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Dielectric Response Function of Electron Liquids. II

——*Static Properties*——

Hiroo TOTSUJI and Setsuo ICHIMARU

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo

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The static properties of the dielectric response function for an electron liquid, obtained previously by one of the authors, is investigated. It is shown that the resulting pair correlation function has the correct long-range and short-range behaviors so that the correlation energy calculated therefrom reproduces the known exact values to the order of ε^2, when an expansion with respect to the plasma parameter ε is carried out. The compressibility sum rule is satisfied to the order of $\varepsilon^2 \ln \varepsilon$. It is found that within the accuracy stated above the short-range behavior of the pair correlation function is not sensitive to the form of the ternary correlation function.

§1. Introduction

In a previous publication,[1] hereafter referred to as I, one of the authors has calculated a dielectric response function for a strongly correlated electron liquid from a solution of the second equation of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.[2] The dielectric response function has been expressed as a functional of both the single-particle distribution function and the static form factor of the plasma. The static form factor is related to the static dielectric response function through the fluctuation-dissipation theorem. One thereby obtains a self-consistent scheme of determining the static form factor or the pair correlation function of an electron liquid in thermodynamic equilibrium.

In the present paper, we extend the foregoing line of approach and carry out a detailed study of the thermodynamic properties of a plasma as described by the dielectric response function of I. We particularly note the existence of a number of exact calculations for such thermodynamic properties[3)~5)] based on expansions with respect to the plasma parameter $\varepsilon=(4\pi n)^{1/2}e^3T^{-3/2}$, where n is the number density of electrons and T is the temperature in energy units. We thus compare our calculations with those exact results by carrying out a similar expansion with respect to the plasma parameter. From such a comparison, we find that the dielectric response function obtained in I reproduces those known results exactly. We then examine the compressibility sum rule;[6)] it will be found analytically that the dielectric response function satisfies this sum rule up to the terms involving $\varepsilon^2 \ln \varepsilon$. These agreements in the analytical expressions enable us to secure a concrete footing of calculations for a nearly ideal plasma; we may now extend the numerical calculations of the thermodynamic quantities into the domain of nonideal plasmas where an expansion with respect to ε is no longer a useful concept. The purpose of the present paper is to show analytically how those exact relationships can be reproduced from the dielectric response function obtained in I.

§ 2. Integral equation for correlation function

We consider a classical system containing n identical particles of charge e and mass m in a box of unit volume. We assume a smeared-out background of opposite charge such that the average space-charge field of the system vanishes. The system can be described by the BBGKY hierarchy; the first two equations of the hierarchy are

$$\left[\frac{\partial}{\partial t}+L(1)\right]F(1)=\int d2\,V(12)[F(1)F(2)+G(12)], \tag{1}$$

$$\begin{aligned}\left[\frac{\partial}{\partial t}+L(1)+L(2)\right]G(12)&=\frac{1}{n}[V(12)+V(21)][F(1)F(2)+G(12)]\\&+\int d3\,V(13)[F(1)G(23)+F(3)G(12)]\\&+\int d3\,V(23)[F(2)G(31)+F(3)G(12)]\\&+\int d3[V(13)+V(23)]H(123).\end{aligned} \tag{2}$$

Here, $F(1)$, $G(12)$ and $H(123)$ denote the first three correlation functions; $i\equiv(\boldsymbol{r}_i, \boldsymbol{v}_i)$ represents the position and the velocity of the i-th particle; and the operators $L(i)$ and $V(ij)$ are defined so that

$$L(i)=\boldsymbol{v}_i\cdot\frac{\partial}{\partial \boldsymbol{r}_i},$$

$$V(ij)=\frac{ne^2}{m}\frac{\partial}{\partial \boldsymbol{r}_i}\frac{1}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}\cdot\frac{\partial}{\partial \boldsymbol{v}_i}.$$

