Diffeomorphic Matching Problems in One Dimension: Designing and Minimizing Matching Functionals

Alain Trouvé and Laurent Younes

LAGA (UMR CNRS 7593), Institut Galilée, Université Paris XIII
Av. J-B. Clément, F-93430 Villetaneuse
email: trouve@math.univ-paris13.fr
CMLA (CNRS, URA 1611), Ecole Normale Supérieure de Cachan
61, avenue du Président Wilson, F-94 235 Cachan CEDEX
email: younes@cmla.ens-cachan.fr

keywords Shape, Shape representation and recognition, Elastic matching, Calculus of variations, dynamic programming

Abstract. This paper focuses on matching 1D structures by variational methods. We provide rigorous rules for the construction of the cost function, on the basis of an analysis of properties which should be satisfied by the optimal matching. A new, exact, dynamic programming algorithm is then designed for the minimization. We conclude with experimental results on shape comparison.

1 Introduction

In signal processing, or image analysis, situations arise when objects of interest are functions θ defined on an interval $I \subset \mathbb{R}$, and taking values in $\mathbb{R}^d$. The interval I may be a time interval (with applications to speech recognition, or to on-line handwritten character recognition), a depth interval (for example to analyze 1D geological data), or arc-length (with direct application to shape recognition, 2D or 3D curve identification and comparison, etc. . .)

Comparing these "functional objects" is an important issue, for identification, for retrieval in a database. Most of the time, the problem is intricately coupled with the issue of *matching* the functions. The matching problem can be described as "finding similar structures appearing at similar places (or similar times)": given two "objects", θ and θ', expressed as functions defined on the same interval I, the issue is to find, for each $x \in I$, some $x' \in I$ such that $x \simeq x'$ and $\theta(x) \simeq \theta'(x')$. If every point in one curve is uniquely matched to some point in the other curve, the matching is a bijection $\phi : I \to I$, and the problem can be formulated as finding such a ϕ such that $\phi \simeq \mathrm{id}$ (where id is the identity function $x \mapsto x$) and $\theta \simeq \theta' \circ \phi$.

Since both constraints may drag the solution to opposite directions, a common approach is to balance them by minimizing some functional $L_{\theta,\theta'}(\phi)$. One

D. Vernon (Ed.): ECCV 2000, LNCS 1842, pp. 573–587, 2000.

simple example is (letting $\dot\phi = \frac{d\phi}{dx}$)

$$L_{\theta,\theta'}(\phi) = \int_I \dot\phi^2 dx + \mu \int_I (\theta(x) - \theta' \circ \phi(x))^2 dx\,. \tag{1}$$

Many functionals which are used in the literature fall into this category, with some variations (see, for example [2]), in spite of the fact that this formulation has the drawback of not being symmetrical with respect to θ and θ' (matching θ to θ' or θ' to θ are distinct operations).

As an example of symmetric matching functional, let us quote [1], with (among other proposals)

$$L_{\theta,\theta'}(\phi) = \int_I |\dot\phi(x) - 1| dx + \mu \int_I |\theta(x) - \dot\phi(x)\theta' \circ \phi(x)| dx \tag{2}$$

or [13], with

$$L_{\theta,\theta'}(\phi) = \text{length}(I) - \int_I \sqrt{\dot\phi(x)} \left|\cos\left(\frac{\theta(x) - \theta' \circ \phi(x)}{2}\right)\right| dx\,. \tag{3}$$

The last two examples provide, after minimization over ϕ, a distance $d(\theta,\theta')$ between the functions θ and θ'.

In this paper, all the matching functionals are associated to a function F defined on $]0,+\infty[\times \mathbb{R}^d \times \mathbb{R}^d$, letting

$$L_{\theta,\theta'}(\phi) = \int_I F(\dot\phi(x), \theta(x), \theta' \circ \phi(x)) dx$$

and the optimal matching corresponds to a minimum of L. To fix the ideas, we also let $I = [0,1]$.

Our first goal is to list some essential properties which must be satisfied by the matching functionals, and see how these properties can constrain their design.

2 Designing Matching Functionals

Let a function F be defined on $]0,+\infty[\times \mathbb{R}^d \times \mathbb{R}^d$. We specify the problem of optimal matching between two functions θ and θ', defined on $[0,1]$, and with values in $\mathbb{R}^d$ as the search of the minimum, among all increasing diffeomorphisms of $[0,1]$, of the functional,

$$L_{\theta,\theta'}(\phi) = \int_0^1 F(\dot\phi(x), \theta(x), \theta' \circ \phi(x)) dx$$

Note that the functional $L_{\theta,\theta'}$ is only altered by the addition of a constant $\lambda + \mu$ if F is replaced by $F + \lambda\xi + \mu$, for some real numbers λ and μ. Since this does not affect the variational problem, all the conditions which are given below on F are implicitly assumed to be true up to such a transform.

2.1 Convexity

The first property we introduce can be seen as technical but is nonetheless essential for the variational problem. It states that F must be a *convex* function of $\dot{\phi}$. From a theoretical point of view, this is almost a minimal condition for the well-posedness of the optimization. It is indeed proved in [4] that this condition is equivalent to the fact that the functional $L_{\theta,\theta'}$ is lower semi-continuous as a function of ϕ (in suitable functional spaces): lower-semi continuity must indeed be considered as a weak constraint for minimization. Of course, this assumption does not imply that $L_{\theta,\theta'}$ is convex in ϕ. We state the convexity condition for future reference:

[Convex] for all $u, v \in \mathbb{R}^d$, $\xi \mapsto F(\xi, u, v)$ is convex on $]0, +\infty[$.

