

PROPOSITION 2. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of operators on an infinite-dimensional Hilbert space \mathcal{H} such that the algebra generated by it is uniformly dense in $B(\mathcal{H})$. Then the cardinality of Λ is at least $2^{\dim \mathcal{H}}$.

PROOF. Since the algebra generated by $\{T_\lambda : \lambda \in \Lambda\}$ is uniformly dense in $B(\mathcal{H})$, the same is true for

$$\mathcal{A}_0 = \text{span}_{\mathbb{Q}}\{T_{\lambda(1)} \dots T_{\lambda(n)} : \lambda(1), \dots, \lambda(n) \in \Lambda; n \in \mathbb{N}\},$$

which has cardinality at most $\max\{\aleph_0, \text{card } \Lambda\}$.

Now choose an orthonormal basis $\{e_i : i \in I\}$ of \mathcal{H} . For every subset J of I let P_J denote the projection onto $\overline{\text{span}}\{e_j : j \in J\}$. For every $J \subset I$ we can find $T_J \in \mathcal{A}_0$ with $\|P_J - T_J\| < 1/2$. Since $\|P_J - P_{J'}\| = 1$ for $J \neq J'$, we deduce that the mapping $J \mapsto T_J$ from the family of all subsets of I into \mathcal{A}_0 is one-to-one, hence $\text{card } \mathcal{A}_0 \geq 2^{\dim \mathcal{H}}$. This implies $\text{card } \Lambda \geq 2^{\dim \mathcal{H}}$.

COROLLARY 2. The C^* -algebra generated by a countable family of operators on a separable Hilbert space \mathcal{H} is always a proper subalgebra of $B(\mathcal{H})$.

Acknowledgements. The second author wants to thank Professor George Maltese and Doctor Rainer Berntzen for their warm hospitality during his visit to the Westfälische Wilhelms-Universität where this paper was written.

References

- [1] J. B. Conway, *A Course in Functional Analysis*, Springer, New York, 1985.
- [2] C. Davis, *Generators of the ring of bounded operators*, Proc. Amer. Math. Soc. 6 (1955), 970–972.
- [3] E. A. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, *Quadratic operators and invariant subspaces*, Studia Math. 88 (1988), 263–268.
- [4] G. K. Pedersen, *Analysis Now*, Springer, New York, 1995.
- [5] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, Berlin, 1973.
- [6] W. Żelazko, *Generation of $B(X)$ by two commutative subalgebras—results and open problems*, in: Functional Analysis and Operator Theory, Banach Center Publ. 30, Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1994, 363–367.

Mathematisches Institut
Westfälische Wilhelms-Universität
Einsteinstraße 62
48149 Münster, Germany
E-mail: berntze@escher.uni-muenster.de

Faculty of Mathematics and Computer Science
Adam Mickiewicz University
ul. Matejki 48/49
60-769 Poznań, Poland
E-mail: asoltys@math.amu.edu.pl

Received December 2, 1996

(3791)

Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces

by

DANIEL AZAGRA (Madrid)

Abstract. We prove that for every infinite-dimensional Banach space X with a Fréchet differentiable norm, the sphere S_X is diffeomorphic to each closed hyperplane in X . We also prove that every infinite-dimensional Banach space Y having a (not necessarily equivalent) C^p norm (with $p \in \mathbb{N} \cup \{\infty\}$) is C^p diffeomorphic to $Y \setminus \{0\}$.

In 1966 C. Bessaga [1] proved that every infinite-dimensional Hilbert space H is C^∞ diffeomorphic to its unit sphere. The key to proving this astonishing result was the construction of a diffeomorphism between H and $H \setminus \{0\}$ which is the identity outside a ball, and this construction was possible thanks to the existence of a C^∞ non-complete norm in H . In [5], T. Dobrowolski developed Bessaga's non-complete norm technique and proved that every infinite-dimensional Banach space X which is linearly injectable into some $c_0(\Gamma)$ is C^∞ diffeomorphic to $X \setminus \{0\}$. More generally, he proved that every infinite-dimensional Banach space X having a C^p non-complete norm is C^p diffeomorphic to $X \setminus \{0\}$. If in addition X has an equivalent C^p smooth norm $\|\cdot\|$ then one can deduce that the sphere $S = \{x \in X : \|x\| = 1\}$ is C^p diffeomorphic to any of the hyperplanes in X . So, regarding the generalization of Bessaga and Dobrowolski's results to every infinite-dimensional Banach space having a differentiable norm (resp. C^p smooth norm, with $p \in \mathbb{N} \cup \{\infty\}$), the following problem naturally arises: does every infinite-dimensional Banach space with a C^p smooth equivalent norm have a C^p smooth non-complete norm? Surprisingly enough, this seems to be a difficult question which still remains unsolved. Without proving the existence of smooth non-complete norms we show that every infinite-dimensional Banach space X with a Fréchet differentiable (resp. C^p smooth) norm $\|\cdot\|$ is diffeomorphic (resp. C^p diffeomorphic) to $X \setminus \{0\}$, and we deduce