We are dealing with a homogeneous, isotropic system in thermodynamic equilibrium. Hence, we may set

$$\begin{aligned}F(1)&=f(v_1),\\G(12)&=F(1)F(2)g(|\boldsymbol{r}_1-\boldsymbol{r}_2|),\\H(123)&=F(1)F(2)F(3)h(|\boldsymbol{r}_1-\boldsymbol{r}_2|,|\boldsymbol{r}_2-\boldsymbol{r}_3|,|\boldsymbol{r}_3-\boldsymbol{r}_1|),\end{aligned} \tag{3}$$

where the single-particle distribution function is a Maxwellian

$$f(v)=\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mv^2}{2T}\right).$$

The static form factor $S(k)$ is related to the radial part of the pair correlation function via

$$g(r)=\frac{1}{n}\sum_{\boldsymbol{k}}[S(k)-1]\exp(i\boldsymbol{k}\cdot\boldsymbol{r}). \tag{4}$$

In I, we have introduced an ansatz for the radial part of the ternary correlation such that

$$\begin{aligned}h(|\boldsymbol{r}_1-\boldsymbol{r}_2|,|\boldsymbol{r}_2-\boldsymbol{r}_3|,|\boldsymbol{r}_3-\boldsymbol{r}_1|)&=g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)g(|\boldsymbol{r}_2-\boldsymbol{r}_3|)\\&+g(|\boldsymbol{r}_2-\boldsymbol{r}_3|)g(|\boldsymbol{r}_3-\boldsymbol{r}_1|)+g(|\boldsymbol{r}_3-\boldsymbol{r}_1|)g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\\&+n\int d\boldsymbol{r}_4 g(|\boldsymbol{r}_1-\boldsymbol{r}_4|)g(|\boldsymbol{r}_2-\boldsymbol{r}_4|)g(|\boldsymbol{r}_3-\boldsymbol{r}_4|).\end{aligned} \tag{5}$$

We arrived at this ansatz, guided by the form of the lowest-order solution for h in the plasma-parameter expansion in the long-range domain.[5] Substituting Eqs. (3) $\sim$ (5) into Eq. (2), we obtain the integral equation for $S(k)$:

$$S(k)-1=-\frac{k_D^2}{k^2+k_D^2 t(k)}\{t(k)+S(k)[w(k)-u(k)]\}, \tag{6}$$

where

$$t(k)=1+u(k)=1+\frac{1}{n}\sum_{\boldsymbol{q}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2}[S(|\boldsymbol{k}-\boldsymbol{q}|)-1], \tag{7}$$

$$w(k)=\frac{1}{n}\sum_{\boldsymbol{q}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2}S(q)[S(|\boldsymbol{k}-\boldsymbol{q}|)-1], \tag{8}$$

$$k_D^2=4\pi ne^2/T.$$

Equation (6) corresponds to a static evaluation of the dielectric response function obtained in I; one applies the fluctuation-dissipation theorem to the static response function in order to derive the integral equation (6) for the pair correlation function.

§ 3. Long-range and short-range behavior

In the long-range domain such that $r\gg e^2/T$ or $k\ll T/e^2$, the solution of Eq. (6) can be expressed to the first order in the plasma-parameter expansion:

$$S(k)-1=-\frac{k_D^2}{k^2+k_D^2}+\frac{\varepsilon}{2}\frac{k^3k_D}{(k^2+k_D^2)^2}\left[\frac{\pi}{2}-\tan^{-1}\left(2\frac{k_D}{k}\right)\right]. \tag{9}$$

In the short-range domain such that $r\ll k_D^{-1}$ or $k\gg k_D$, such an expansion is not applicable. Instead, we rewrite Eq. (6) into the form

$$[S(k)-1]\left\{1+\frac{k_D^2}{k^2+k_D^2t(k)}[w(k)-u(k)]\right\}=-\frac{k_D^2}{k^2+k_D^2}t(k)$$
$$\times\left\{1+\frac{w(k)-u(k)}{t(k)}\right\}\left\{1+\frac{k_D^2}{k^2}-\frac{k_D^2(k^2+k_D^2)}{k^2[k^2+k_D^2t(k)]}\right\}.$$

We then note that in the short-range domain the relationships

$$\left|\frac{k_D^2}{k^2}t(k)\right|\leq\frac{k_D^2}{k^2}\ll 1\,, \tag{10}$$

$$|w(k)-u(k)|\ll|t(k)| \tag{11}$$

hold true; these are proved in Appendix A. The integral equation for the correlation function therefore reduces to

$$S(k)-1=-\frac{k_D^2}{k^2+k_D^2}t(k). \tag{12}$$

The solution of this equation appropriate to the short-range domain has been obtained by O'Neil and Rostoker;[5] it is

$$g(r)=-1+\exp\left[-\frac{e^2}{Tr}\exp(-k_Dr)\right], \tag{13}$$

whence

$$S(k)-1=\varepsilon\left(\frac{k_D}{k}\right)\int_0^\infty dxx\left\{\exp\left[-\frac{1}{x}\exp(-\varepsilon x)\right]-1\right\}\sin\left(\frac{\varepsilon kx}{k_D}\right). \tag{14}$$