We shall see later that this assumption also has interesting numerical consequences, in particular when the functions θ and θ' are piecewise constant.

2.2 Symmetry

The next property we introduce is symmetry. In most of the applications, there are no reasons to privilege one object rather than the other, which implies that the optimal matching should not depend upon the order in which the functions θ and θ' are considered. We thus aim at the property that, for any functions θ and θ',

$$\phi = \text{argmin} L_{\theta,\theta'} \Leftrightarrow \phi^{-1} = \text{argmin} L_{\theta',\theta}$$

Since

$$L_{\theta',\theta}(\phi^{-1}) = \int_0^1 \dot{\phi}(x) F\left(\frac{1}{\dot{\phi}(x)}, \theta' \circ \phi(x), \theta(x)\right) dx$$

A sufficient condition for symmetry is

[Symmetry] For all $(\xi, u, v) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$, one has $F(\xi, u, v) = \xi F(1/\xi, v, u)$.

It is very important to check that this condition is compatible with the first one. This fact is a consequence of the next lemma

Lemma 1. *A mapping $f :]0, +\infty[\to \mathbb{R}$ is convex if and only if $f^* : \xi \mapsto \xi f(1/\xi)$ is convex*

Proof. We know that a function is convex if and only if it can be expressed as the supremum of some family of affine functions: $f(x) = \sup_i \{f_i(x)\}$ where each f_i is affine, $f_i(x) = \alpha_i x + \beta_i$. Then, for all $x > 0$, $f^*(x) := x f(1/x) = \sup_i \{\alpha_i + \beta_i x\}$, which proves that f^* is convex.

In the general case, we let $F^*(\xi, u, v) = \xi F(1/\xi, v, u)$, so that the symmetry condition becomes $F = F^*$. We let F^s be the symmetrized version of F, $F^s = F + F^*$. Lemma 1 implies that F^s satisfies **[Convex]** as soon as F satisfies it. Returning to example (1), for which

$$F(\xi, u, v) = \xi^2 + \mu(u - v)^2$$

we have

$$F^s(\xi, u, v) = \xi^2 + \frac{1}{\xi} + \mu(1+\xi)(u-v)^2$$

and the symmetrized matching functional is

$$L_{\theta,\theta'}(\phi) = \int_0^1 \left(\dot{\phi}(x)^2 + \frac{1}{\dot{\phi}(x)}\right) dx + \int_0^1 (1+\dot{\phi}(x))(\theta(x) - \theta\circ\phi(x))^2 dx$$

2.3 Consistent Self-Matching

Another natural condition is that, when comparing a function θ with itself, the optimal ϕ should be $\phi = \mathrm{id}$. In other terms, one should have, for all functions θ, and all diffeomorphisms ϕ

$$\int_0^1 F(\dot{\phi}(x), \theta(x), \theta\circ\phi(x))dx \geq \int_0^1 F(1, \theta(x), \theta(x))dx \tag{4}$$

Making the change of variable $y = \phi(x)$ in the first integral and letting $\psi = \phi^{-1}$, (4) yields, for any diffeomorphism ψ, and for all θ

$$\int_0^1 F^*(\dot{\psi}(x), \theta(x), \theta\circ\psi(x))dx \geq \int_0^1 F(1, \theta(x), \theta(x))dx$$

so that, if (4) is true for F, it is also true for F^* and thus for F^s (one has $F^*(1,u,u) = F(1,u,u)$ for all u). This shows that our conditions are compatible.

We use a more convenient, almost equivalent, form of (4):

[Self-matching] There exists a measurable function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that, for all $\xi > 0, u, v \in \mathbb{R}^d$,

$$F(\xi, u, v) \geq F(1, u, u) + \lambda(v)\xi - \lambda(u)$$

We have

Proposition 1. *If F satisfies* **[Self-matching]**, *then inequality (4) is true for all ϕ, θ.*

Conversely, if inequality (4) is true for all ϕ and θ, and if F is differentiable with respect to its first variable at $\xi = 1$, then **[Self-matching]** *is true.*

The first assertion is true by the sequence of inequalities:

$$\begin{aligned}\int_0^1 F(\dot{\phi}(x), \theta(x), \theta\circ\phi(x))dx &\geq \int_0^1 F(1, \theta(x), \theta(x))dx \\ &\quad + \int_0^1 \dot{\phi}(x)\lambda(\theta\circ\phi(x))dx - \int_0^1 \lambda(\theta(x))dx \\ &= \int_0^1 F(1, \theta(x), \theta(x))dx\end{aligned}$$

since $\int_0^1 \dot{\phi}(x)\lambda(\theta\circ\phi(x))dx = \int_0^1 \lambda(\theta(x))dx$ by change of variables. The proof of the converse is given in the appendix.

2.4 Focus Invariance

Additional constraints may come from invariance properties which are imposed on the matching. Whereas **[Convex]**, **[Symmetry]** and **[Self-matching]** have some kind of universal validity, the next ones have to be, to some extent, application dependent.