1991 Mathematics Subject Classification: 58B99, 46B20.

Key words and phrases: C^p smooth norm, spheres and hyperplanes in Banach spaces.

Supported in part by DGICYT PB 93/0452.

that the sphere $S_X = \{x \in X : \|x\| = 1\}$ is (C^p) diffeomorphic to any of the closed hyperplanes H in X . We also prove that every infinite-dimensional Banach space Y having a (not necessarily equivalent) C^p smooth norm is C^p diffeomorphic to $Y \setminus \{0\}$. Our method of defining deleting diffeomorphisms can be viewed, in a sense, as an analytical adaptation of Klee's geometrical approach in [14], which was rediscovered and simplified in [10], where a recipe for a construction of homeomorphisms removing convex bodies from non-reflexive Banach spaces is given.

Let us formally state our main result. Recall that a norm in a Banach space X is said to be *Fréchet differentiable* (resp. *C^p smooth*) if it is so in $X \setminus \{0\}$.

THEOREM 1. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a C^p smooth norm $\|\cdot\|$, and let S_X be its unit sphere. Then, for every closed hyperplane H in X , there exists a C^p diffeomorphism between S_X and H .*

The argument in the proof of this result is a modification of that in [1], changing the non-complete norm and the use of Banach's contraction principle for a different kind of *non-complete convex function* and the following *fixed point lemma*:

LEMMA 2. *Let $F : (0, \infty) \rightarrow [0, \infty)$ be a continuous function such that, for every $\beta \geq \alpha > 0$,*

$$F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha) \quad \text{and} \quad \limsup_{t \rightarrow 0^+} F(t) > 0.$$

Then there exists a unique $\alpha > 0$ such that $F(\alpha) = \alpha$.

Proof. Note that $\lim_{\beta \rightarrow \infty} [F(\beta) - \beta] \leq \lim_{\beta \rightarrow \infty} [F(1) + \frac{1}{2}(\beta - 1) - \beta] = -\infty$, while $\limsup_{\beta \rightarrow 0^+} [F(\beta) - \beta] > 0$. Then, from Bolzano's theorem we get an $\alpha > 0$ such that $F(\alpha) = \alpha$. Moreover, the first condition in the statement implies that the function $\beta \rightarrow F(\beta) - \beta$ is strictly decreasing, which yields the uniqueness of this α .

The key to the proof of Theorem 1 is the following

PROPOSITION 3. *Let $(X, \|\cdot\|)$ be a non-reflexive infinite-dimensional Banach space with a C^p smooth norm $\|\cdot\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.*

Proof. Since X is not reflexive, according to James' theorem [13], there exists a continuous linear functional $T : X \rightarrow \mathbb{R}$ such that T does not attain its norm. We may assume $\|T\| = 1$, so that $T(x) < \|x\|$ for every $x \neq 0$, and there exists a sequence (y_k) of vectors such that $\|y_k\| = 1$ and

$$\|y_k\| - T(y_k) = 1 - T(y_k) \leq 1/4^{k+1}$$

for every $k \in \mathbb{N}$. Define $\omega : X \rightarrow \mathbb{R}$ by

$$\omega(x) = \|x\| - T(x).$$

Note that $\omega(x) = 0$ if and only if $x = 0$, $\omega(x+y) \leq \omega(x) + \omega(y)$ and $\omega(rx) = r\omega(x)$ for each $r > 0$, although ω is not a norm in X because $\omega(x) \neq \omega(-x)$ in general. Now, let $\gamma : [0, \infty) \rightarrow [0, 1]$ be a non-increasing C^∞ function such that $\gamma = 1$ in $[0, 1/2]$, $\gamma = 0$ in $[1, \infty)$ and $\sup\{|\gamma'(t)| : t \in [0, \infty)\} \leq 4$, and define the following deleting path $p : (0, \infty) \rightarrow X$:

$$p(t) = \sum_{k=1}^{\infty} \gamma(2^{k-1}t)y_k.$$

It is quite clear that p is a well defined C^∞ path such that $p(t) = 0$ for $t \geq 1$. Let y be an arbitrary vector in X and let $F : (0, \infty) \rightarrow [0, \infty)$ be defined by $F(\alpha) = \omega(y - p(\alpha))$ for $\alpha > 0$. Let us see that $F(\alpha)$ satisfies the conditions of Lemma 2. If $\beta \geq \alpha$ then $\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta) \geq 0$ because γ is non-increasing, and also $\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta) \leq 4|2^{k-1}\alpha - 2^{k-1}\beta|$ because $\sup\{|\gamma'(t)| : t \in [0, \infty)\} \leq 4$. Note also that the property $\omega(z + y) \leq \omega(z) + \omega(y)$ implies that $\omega(x) - \omega(y) \leq \omega(x - y)$, as well as $\omega(\sum_{k=1}^{\infty} z_k) \leq \sum_{k=1}^{\infty} \omega(z_k)$ for every convergent series $\sum_{k=1}^{\infty} z_k$. Taking this into account and recalling the positive homogeneity of ω we may deduce

$$\begin{aligned} F(\beta) - F(\alpha) &= \omega(y - p(\beta)) - \omega(y - p(\alpha)) \\ &\leq \omega((y - p(\beta)) - (y - p(\alpha))) = \omega(p(\alpha) - p(\beta)) \\ &= \omega\left(\sum_{k=1}^{\infty} (\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))y_k\right) \\ &\leq \sum_{k=1}^{\infty} \omega((\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))y_k) \\ &= \sum_{k=1}^{\infty} (\gamma(2^{k-1}\alpha) - \gamma(2^{k-1}\beta))\omega(y_k) \\ &\leq \sum_{k=1}^{\infty} 4|2^{k-1}\alpha - 2^{k-1}\beta|\omega(y_k) \\ &= \sum_{k=1}^{\infty} 2^{k+1}\omega(y_k)|\beta - \alpha| \leq \sum_{k=1}^{\infty} 2^{k+1} \frac{1}{4^{k+1}} |\beta - \alpha| = \frac{1}{2}(\beta - \alpha) \end{aligned}$$

for every $\beta \geq \alpha$, so that the first condition in Lemma 2 is satisfied. Let us check that F also satisfies the second condition. Let $M > 0$ and choose $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^{k_0} T(y_j) > M + T(y)$ (this is clearly possible, as $T(y_k) \rightarrow 1$ when $k \rightarrow \infty$). Then, if $0 < \alpha < 1/2^{k_0}$, $\gamma(2^{j-1}\alpha) = 1$ for

$j = 1, \dots, k_0$, which implies

$$\begin{aligned} F(\alpha) &= \omega(y - p(\alpha)) = \|y - p(\alpha)\| - T(y) + T(p(\alpha)) \\ &\geq -T(y) + T(p(\alpha)) = -T(y) + \sum_{k=1}^{\infty} \gamma(2^{k-1}\alpha)T(y_k) \\ &\geq -T(y) + \sum_{j=1}^{k_0} \gamma(2^{j-1}\alpha)T(y_j) = -T(y) + \sum_{j=1}^{k_0} T(y_j) \\ &> -T(y) + M + T(y) = M \end{aligned}$$

for every $\alpha > 0$ such that $\alpha < 1/2^{k_0}$. This proves that

$$\lim_{t \rightarrow 0^+} F(t) = +\infty.$$

So, according to Lemma 2, the equation $F(\alpha) = \alpha$ has a unique solution. This means that for any $y \in X$, a number $\alpha(y) > 0$ with the property

$$\omega(y - p(\alpha(y))) = \alpha(y)$$

is uniquely determined. This implies that the mapping

$$\psi(x) = x + p(\omega(x))$$

is one-to-one from $X \setminus \{0\}$ onto X , with

$$\psi^{-1}(y) = y - p(\alpha(y)).$$

As ω and p are C^p , so is ψ . Let $\Phi(y, \alpha) = \alpha - \omega(y - p(\alpha))$. Since for any $y \in X$ we have $y - p(\alpha(y)) \neq 0$, the mapping Φ is differentiable on a neighbourhood of any point $(y_0, \alpha(y_0))$ in $X \times (0, \infty)$. On the other hand, since $F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ for $\beta \geq \alpha > 0$, it is clear that $F'(\alpha) \leq \frac{1}{2}$ for every α in a neighbourhood of $\alpha(y)$, and so

$$\frac{\partial \Phi(y, \alpha)}{\partial \alpha} = 1 - F'(\alpha) \geq 1 - \frac{1}{2} > 0.$$