The short-range behavior of the pair correlation function is, in fact, quite insensitive to the choice of the ansatz for the ternary correlation function. For example, in place of (5), we may choose

$$\begin{aligned}h(|\boldsymbol{r}_1-\boldsymbol{r}_2|,|\boldsymbol{r}_2-\boldsymbol{r}_3|,|\boldsymbol{r}_3-\boldsymbol{r}_1|)&=g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)g(|\boldsymbol{r}_2-\boldsymbol{r}_3|)\\&+g(|\boldsymbol{r}_2-\boldsymbol{r}_3|)g(|\boldsymbol{r}_3-\boldsymbol{r}_1|)+g(|\boldsymbol{r}_3-\boldsymbol{r}_1|)g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\\&+g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)g(|\boldsymbol{r}_2-\boldsymbol{r}_3|)g(|\boldsymbol{r}_3-\boldsymbol{r}_1|),\end{aligned} \tag{15}$$

or even

$$h(|\boldsymbol{r}_1-\boldsymbol{r}_2|,|\boldsymbol{r}_2-\boldsymbol{r}_3|,|\boldsymbol{r}_3-\boldsymbol{r}_1|)=0. \tag{16}$$

These ansatz lead correspondingly to

$$[S(k)-1]\left\{1+\frac{k_D^2}{k^2+k_D^2t(k)}[w(k)-u(k)]\right\}$$
$$=-\frac{k_D^2}{k^2+k_D^2}t(k)\left[1+\frac{v(k)}{t(k)}\right]\left\{1+\frac{k_D^2}{k^2}-\frac{k_D^2(k^2+k_D^2)}{k^2[k^2+k_D^2t(k)]}\right\} \tag{17}$$

or

$$S(k)-1=-\frac{k_D^2}{k^2+k_D^2}t(k), \tag{18}$$

where

$$v(k)=\frac{1}{n}\sum_q \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2}t(q)[S(q)-1]S(|\boldsymbol{k}-\boldsymbol{q}|).$$

Equation (18) is identical to Eq. (12). Equation (17) also reduces to Eq. (12), because

$$|v(k)|\ll|t(k)| \tag{19}$$

is proved in Appendix A. The long-range behavior of the pair correlation function obtained from the ansatz (15) or (16) is, however, significantly different from the correct values given by Eq. (9).

§ 4. Correlation energy

The density E_c of the correlation energy and the pressure P of the system may be calculated according to the following formula, as soon as the radial part $g(r)$ of the pair correlation function or the static form factor $S(k)$ is known:

$$\begin{aligned}\frac{E_c}{nT}&=3\left(\frac{P}{nT}-1\right)\\&=\frac{k_D^2}{2}\int_0^\infty dr r g(r)=\frac{\varepsilon}{\pi}\int_0^\infty\frac{dk}{k_D}[S(k)-1].\end{aligned} \tag{20}$$

Abe[3] and O'Neil and Rostoker[5] calculated these quantities by dividing the radial integration into two parts. Here, we alternatively carry out the k integration with the aid of the results obtained in the previous section.

To do so, we arbitrarily select a wave number k_1 such that

$$\varepsilon\ll k_D/k_1\ll 1\,. \tag{21}$$

The domain of the k integration is then divided into two parts: $0\le k\le k_1$ and $k_1\le k$. In the former domain, we may use the long-range expression (9); in the latter, we may use Eq. (14). After a series of calculations described in Appendix B, we obtain

$$\frac{1}{\pi}\int_0^{k_1}\frac{dk}{k_D}[S(k)-1]=-\frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_D}\right)+\varepsilon\left[\frac{1}{4}\ln\left(\frac{k_1}{3k_D}\right)-\frac{1}{24}\right], \tag{22}$$

$$\frac{1}{\pi}\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)-1]=-\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_D}\right)-\varepsilon\left[\frac{1}{4}\ln\left(\frac{k_1}{k_D}\varepsilon\right)+\frac{\gamma}{2}-\frac{3}{8}\right], \tag{23}$$

where $\gamma=0.57721\cdots$ is Euler's constant. Substitution of (22) and (23) into Eq. (20) yields the correct expression for the correlation energy to the order of ε^2:

$$\frac{E_c}{nT}=-\frac{\varepsilon}{2}-\varepsilon^2\left[\frac{1}{4}\ln 3\varepsilon+\frac{\gamma}{2}-\frac{1}{3}\right]. \tag{24}$$