The invariance property we consider in this section will be called "focus invariance". It states that the matching remains stable when the problem is refocused on a sub-interval of $[0,1]$.

To describe this, consider θ and θ' as signals, defined on $[0,1]$, and assume that they have been matched by some function ϕ^*. Let $[a,b]$ be a sub-interval of $[0,1]$ and set $[a',b'] = [\phi^*(a), \phi^*(b)]$. To refocus the matching on these intervals, rescale the functions θ and θ', to get new signals defined on $[0,1]$, which can be matched with the same procedure. Focus invariance states that this new matching is the same as the one which has been obtained initially.

Let us be more precise. To rescale θ (resp. θ'), define $\theta_{ab}(x) = \theta(a+(b-a)x)$ (resp. $\theta'_{a'b'}(x) = \theta'(a' + (b'-a')x))$, $x \in [0,1]$. Comparing these signals with the functional F yields an optimal matching which, if it exists, minimizes

$$\int_0^1 F(\dot{\phi}(x), \theta_{a,b}(x), \theta'_{a',b'} \circ \phi(x))dx \tag{5}$$

The original optimal matching between the functions θ and θ' clearly minimizes

$$\int_a^b F(\dot{\phi}(y), \theta(y), \theta' \circ \phi(y))dy$$

with the constraints $\phi(a) = a'$ and $\phi(b) = b'$. Making the change of variables $y = a + (b-a)x$, setting $\psi(x) = (\phi(y) - a')/(b' - a')$, this integral can be written

$$(b-a)\int_0^1 F(\lambda\dot{\psi}(x), \theta_{a,b}(x), \theta'_{a',b'} \circ \psi(x))dx \tag{6}$$

with $\lambda = \dfrac{b'-a'}{b-a}$. We say that F satisfies a focus invariance property if, for any θ and θ', the minimizer of (5) is the same as the minimizer of (6).

One possible condition ensuring such a property is that F is itself (relatively) invariant under the transformation $(\xi, u, v) = (\lambda\xi, u, v)$, that is, for some $\alpha > 0$, for all $\xi > 0$, $u, v \in \mathbb{R}^d$,

$$F(\lambda\xi, u, v) = \lambda^\alpha F(\xi, u, v)$$

or $F(\xi, u, v) = \xi^\alpha F(1, u, v)$. We state this condition

[Focus] For some $\alpha > 0$, F takes the form, for some function F_1 defined on $\mathbb{R}^d \times \mathbb{R}^d$: $F(\xi, u, v) = -\xi^\alpha F_1(u, v)$.

For such a function, **[Convex]** is true if and only if, either $\alpha = 1$, or $\alpha \in]0,1[$ and $F_1 \geq 0$, or $\alpha \in]-\infty, 0[\cup]1, +\infty[$ and $F_1 \leq 0$. To ensure **[Symmetry]**, one needs $\alpha = 1/2$ and F_1 symmetrical. We thus get that

Proposition 2. *The only matching functionals which satisfy* [**Symmetry**] *and* [**Focus**] *take the form*

$$F(\xi, u, v) = -\sqrt{\xi} F_1(u, v) \tag{7}$$

with $F_1(u, v) = F_1(v, u)$.

Such a function F *satisfies* [**Convex**] *if and only if* $F_1(u, v) \geq 0$ *for all* u *and* v.

It satisfies [**Self-matching**] *if, for all* $u, v \in I\!R^d$,

$$F_1(u, v) \leq \sqrt{F_1(u, u) F_1(v, v)} \tag{8}$$

Proof. It remains to prove the last assertion. For [**Self-matching**], we must have, for some function λ,

$$-\lambda(u) - F_1(u, u) = \min_{v,\xi} (-\lambda(v)\xi - \sqrt{\xi} F_1(u, v))$$

For a fixed v, $-\lambda(v)\xi - \sqrt{\xi} F_1(u, v)$ has a finite minimum in two cases: first, if $\lambda(v) < 0$, and this minimum is given by $F_1(u, v)^2/(4\lambda(v))$ and second, if $\lambda(v) = F_1(u, v) = 0$. In the first case, we have

$$-\lambda(u) - F_1(u, u) = \min_{v, \lambda(v) > 0} \frac{F_1(u, v)^2}{4\lambda(v)} \tag{9}$$

In particular, taking $v = u$, one has, if $\lambda(u) > 0$,

$$F_1(u, u)^2 + 4\lambda(u) F_1(u, u) + 4(\lambda(u))^2 \leq 0$$

which is possible only if $F_1(u, u) = -2\lambda(u)$. Given this fact, which is true also if $\lambda(u) = 0$, (9) clearly implies (8).

