistic logic, based on an analysis of argumentative processes as codified in the concepts of a *dialogue* and a *strategy* for dialogues. This work is presented in Section 3. A general historical introduction is given in Section2. Since already there the reader will need to know exactly what a dialogue and a strategy shall be, these basic concepts are defined in the (purely technical) Section 1.

1 BASIC CONCEPTS: DIALOGUES AND STRATEGIES

I consider a first-order language, built with variables *x,y,...* and terms *t*; formulas shall be constructed from atomic formulas with the propositional connectives A,V,-»,-i and the quantifiers V, 3; I shall also consider V,Ai,A-2,3 as *special symbols* in their own right. By an *expression* I understand either a term or a formula or a special symbol. I introduce two further symbols *P* and *Q* ; taking two new (and disjoint) copies of the set of expressions, I form for every expression e two new expressions *Pe* and *Qe*, the *P-signed* and the *Q-signed version* of the expression e.

The symbols *P, Q* shall symbolise two persons engaged in an argument or in a dialogue; I shall use *X, Y* as variables for *P, Q* and shall assume *X* ^ *Y*. An *argumentation form* is a schematic presentation of an argument, concerning a logically composite assertion; it describes how a composite assertion made by *C* may be *attacked* by *Y* and how, if possible, this attack may be *answered* by A'. As the logical form of the composite assertion shall completely determine the argument, each of the four propositional connectives and each of the two quantifiers determines an argumentation form:

DM. Gabbay and F. Guenthner (eds.),
Handbook of Philosophical Logic, 2nd Edition, Volume 5, 115-145.

$\wedge$:	assertion:	$X w_1 \wedge w_2$	
	attack:	$Y \wedge_i$	(i.e., Y chooses $i = 1$ or $i = 2$)
	answer:	$X w_i$	
$\vee$:	assertion:	$X w_1 \vee w_2$	
	attack:	$Y \vee$	
	answer:	$X w_i$	(i.e., X chooses $i = 1$ or $i = 2$)
$\rightarrow$:	assertion:	$X w_1 \rightarrow w_2$	
	attack:	$Y w_1$	
	answer:	$X w_2$	
$\neg$:	assertion:	$X \neg w$	
	attack:	$Y w$	
	answer:	*no answer possible*	
$\forall$:	assertion:	$X \forall x w$	
	attack:	$Y t$	(i.e., Y chooses the term t)
	answer:	$X w(t)$	
$\exists$:	assertion:	$X \exists x w$	
	attack:	$Y \exists$	
	answer:	$X w(t)$	(i.e., X chooses the term t).

In the last two answers I have written $w(t)$ for the substitution instance obtained from w if the term t is substituted for the variable x.

A dialogue shall be a (finite or infinite) sequence δ of statements, i.e., signed expressions, stated alternatingly by P and Q and progressing in accordance with the argumentation forms; I shall consider only such dialogues which are begun by P. Since it is necessary to distinguish carefully between attacks, answers and the assertions they refer to, I shall introduce besides δ an accompanying sequence η of references, and there I shall use the symbols A for *attack* and D for *answer (defense)*. For notational convenience, I shall assume that a natural number is the set of all smaller natural numbers (whence 0 is the first natural number), and a *sequence* shall always be a function, defined on either a natural number or on the set ω of all natural numbers. The precise definition then reads as follows:
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A *dialogue* δ, η consists of two sequences such that

δ is a sequence of signed expressions,

η is a function defined on the *positive* members of def(δ), and if n in def(η) is an ordered pair $[m, Z]$ such that m is a natural number less than n and Z is either A or D ,

satisfying the properties (D00)-(D02):

(DOO) *6(n)* is F-signed if *n* is even and Q-signed if *n* is odd; 6(0) is a composite formula.

(D01) If *T](n) = [m, A*] then *6(m)* is a composite formula and *6(n)* is attack upon *6(m)* according to the appropriate argumentation form.

(D02) If *r}(p) = [n,D]* then 77(71) = [m,^4] and *6(p)* is the answer to the attack *6(n)* according to the appropriate argumentation form.

The signed formulas occurring as values of *6* are called the *assertions* of the dialogue while the remaining values of *6* are *symbolic statements* or, more correctly, *symbolic attacks.* The numbers in *dei(6)* are called the *positions* or *places* of the dialogue. If *Pv* is the assertion *6(0),* the dialogue is said to be a dialogue *for* the formula *v* (or, sometimes, for *Pv).*

Assume now that a particular class *H* of dialogues is given, defined maybe by additional conditions, which has the property that, for every position *n* of an //-dialogue *6, 77,* the *restrictions* of *S,* 77 to positions *i* such that *i* < *n* form an //-dialogue again. Assume further hat a subclass of *H* has been defined, consisting of certain *finite* //-dialogues which then are said to be the //-dialogues *won* by *P.* Let *v* be a composite formula; to say that *P* has an //-strategy shall mean that *P* is in possession of a system of information, consisting of possible choices of F-statements in dialogues, such that every //-dialogue for *v* is won by *P* if only *P* chooses, after every statement made by *Q,* its own statement from this system of information. In order to formulate a more precise definition, recall that a *tree* 5 is a partially ordered set of elements called *nodes* with the following properties: there exists a *largest* element es (the top node), and for every node e the number ||e|| of nodes / such that e < / < es is *finite;* every node except es has exactly one *upper neighbour* but may have arbitrarily many *lower neighbours* (i.e., the tree is branching downwards). A *path* in *S* is a linearly ordered subset of nodes which, together with each of its elements e, contains all the preceding nodes / with e < /; a *branch* is a path which is maximal. If *A* is a branch of *S,* let *a A* be the unique order-preserving bijection which maps either a natural number or all of *u* onto *A,* i.e. ||a>i(i)|| = *i* holds for every node a^(i) in *A.* Consider now a tree 5 and functions *6,r/* where *6* is defined on all nodes of 5 and 77 on the nodes different from es; for every branch *A* define *6 A = S • a A ,T\A = V ' ^a^A-* The triplet *S,6,TJ* then is an *H-strategy* for *v* if