§ 5. Compressibility sum rule

According to the compressibility sum rule,[6] the frequency and wave-vector dependent dielectric response function $\varepsilon(\boldsymbol{k}, \omega)$ is related to the isothermal sound velocity c of the plasma via

$$\lim_{k\to 0}\left(\frac{k}{k_D}\right)^2[\varepsilon(\boldsymbol{k}, 0)-1]=\frac{T/m}{c^2}. \tag{25}$$

Equivalently, with the aid of the fluctuation-dissipation theorem, Eq. (25) may be transformed as

$$\lim_{k\to 0}\left(\frac{k_D}{k}\right)^4\left\{\left(\frac{k}{k_D}\right)^2-S(k)\right\}=\frac{c^2}{T/m}. \tag{26}$$

On the other hand, one calculates the sound velocity thermodynamically from the isothermal compressibility of the system:

$$\begin{aligned} mnc^2&=-V\left(\frac{\partial P}{\partial V}\right)_T \\ &=nT\left\{1+\frac{1}{2\pi}\varepsilon\left(1+\frac{1}{3}\varepsilon\frac{d}{d\varepsilon}\right)\int_0^\infty\frac{dk}{k_D}[S(k)-1]\right\}. \end{aligned} \tag{27}$$

It then becomes important to see if the two mutually-independent evaluations of the sound velocity, Eqs. (26) and (27), would in fact agree with each other. The calculation of c in Eq. (26) involves only the values of $S(k)$ in the limit of $k\to 0$, while Eq. (27) calls for an integrated value of $S(k)$ over the entire k space.

Substituting (22) and (23) into Eq. (27), we first obtain

$$\frac{c^2}{T/m}=1-\frac{1}{4}\varepsilon-\varepsilon^2\left(\frac{1}{6}\ln 3\varepsilon+\frac{\gamma}{3}-\frac{13}{72}\right). \tag{28}$$

This, therefore, is the exact expression for the sound velocity to the order ε^2 in the plasma-parameter expansion.

The calculation of the sound velocity from Eq. (26) is facilitated by noticing the behavior of $w(k)$ in the limit of small k:

$$w(k)\to\frac{1}{3\pi}\varepsilon\left(\frac{k}{k_D}\right)^2\int_0^\infty\frac{dq}{k_D}[S(q)^2-1].$$

Equation (26) then becomes

$$\frac{c^2}{T/m}=1+\frac{\varepsilon}{3\pi}\int_0^\infty\frac{dk}{k_D}[S(k)^2-1]. \tag{29}$$

The calculation of this integral is shown in Appendix C. The result is

$$\frac{c^2}{T/m}=1-\frac{1}{4}\varepsilon-\varepsilon^2\left(\frac{1}{6}\ln 3\varepsilon+\frac{\gamma}{3}-\frac{11}{54}\right). \tag{30}$$

We here see that the compressibility sum rule is satisfied in our formalism up to the terms of the order of $\varepsilon^2 \ln \varepsilon$ in the plasma-parameter expansion.

Recently, Vashishta and Singwi[7] introduced a dielectric response function which satisfies the compressibility sum rule in the sense that the sound velocities calculated from Eqs. (26) and (27) agree with each other. The values of the sound velocity so obtained in their formalism, however, differ from the exact values of Eq. (28) in the terms involving $\varepsilon^2 \ln \varepsilon$.