2.5 Scale Invariance for Shape Comparison

Focus invariance under the above form is not a suitable constraint for every matching problem. Let us restrict to the comparison of plane curves, which has initially motivated this paper. In this case, the functions θ typically are geometrical features computed along the curve, expressed in function of the arc-length. In such a context, focusing should rather be interpreted from a geometrical point of view, as rescaling (a portion of) a plane curve so that it has, let's say, length 1. But applying such a scale change may have some impact not only on the variable x (which here represents the length), but also on the *values* of the geometric features θ. In [13], for example, the geometric features were the orientations of the tangents, which are not affected by scale change, so that focus invariance is in this case equivalent to geometric scale invariance. Letting κ be the curvature computed along the curve, the same invariance would be true if we had taken $\theta = \kappa'/\kappa^2$ (which is the "curvature" which characterizes curves up to similitudes). But if we had chosen to compare precisely Euclidean curvatures, the invariance constraints on the matching would be different: since curvatures are scaled by λ^{-1} when a curve is scaled by λ, the correct condition should be (instead of [**Focus**]):

$$F(\lambda\xi, \lambda u, v) = \lambda^\alpha F(\xi, u, v)$$

This comes from rescaling only the first curve. Rescaling the second curve yields

$$F(\lambda\xi, u, v/\lambda) = \lambda^\beta F(\xi, u, v)$$

Note that, if the symmetry condition is valid, we must have $\beta = 1 - \alpha$, which we assume hereafter.

One can solve this identity, and compute all the (continuously differentiable) functions which satisfy it. This yields functions F of the kind

$$F(\xi, u, v) = H(\xi \frac{v}{u}) u^\alpha v^{\alpha - 1}$$

Note that, since F should be convex as a function of ξ, H itself shoud be convex. The symmetry condition is ensured as soon as $xH(1/x) = H(x)$ for all x. One choice can be

$$F(\xi, u, v) = |\xi v - u|\,.$$

which satisfies [**Convex**], [**Symmetry**] and [**Self-matching**].

Many variations can be done on these computations. The first chapters of [8] contain information on how devising functionals which satisfy given criteria of invariance.

2.6 Remark

A similar, "axiomatic" approach has been taken in [1], in which a set of constraints has been proposed in the particular case of matching curvatures for shape comparison. They have introduced a series of conditions, in this context (which turned out, however, to be incompatible). The only common condition with our paper is the symmetry, since we have chosen not to discuss the triangular inequality. Note, also, that scale invariance is not taken into account in [1].

3 Existence Results

One essential issue, for the matching problem, is to know by advance that the associated variational problem has a solution. It is also interesting to be able to analyze *a priori* some properties of the optimal matching. These problems have been addressed in [12], in the particular case of focus invariant symmetric matching, that is, for F of the kind $F(\xi, u, v) = -\sqrt{\xi} F_1(u, v)$ (the objective was in particular to be able to deal with functionals like (3)). However, we believe that, with a not so large effort, the results can be extended to a wider range of cases.

In general, it is (relatively) easy to prove that the variational problem has a solution in a larger space than only the diffeomorphisms of $[0, 1]$. In [12] (see also

[10]), we have extended the functional to the set of all probability measures on $[0,1]$, replacing $\dot{\phi}$ by the Radon-Nicodym derivative with respect to Lebesgue's measure. Using a direct method (cf. [4]), a minimizer of the extended functional could be shown to exist. The hardest part of the study is then to give conditions under which this minimizer yields a correct matching, in the sense that it provides at least a homeomorphism of $[0,1]$. We now state the results, in the case when $F(\xi,u,v) = -\sqrt{\xi}F_1(u,v)$. Fixing θ and θ', we let $f(x,y) = F_1(\theta(x),\theta'(y))$, and

$$U_f(\phi) = -\int_0^1 \sqrt{\dot{\phi}(x)} f(x,\phi(x))dx$$

where $\dot{\phi}$ should be understood as a Radon-Nicodym derivative of the measure defined by $\mu([0,x[) = \phi(x)$. To simplify, we assume that

Notation For $a,b \in [0,1]^2$ denote by $[a,b]$ the closed segment $\{a+t(b-a), 0 \leq t \leq 1\}$, and by $]a,b[$ the open segment $[a,b] \setminus \{a,b\}$. A segment is horizontal (respectively vertical) if $a_2 = b_2$ (respectively $a_1 = b_1$), where $a = (a_1,a_2)$ and $b = (b_1,b_2)$.

Notation We let $\Delta_f = \int_0^1 f(x,x)dx$, and Ω_f be the set

$$\Omega_f = \left\{ (x,y) \in [0,1]^2 \ : \ |x-y| \leq \sqrt{1 - \left(\frac{\Delta_f}{\|f\|_\infty}\right)^2} \right\}$$

We have

Theorem 1. *Assume that $f \geq 0$ is bounded, upper semi-continuous, and*

- *there exists a finite family of closed segments $([a_j,b_j])_{j\in J}$ such that each of them is horizontal or vertical and f is continuous on $[0,1]^2 \setminus F$ where $F = \bigcup_{j\in J}[a_j,b_j]$.*
- *there does not exist any non empty open vertical or horizontal segment $]a,b[$ such that $]a,b[\subset \Omega_f$ and f vanishes on $]a,b[$.*

Then there exists $\phi^* \in Hom^+$ such that $U_f(\phi^*) = \min\{U_f(\phi), \phi \in Hom^+\}$. Moreover, if ϕ is a minimizer of U_f, one has, for all $x \in [0,1]$, $(x,\phi(x)) \in \Omega_f$.

We have denoted by Hom^+ the set of (strictly) increasing homeomorphisms on $[0,1]$. We now pass to conditions under which the optimal matching satisfies some smoothness properties.