(50) For every branch *A* of *S* the pair *6A, VA* is an //-dialogue for *v* which is won by *P.*

(51) For every node e of 5 the following is the case. If ||e|| is odd then *S* does not branch at e. If ||e|| is even then e has as many lower neighbours as *Q* has possibilities to extend, by adding a new position, to an //-dialogue the (restricted) dialogue leading to e,

and *8, r)* assign these lower neighbours the values which realise these possibilities.

The general definitions having been established, particular classes of dialogues can be introduced. To do so, I shall need the following terminology. Let *S, rj* be a dialogue, and let *S(n)* be one of its attacks. The attack *5(n)* will be said to be *open at a position k* with *n* < *k* if there is no position *n'* with *n<n'* <_ *k* which carries an answer *5{n')* to that attack. In particular, an attack upon a formula *X-<v* remains open at all later places. A *D-dialogue* shall be a dialogue <S, 77 satisfying the following properties *(D10)~(D13)* :

(D10) *P* may assert an atomic formula only after it has been asserted by *Q* before: if *5(n)* = *Pa* and *a* is atomic then there exists *m* such that *m<n* and *S(m)* = *Qa.*

{Dll) If, at a position *p—1,* there are several open attacks suitable to be answered at p, then only the *latest* of them may be answered at *p* : if *vip)* — K-C] a n d if *n<n'* <*p,n'-n* =_0 (mod 2), 77(71') = [m',A] then there exists *p'* such that *n'* <*p'* <*p ,f](p')* — *[n',D].*

(D12) An attack may be answered at most once: for every *n* there exists at most one *p* such that *r)(p)* = *[n,D\.*

(Z)13) A P-formula may be attacked at most once: if *m* is even then there exists at most one n such that 77(71) = *[m, A].*

A D-dialogue is said to be *won by P* if it is finite, ends with an even position and if the rules do not permit *Q* to continue with another attack or answer. In that case the last position carries an atomic formula asserted by *P.*

The importance of Z?-dialogues rests in the fact that the formulas for which there exist £>-strategies are precisely those provable in intuitionistic logic. This follows from the following, stronger

EQUIVALENCE THEOREM. *There exist recursive algorithms which, for every formula v, transform a proof of the sequent => v in Gentzen's calculus* LJ *(for intuitionistic logic) into a D-strategy — and vice versa.*

Contrary to first appearances, a proof of this theorem is by no mean obvious; it cannot be pursued here and may be found in Felscher [1981; 1985].

An *E-dialogue* shall be a Z?-dialogue satisfying the additional condition that *Q* can react only upon the immediately preceding utterance of *P:*

(E) For every *n* in def(<5): if *n* is odd then *5(n)* is either attack upon *6(n* — 1) or answer to *S(n*— 1).

An i?-dialogue is said to be *won by P* if, again, it is finite, ends with an even position and if now the rules for ^-dialogues do not permit *Q* to continue with either an attack or an answer. There will be occasion to refer to the following result which is auxiliary to the proof of the Equivalence Theorem.

EXTENSION LEMMA. *There is a recursive algorithm by which every E-strategy can be embedded into a D-strategy.*

It follows from this lemma that the Equivalence theorem holds also for *E*-strategies in place of D-strategies.

Readers not familiar with the use of dialogues may appreciate the following *examples* in which *a,b,...* are assumed to be atomic formulas.

(la)

0.	*P(a* A *b) -> (a* A *b)*	
1.	Q(aA6)	[0,4
2.	PAi	[i>4
3.	Qa	[2,0]
4.	FA2	11 *A*
5.	*Qb*	[4,0]
6.	*P(a A b)*	[1-0]

7.	QA!	[6,Q]		7.	QA2	[6,Q]
8.	*Pa*	[7,0]		8.	P6	[7,0]

(lb)

0.	*P(a A b)* -> (a A 6)	
1.	*Q(aAb)*	[0,4
2.	*P{a A b)*	[1.0]

3.	QAi	[2,4		3.	QA2	[2,4
4.	PAi	[i,4		4.	PA2	[i.4
5.	*Qa*	[4,0]		5.	*Qb*	[4,0]
6.	Pa	[3,0]		6.	*Pb*	[3,0]

Here we have two different Z?-strategies for the same formula.

(2a)

0.	*P(a* —> —a)	
1.	*Qa*	[o,4
2.	*P-i-ya*	[i,0]
3.	*Q-<a*	[2,4
4.	*Pa*	[3,4

(2b)

0.	*P(^a* -> a)	
1.	Q-.-.a	[0,4
2.	*P^a*	[i>4
3.	*Qa*	[3,4