§ 6. Concluding remark

We have thus shown that the dielectric response function

$$\varepsilon(\boldsymbol{k},\omega)=1-\frac{\phi(k)\chi(\boldsymbol{k},\omega)}{1-\phi(k)w(k)\chi(\boldsymbol{k},\omega)} \tag{31}$$

with

$$\phi(k)=4\pi e^2/k^2,$$

$$\chi(\boldsymbol{k},\omega)=-\frac{n}{m}\int d\boldsymbol{v}\frac{1}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i0}\boldsymbol{k}\cdot\frac{\partial f}{\partial \boldsymbol{v}}$$

reproduces the rigorous calculations of the thermodynamic quantities in a very satisfactory manner. The utility of the dielectric response function (31), as was remarked in I, lies in its ability to treat those plasmas for which the plasma parameter may no longer be considered as a small expansion parameter. Numerical calculations of thermodynamic quantities for such nonideal plasmas are in progress.

Appendix A

Proof of the inequalities (10), (11) and (19)

We substitute (14) into (12) to find an expression for $t(k)$. For $T/e^2 \gg k \gg k_D$, we find after partial integrations

$$t(k)\simeq\int_0^\infty \frac{dx}{x^3}\exp\left(-\frac{1}{x}\right)=1\,. \tag{A·1}$$

As k increases, the value of $t(k)$ decreases because of the oscillatory behavior of the integrand. Hence, the inequality (10) has been proved.

In order to prove the inequalities (11) and (19), we construct the integrals

$$I_m\equiv\frac{1}{nt(k)}\sum_{\boldsymbol{q}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2}t^m(q)[S(|\boldsymbol{k}-\boldsymbol{q}|)-1][S(q)-1]. \tag{A·2}$$

Since $I_0=[w(k)-u(k)]/t(k)$ and $I_1=v(k)/t(k)$, the proof will be completed as soon as $|I_m|\ll 1$ is shown for $m=0$ and $m=1$.

For an estimation of the magnitudes of I_m, we may use Eq. (12) in place of $S(k)-1$ on the right-hand side of Eq. (A·2). For the solution of Eq. (12) yields correct values of the correlation function in the short-range domain, while it is accurate to the order of ε^0 in the long-range domain. Transforming the $\boldsymbol{q}$ summation into integration, we have

$$I_m=\frac{1}{2\pi}\frac{\varepsilon}{t(k)}\frac{k_D}{k}[I_m^{(1)}+I_m^{(2)}], \tag{A·3}$$

$$I_m^{(1)}=\int_0^\infty dx\frac{t^{m+1}(xk_D)}{x(x^2+1)}\left\{\frac{k^2}{k_D^2}J_1(k,x)-J_3(k,x)\right\}, \tag{A·4}$$

$$I_m^{(2)}=\int_0^\infty dx\frac{xt^{m+1}(xk_D)}{x^2+1}J_1(k,x), \tag{A·5}$$

where

$$J_l(k,x)=\int_{|k/k_D-x|}^{k/k_D+x}dy\frac{y^l}{y^2+1}t(yk_D). \tag{A·6}$$

In (A·4), we may regard $k/k_D\gg x$, because $k/k_D\gg 1$ and the integrand involves a weighting function $[x(x^2+1)]^{-1}$; hence

$$J_l(k,x)\simeq 2x\frac{(k/k_D)^l}{(k/k_D)^2+1}t(k).$$

We thus find

$$|I_m^{(1)}|\ll\left|\frac{k}{k_D}t(k)\int_0^\infty dx\frac{t^{m+1}(xk_D)}{x^2+1}\right|\sim\left|\frac{k}{k_D}t(k)\right|. \tag{A·7}$$

The integral $I_m^{(2)}$ in (A·5) may be estimated in the following way. For $k/k_D\gg 1/\varepsilon$, we can assume $k/k_D\gg x$, since the x integration has a cutoff at $x\simeq\varepsilon^{-1}$ arising from the factor $[t(xk_D)]^{m+1}$; we obtain

$$|I_m^{(2)}|\simeq\left|2\frac{k_D}{k}t(k)\int_0^\infty dx\frac{x^2t^{m+1}(xk_D)}{x^2+1}\right|\lesssim\left|\frac{1}{\varepsilon}\frac{k_D}{k}t(k)\right|. \tag{A·8}$$

For $1/\varepsilon\gtrsim k/k_D\gg 1$, $t(k)\simeq 1$; we may approximate $t(yk_D)=1$ in J_1. We thus find

$$|I_m^{(2)}|\lesssim\left|\frac{1}{2}\int_0^\infty dx\frac{xt^{m+1}(xk_D)}{x^2+1}\ln\left\{\frac{[(k/k_D)+x]^2+1}{[(k/k_D)-x]^2+1}\right\}\right|. \tag{A·9}$$

The right-hand side of (A·9) is a quantity of the order of unity. Combining the results of (A·7), (A·8) and (A·9) with (A·3), we see that $|I_m|\ll 1$ for $m=0$ and $m=1$.