Definition 1. *We say that $f : [0,1]^2 \to \mathbb{R}$ is Hölder continuous at (y,x) if there exist $\alpha > 0$ and $C > 0$ such that*

$$|f(y',x') - f(y,x)| \leq C \max(|y'-y|^\alpha, |x'-x|^\alpha) \qquad (10)$$

for any $(y',x') \in [0,1]^2$ such that $(y',x') \neq (y,x)$.

We say that f is locally uniformly Hölder continuous at (y_0,x_0) if there exists a neighborhood V of (y_0,x_0) such that, f is Hölder continuous at all $(y,x) \in V$, with constants C and α which are uniform over V.

Theorem 2. *Let f be a non-negative real-valued mesurable function on* $[0,1]^2$ *and assume that U_f reaches its minimal value on Hom^+ at ϕ^*. Then for any $x_0 \in [0,1]$, if $f((\phi(x_0), x_0) > 0$ and if f is Hölder continuous at $(x_0, \phi(x_0))$, then ϕ^* is differentiable at x_0 with strictly positive derivative.*

Moreover, if f is locally uniformly Hölder continuous, then $\dot{\phi}^*$ is continuous in a neighborhood of x_0.

Theorem 3. *Assume that f is continuously differentiable in both variables. Let $\phi \in Hom^+$ be such that $U_f(\phi) = \min\{U_f(\psi) \,|\, \psi \in Hom^+\}$ and that, for all $x \in [0,1]$, one has $f(x, \phi(x)) > 0$. Then, ϕ is twice continuously differentiable.*

4 Numerical Study

We now study the numerical problem of minimizing a functional of the kind

$$U(\phi) = \int_0^1 F(\dot{\phi}(x), \theta(x), \theta' \circ \phi(x))dx$$

in ϕ, for two functions θ and θ' defined on $[0,1]$, with values in $\mathbb{R}^d$; ϕ is an increasing diffeomorphism of $[0,1]$.

We assume [**Convex**], so that F is convex in its first variable. This condition will allow us to devise a dynamic programming algorithm to compute exactly the optimal matching in the case of discretized functions θ and θ'.

We recall the notation, for all $\xi > 0, u, v \in \mathbb{R}^d$: $F^*(\xi, u, v) = \xi F(1/\xi, v, u)$.

We start with a very simple lemma, which is implied by the first assumption:

Lemma 2. *Assume* [**Convex**]. *Let $0 \le a < b \le 1$, and $0 \le a' < b' \le 1$. Fix $u, v \in \mathbb{R}^d$. Then the minimum in ϕ of*

$$\int_a^b F(\dot{\phi}(x), u, v)dx$$

with constraints $\phi(a) = a'$ and $\phi(b) = b'$, is attained for ϕ linear: $\phi(x) = a' + (x-a)(b'-a')/(b-a)$.

Moreover, F is convex in $\dot{\phi}$ if and only if F^* is convex in $\dot{\phi}$

Proof. For any convex function G, and for any ϕ such that $\phi(a) = a'$ and $\phi(b) = b'$, the fact that

$$\int_a^b G(\dot{\phi}(x))dx \ge (b-a)G(\frac{b'-a'}{b-a}).$$

is a consequence of Jensen's inequality.

The second assertion is lemma 1.

Now, we assume that θ and θ' are piecewise constant. This means that there exist subdivisions of $[0,1]$, $0 = s_0 < s_1 < \cdots < s_{m-1} < s_m = 1$ and $0 = s'_0 < s'_1 < \cdots < s'_{n-1} < s'_n = 1$, and some constant values $\theta_1, \ldots, \theta_m$, $\theta'_1, \ldots, \theta'_n$ in $\mathbb{R}^d$, such that $\theta(x) \equiv \theta_i$ on $[s_{i-1}, s_i[$ and $\theta'(x) \equiv \theta'_i$ on $[s'_{i-1}, s'_i[$.

We now get a new expression for $U(\phi)$ in this situation. For this, we let $\psi = \phi^{-1}$. We also denote $\tau_i = \phi(s_i)$ and $\tau'_i = \psi(s'_i)$. We have

$$U(\phi) = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} F(\dot\phi(x), \theta_i, \theta' \circ \phi(x)) dx = \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} \dot\psi(x') F(1/\dot\psi(x'), \theta_i, \theta'(x')) dx'$$

$$= \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} F^*(\dot\psi(x'), \theta'(x'), \theta_i) dx' = \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{\tau_{i-1} \vee s'_{j-1}}^{\tau_i \wedge s'_j} F^*(\dot\psi(x'), \theta'_j, \theta_i) dx'$$

Thus, by lemma 2, the minimizer of U can be searched over all *piecewise linear* ϕ. Moreover, ϕ has to be linear on every interval of the kind $]\tau_{i-1} \vee s'_{j-1}, \tau_i \wedge s'_j[$. For such a ϕ, we have

$$U(\phi) = \sum_{i=1}^{m} \sum_{j=1}^{n} (\tau_i \wedge s'_j - \tau_{i-1} \vee s'_{j-1}) F^*\left(\frac{\tau'_j \wedge s_i - \tau'_{j-1} \vee s_{i-1}}{\tau_i \wedge s'_j - \tau_{i-1} \vee s'_{j-1}}, \theta'_j, \theta_i\right)$$

So that U is only a function of $\tau := (\tau_1, \ldots, \tau_{m-1})$ and $\tau' := (\tau'_1, \ldots, \tau'_{n-1})$, and the numerical procedure has to compute their optimal values. With a slight abuse of notation, we write $U(\phi) = U(\tau, \tau')$.