Appendix B

Calculations of Eqs. (22) and (23)

We begin with the calculation of Eq. (22). With the aid of Eq. (9), we have

$$I_2 \equiv \frac{1}{\pi}\int_0^{k_1}\frac{dk}{k_D}[S(k)-1]$$
$$= -\frac{1}{\pi}\int_0^{k_1/k_D}\frac{1}{x^2+1}dx + \frac{\varepsilon}{4}\int_0^{k_1/k_D}\frac{x^3}{(x^2+1)^2}dx$$
$$-\frac{\varepsilon}{2}\int_0^{\infty}\frac{x^3}{(x^2+1)^2}\tan^{-1}\left(\frac{2}{x}\right)dx + \frac{\varepsilon}{2}\int_{k_1/k_D}^{\infty}\frac{x^3}{(x^2+1)^2}\tan^{-1}\left(\frac{2}{x}\right)dx\,.$$

Noting that

$$\int_0^{\infty}dx\frac{x^3}{(x^2+1)^2}\tan^{-1}\left(\frac{2}{x}\right) = \pi\left(\ln 3 - \frac{1}{3}\right),$$

we obtain

$$I_2 = -\frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_D}\right) + \varepsilon\left[\frac{1}{4}\ln\left(\frac{k_1}{k_D}\right) - \frac{1}{4}\ln 3 - \frac{1}{24}\right]. \qquad (B\cdot1)$$

Next, we consider

$$I_3 \equiv \frac{1}{\pi}\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)-1]. \qquad (B\cdot2)$$

Since Eq. (14) can be rewritten as

$$S(k)-1 = -\frac{k_D^2}{k^2+k_D^2} + \frac{k_D}{\varepsilon k}\int_0^{\infty}dx\,x\left\{\exp\left[-\frac{\varepsilon}{x}\exp(-x)\right]\right.$$
$$\left.-1+\frac{\varepsilon}{x}\exp(-x)\right\}\sin\left(\frac{k}{k_D}x\right),$$

Eq. (B·2) becomes

$$I_3 = -\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_D}\right)$$
$$-\frac{\varepsilon}{\pi\delta^2}\int_0^{\infty}dt\,t\left\{\exp\left[-\frac{\delta}{t}\exp\left(-\frac{\varepsilon}{\delta}t\right)\right] - 1 + \frac{\delta}{t}\exp\left(-\frac{\varepsilon}{\delta}t\right)\right\}\mathrm{si}(t), \qquad (B\cdot3)$$

where $\delta \equiv \varepsilon k_1/k_D \ll 1$ and

$$\mathrm{si}(t) = -\int_t^{\infty}dx\frac{\sin x}{x}\,.$$

In the integrand of the last term in Eq. (B·3), we may let $\varepsilon/\delta = k_D/k_1 \to 0$ by virtue of (21).

We calculate the integral in (B·3) by dividing the range of integration into two parts: $0 \leq t \leq 1$ and $1 \leq t$. The contribution from the first part can then be rewritten as

$$-\frac{\varepsilon}{\pi\delta^2}\int_\Delta^1 dt\frac{1}{2}\frac{\delta^2}{t}\mathrm{si}(t)-\frac{\varepsilon}{\pi\delta^2}\int_\Delta^1 dtt\left[\exp\left(-\frac{\delta}{t}\right)-1+\frac{\delta}{t}-\frac{1}{2}\frac{\delta^2}{t^2}\right]\mathrm{si}(t).$$

We eventually let $\Delta \to 0$. The second term is calculated explicitly with the aid of the series expansion

$$\mathrm{si}(t)=-\frac{\pi}{2}+t-\frac{t^3}{3\cdot 3!}+\frac{t^5}{5\cdot 5!}-\cdots$$

and partial integrations; to the lowest-order terms in δ, we obtain

$$\int_\Delta^1 dtt\left[\exp\left(-\frac{\delta}{t}\right)-1+\frac{\delta}{t}-\frac{1}{2}\frac{\delta^2}{t^2}\right]\mathrm{si}(t)=-\frac{3\pi}{8}\delta^2-\frac{\pi}{4}\delta^2\int_\Delta^1 dt\frac{\exp(-\delta/t)-1}{t}.$$

Hence, we have

$$\frac{\varepsilon}{\pi\delta^2}\int_0^1 dtt\left[\exp\left(-\frac{\delta}{t}\right)-1+\frac{\delta}{t}\right]\mathrm{si}(t)$$

$$=-\frac{3}{8}\varepsilon-\frac{\varepsilon}{4}\int_0^1\frac{dt}{t}\left[\exp\left(-\frac{\delta}{t}\right)-1-\frac{2}{\pi}\mathrm{si}(t)\right]. \qquad \text{(B·4)}$$

For the domain of integration $1 \leq t$, we may use

$$\exp\left(-\frac{\delta}{t}\right)-1+\frac{\delta}{t}=\frac{1}{2}\frac{\delta^2}{t^2}.$$

Thus, we have

$$\frac{\varepsilon}{\pi\delta^2}\int_1^\infty dtt\left[\exp\left(-\frac{\delta}{t}\right)-1+\frac{\delta}{t}\right]\mathrm{si}(t)=\frac{\varepsilon}{2\pi}\int_1^\infty dt\frac{\mathrm{si}(t)}{t}. \qquad \text{(B·5)}$$

Substituting (B·4) and (B·5) into (B·3) and noting

$$\int_0^\infty dx\frac{\sin x}{x}\ln x=-\frac{\pi}{2}\gamma,$$

we finally obtain

$$I_3=-\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_\mathrm{D}}\right)+\varepsilon\left[\frac{1}{4}\int_\delta^\infty\frac{\exp(-x)}{x}dx-\frac{\gamma}{4}+\frac{3}{8}\right]$$

$$=-\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{k_1}{k_\mathrm{D}}\right)-\varepsilon\left[\frac{1}{4}\ln\left(\varepsilon\frac{k_1}{k_\mathrm{D}}\right)+\frac{\gamma}{2}-\frac{3}{8}\right]. \qquad \text{(B·6)}$$

Appendix C

Calculation of Eq. (30)

With the aid of the wave number k_1 introduced in (21), we divide the domain of the k integration in Eq. (29) into two parts: $0\le k\le k_1$ and $k_1\le k$. In the former domain, we may use Eq. (9) to calculate

$$S(k)^2-1\simeq\frac{k^4}{(k^2+k_D{}^2)^2}-1+\varepsilon\frac{k^5k_D}{(k^2+k_D{}^2)^3}\left\{\frac{\pi}{2}-\tan^{-1}\left(2\frac{k_D}{k}\right)\right\}.$$

We thus obtain

$$\int_0^{k_1}\frac{dk}{k_D}[S(k)^2-1]=-\frac{3}{2}\tan^{-1}\left(\frac{k_1}{k_D}\right)+\frac{1}{2}\frac{k_1k_D}{k_1{}^2+k_D{}^2}+\frac{\pi}{2}\varepsilon\left[\ln\left(\frac{k_1}{k_D}\right)-\ln 3-\frac{5}{18}\right]. \tag{C·1}$$

For $k_1\le k$, we rewrite the integral as

$$\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)^2-1]=2\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)-1]+\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)-1]^2.$$

In the light of (A·1), we calculate

$$\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)^2-1]=2\int_{k_1}^{\infty}\frac{dk}{k_D}[S(k)-1]+\int_{k_1}^{\infty}\frac{dk}{k_D}\left(\frac{k_D{}^2}{k^2+k_D{}^2}\right)^2$$
$$=-\frac{3}{4}\pi+\frac{3}{2}\tan^{-1}\left(\frac{k_1}{k_D}\right)-\frac{1}{2}\frac{k_1k_D}{k_1{}^2+k_D{}^2}-\frac{\pi}{2}\varepsilon\left[\ln\left(\varepsilon\frac{k_1}{k_D}\right)+2\gamma-\frac{3}{2}\right]. \tag{C·2}$$

Summation of (C·1) and (C·2) yields Eq. (30).

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