The function $U(\tau, \tau')$ can be minimized by dynamic programming. Let us give some details about the procedure. To have some idea on the kind of functions ϕ which are searched for, place, on the unit square $[0,1]^2$, the grid G which contains all the points $m = (s, s')$ such that either $s = s_i$ for some i, or $s' = s'_j$ for some j. We are looking for continuous, increasing mappings ϕ which are linear on every portion which does not meet G (see figure 1).

On the set G, let H_{ij} be the horizontal segment $s' = s'_j, s_{i-1} \leq s < s_i$. Similarly, let V_{ij} be the vertical segment $s = s_i, s'_{j-1} \leq s' < s'_j$. Let $G_{ij} = H_{ij} \cup V_{ij}$. If $M \in G$, denote by i_M, j_M the pair i, j such that $M \in G_{ij}$

If $M = (s, s')$ and $P = (t, t')$ in G, write $M < P$ if $s < t$ *and* $s' < t'$. For $M \in G$, let $\mathcal{P}(M)$ be the set of points $M' \in G$ such that $M' < M$ and if $M \in G_{ij}$ for some i, j then $M' \in V_{ij-1} \cup H_{i-1j}$. Finally, for $M = (s, s') \in G$ and $P = (t, t') \in \mathcal{P}(M)$, let

$$V(P, M) = (t' - s') F^*\left(\frac{t - s}{t' - s'}, \theta'_{j_M}, \theta_{i_M}\right) = (t - s) F\left(\frac{t' - s'}{t - s}, \theta_{i_M}, \theta'_{j_M}\right)$$

We can reformulate the problem of minimizing U into the problem of finding an integer p and a sequence $M_0 = (0,0), M_1, \ldots, M_{p-1}, M_p = (1,1) \in G$ such that, for all i, $M_{i-1} \in \mathcal{P}(M_i)$, which minimizes

$$L(M_0, \ldots, M_p) := \sum_{i=1}^{p} V(M_{i-1}, M_i)\,.$$

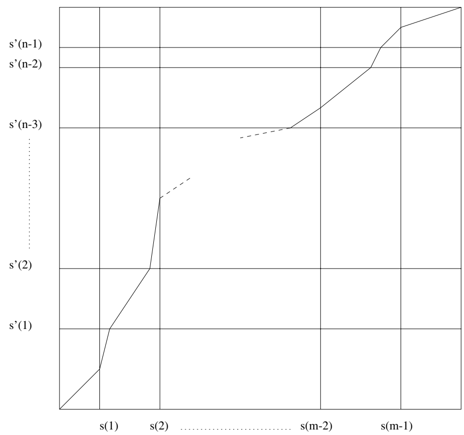

Fig. 1. Piecewise linear function ϕ on the grid G

For $M \in G$, denote by $\mathcal{S}(M)$ the set of all sequences $M_0 = (0,0), M_1, \ldots, M_{p-1}$, $M_q = M$ such that for all i, $M_{i-1} \in \mathcal{P}(M_i)$. Denote by $W(M)$ the infimum of L over $\mathcal{S}(M)$. The dynamic programming principle writes, in our case:

$$W(M) = \sup_{P \in \mathcal{P}(M)} (W(P) + V(P, M)) \tag{11}$$

This formula enables one to compute by induction the function W for all $M \in G$. We have $W(0,0) = 0$, and for all $k > 0$, if $W(M)$ has been computed for all M such that $i_M + j_M \leq k$, then using (11), one can compute $W(M)$ for all M such that $i_M + j_M = k + 1$.

Dynamic programming has been widely used for speech recognition ([11]), or for contour matching ([5], [9], [6], [3]). As presented above our method involves no pruning, no constraint on the slope of the matching functional, unlike most of the applications in dynamic time warping. If $\delta = 1/N$ is the grid step for the discrete representation of $[0,1]^2$ (the points $M_0, \ldots, M_p$ will be assumed to have coordinates of the kind $(k/N, l/N)$ for integer k, l between 0 and N), one can check that the complexity of the algorithm is of order N^2, *for a complete global minimization.*

5 Experiments

We present experimental results of curve comparison. The matched functions are the orientations of the angles of the tangents plotted versus the Euclidean arc-length of the curves (which are assumed to have length 1). The curves are closely approximated by polygons, and the algorithm of section 4 is used. Thus, we are dealing with piecewise constant functions $\theta : [0,1] \mapsto [0, 2\pi[$.

Since our representation is not rotation invariant, and since, for closed curves, the origins of the arc-length coordinates are not uniquely defined, we have first used a rigid alignment procedure similar to the one presented in [7]. Matching is then performed on the aligned shapes.

We have used several matching functionals. They are described in the captions of the figures.

The shapes have been extracted from a database composed with 14,000 geographical contours from a map of a region of Mali. These contours are hand-drawn natural boundaries of topographical sites, and the comparison procedure is part of an application aiming at providing an automatic classification of the regions.

For each pair of compared shapes, the results are presented in two parts: we first draw the (piecewise linear) optimal matching on $[0,1]^2$, the grey-levels corresponding to the values of $F(1,u,v)$ (or $G_\lambda(1,u,v)$). On the second picture, we draw both shapes in the same frame, with lines joining some matched points. One of the shapes has been shrinked for clarity.

Fig. 2. Two comparisons within a set of six shapes from the database (each line ordered by similarity). The distance is the minimum of $L_{\theta,\theta'}$ using $F(\xi,u,v) = 1 - \sqrt{\xi}\left|\cos\left(\frac{u-v}{2}\right)\right|$.

6 Appendix

We finish the proof of proposition 1, and show that (4) and the fact that F has a partial derivative in ξ at $\xi = 1$ imply **[Self-matching]**. For this, we consider a particular case. Take numbers $0 < \gamma < \beta \leq 1$ and assume that θ is constant on $[0,\gamma[$, equal to $u \in I\!R^d$ and on $[\gamma,\beta]$ (equal to v). Let $\phi(x) = x$ for $x \in [\beta,1]$, and ϕ be piecewise linear on $[0,\beta]$. More precisely, we fix $\gamma^* < \gamma$ and let ϕ be linear

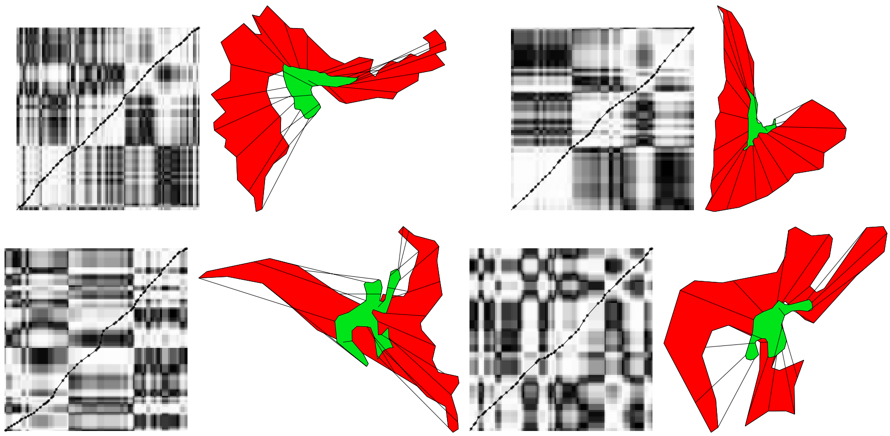

Fig. 3. Shape comparisons using $F(\xi, u, v) = 1 - \sqrt{\xi}\left|\cos\left(\frac{u-v}{2}\right)\right|$.

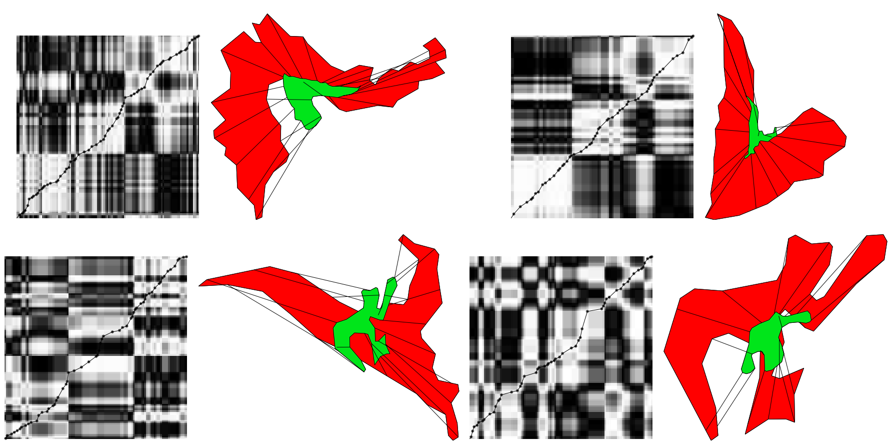

Fig. 4. Shape comparisons using $G_\lambda(\xi, u, v) = \xi^2 + \frac{1}{\xi} + \lambda(1+\xi)\sin^2\left(\frac{u-v}{2}\right)$ with $\lambda = 100$

on $[0,\gamma^*]$, $[\gamma^*,\gamma]$ and $[\gamma,\beta]$. We also let $\tilde\gamma = \phi(\gamma)$ and impose that $\gamma = \phi(\gamma^*)$ (see fig. 5).

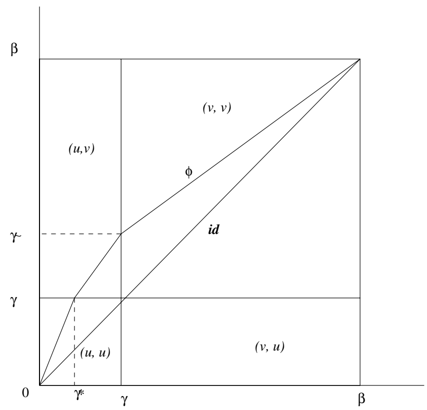

Fig. 5. Analysis of (4) in a particular case

We thus have, on $[0,\beta]$:

$$\dot\phi(x) = \begin{cases} \dfrac{\gamma}{\gamma^*} \text{ on }]0,\gamma^*[\\ \dfrac{\tilde\gamma-\gamma}{\gamma-\gamma^*} \text{ on }]\gamma^*,\gamma[\\ \dfrac{\beta-\tilde\gamma}{\beta-\gamma} \text{ on }]\gamma,\beta[\end{cases}$$

If we apply (4), we get the inequality

$$\gamma^* F(\frac{\gamma}{\gamma^*},u,u)+(\gamma-\gamma^*)F(\frac{\tilde\gamma-\gamma}{\gamma-\gamma^*},u,v)+(\beta-\gamma)F(\frac{\beta-\tilde\gamma}{\beta-\gamma},v,v) \geq \gamma F(1,u,u)+(\beta-\gamma)F(1,v,v)$$

which yields

$$(\gamma-\gamma^*)F(\frac{\tilde\gamma-\gamma}{\gamma-\gamma^*},u,v) \geq \gamma[F^*(1,u,u)-F^*(\frac{\gamma^*}{\gamma},u,u)]+(\beta-\gamma)[F(1,v,v)-F(\frac{\beta-\tilde\gamma}{\beta-\gamma},v,v)]$$

For $\xi \neq 1$, let $G(\xi,u,v) = (F(\xi,u,v) - F(1,u,v))/(\xi-1)$ and $G^*(\xi,u,v) = (F^*(\xi,u,v) - F^*(1,u,v))/(\xi-1)$. We have

$$\begin{aligned}(\gamma-\gamma^*)F(\frac{\tilde\gamma-\gamma}{\gamma-\gamma^*},u,v) &\geq \gamma(1-\frac{\gamma^*}{\gamma})G^*(\frac{\gamma^*}{\gamma},u,u) + (\beta-\gamma)(1-\frac{\beta-\tilde\gamma}{\beta-\gamma})G(\frac{\beta-\tilde\gamma}{\beta-\gamma},v,v) \\ &= (\gamma-\gamma^*)G^*(\frac{\gamma^*}{\gamma},u,u) + (\tilde\gamma-\gamma)G(\frac{\beta-\tilde\gamma}{\beta-\gamma},v,v)\end{aligned}$$

or, letting $\xi = (\tilde{\gamma} - \gamma)/(\gamma - \gamma^*)$, $\xi_1 = \gamma^*/\gamma$, $\xi_2 = (\beta - \tilde{\gamma})/(\beta - \gamma)$,

$$F(\xi, u, v) \geq G^*(\xi_1, u, u) + \xi G(\xi_2, v, v)$$

and one can check that, by suitably choosing the values of $\gamma, \tilde{\gamma}, \gamma^*$, this inequality is true for any $\xi > 0$, and $\xi_1 < 1, \xi_2 < 1$. Assume now that F is differentiable with respect to its ξ variable at $\xi = 1$. For $u \in I\!R^d$, denote by $\lambda(u)$

$$\lambda(u) = \frac{\partial F}{\partial u}(\xi, u, u)$$

This implies that G and G^* can be extended by continuity to $\xi = 1$, and, since $G^*(\xi, u, u) = F(1/\xi, u, u) - \xi G(1/\xi, u, u)$, one gets, in letting ξ_1 and ξ_2 tend to 1:

$$F(\xi, u, v) \geq F(1, u, u) + \xi\lambda(v) - \lambda(u)$$

References

1. R. Basri, L. Costa, D. Geiger, and D. Jacobs, *Determining the similarity of deformable shapes*, in IEEE Workshop on Physics based Modeling in Computer Vision, 1995, pp. 135–143. 574, 579, 579
2. I. Cohen, N. Ayache, and P. Sulger, *Tracking points on deformable objects using curvature information*, in Computer Vision - ECCV'92, G. Sandini, ed., 1992, pp. 458–466. 574
3. S. Cohen, G. Elber, and B. Bar-Yehuda, *Matching of freeform curves*, Computer-Aided design, 29 (1997), pp. 369–378. 583
4. B. Dacorogna, *Direct methods in the calculus of variations*, Springer, 1989. 575, 580
5. D. Geiger, A. Gupta, A. Costa, L, and J. Vlontzos, *Dynamic programming for detecting, tracking and matching deformable contours*, IEEE PAMI, 17 (1995), pp. 295–302. 583
6. W. Gorman, J, R. Mitchell, and P. Kuel, F, *Partial shape recognition using dynamic programming*, IEEE PAMI, 10 (1988). 583
7. S. Marques, J and J. Abrantes, A, *Shape alignment - optimal initial point and pose estimation*, Pattern Recogn. Letters, (1997). 584
8. P. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, 1995. 579
9. A. Picaz and I. Dinstein, *Matching of partially occulted planar curves*, Pattern Recognition, 28 (1995), pp. 199–209. 583
10. M. Piccioni, S. Scarlatti, and A. Trouv, *A variational problem arising from speech recognition*, SIAM Journ. Appl. Math., 58 (1998), pp. 753–771. 580
11. H. Sakoe and S. Chiba, *Dynamic programming algorithm optimization for spoken word recognition*, IEEE Trans. Accoustic, Speech and Signal Proc., 26 (1978), pp. 43–49. 583
12. A. Trouvé and L. Younes, *On a class of diffeomorphic matching problems in one dimension*, tech. rep., LAGA (UMR CNRS 7593), Institut Galilée, Université Paris XIII, 1998. To appear in SIAM J. Optimization and Control. 579, 579
13. L. Younes, *Computable elastic distances between shapes*, SIAM J. Appl. Math, 58 (1998), pp. 565–586. 574, 578