Thus, using the implicit function theorem we deduce that the map $y \rightarrow \alpha(y)$ is of class C^p and therefore $\psi : X \setminus \{0\} \rightarrow X$ is a C^p diffeomorphism. Let $h : X \rightarrow X \setminus \{0\}$ be the inverse of ψ . It should be noted that $h(x) = x$ whenever $\omega(x) = \|x\| - T(x) \geq 1$. In order to conclude the proof we only need to compose h with a C^p diffeomorphism $g : X \rightarrow X$ transforming the set $\{x \in X : \|x\| \leq 1\}$ onto $\{x \in X : \omega(x) \leq 1\}$. The existence of such a diffeomorphism is ensured by the following lemma, which is a restatement of Lemma 2 in [7]; see also [2]. So define $\varphi = g^{-1} \circ h \circ g$. It is clear that φ is a C^p diffeomorphism from X onto $X \setminus \{0\}$ such that φ is the identity outside the unit ball of X .

LEMMA 4. Let X be a Banach space, and let U_1, U_2 be C^p smooth closed convex bodies containing no ray emanating from the origin, and such that

the origin is an interior point of both U_1 and U_2 . Then there exists a C^p diffeomorphism $g : X \rightarrow X$ such that $g(0) = 0$, $g(U_1) = U_2$, and $g(\partial U_1) = \partial U_2$, where ∂U_j stands for the boundary of U_j . Moreover, $g(x) = \lambda(x)x$, where $\lambda : X \rightarrow [0, \infty)$, and hence g takes each of the rays emanating from the origin onto itself.

In the case when X is a reflexive infinite-dimensional Banach space the problem was solved quite a long time ago. We can recall the results of T. Dobrowolski [5] to state the following

PROPOSITION 5. Let $(X, \|\cdot\|)$ be a reflexive infinite-dimensional Banach space with a C^p smooth norm $\|\cdot\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.

Proof. Since X is reflexive, X can be linearly injected into some $c_0(\Gamma)$ and, according to Proposition 5.1 of [5], X admits a C^∞ non-complete norm ω (which may be assumed to satisfy $\omega(x) \leq \|x\|$). Then, using Proposition 3.1 of [5], we get a C^∞ diffeomorphism $h : X \rightarrow X \setminus \{0\}$ such that $h(x) = x$ if $\omega(x) \geq 1$. An application of Lemma 4 as at the end of the proof of Proposition 3 gives us the desired diffeomorphism φ .

Combining Propositions 3 and 5 we get the following

THEOREM 6. Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a C^p smooth norm $\|\cdot\|$. Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$.

In fact, this result can be viewed as a corollary to the following more general result. Recall that a (not necessarily equivalent) norm ρ in a Banach space $(X, \|\cdot\|)$ is said to be C^p smooth if it is so with respect to $\|\cdot\|$, which in principle does not imply the differentiability of ρ with respect to itself.

THEOREM 7. Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space having a (not necessarily complete) C^p smooth norm ρ . Then there exists a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\rho(x) \geq 1$. If in addition the extension of ρ to the completion of the normed space (X, ρ) is C^p differentiable (with respect to itself), then there exists a bijection φ between X and $X \setminus \{0\}$ which is a C^p diffeomorphism in each of the norms $\|\cdot\|$ and ρ and such that $\varphi(x) = x$ whenever $\rho(x) \geq 1$.

Proof. If ρ is complete then it is an equivalent C^p smooth norm on X , and we can deduce that X and $X \setminus \{0\}$ are C^p diffeomorphic from Propositions 3 and 5. If, on the contrary, ρ is not complete, we can use Proposition 3.1 of [5] to conclude that X and $X \setminus \{0\}$ are C^p diffeomorphic.

Let us complete the proof of Theorem 1. We will do nothing but adapt the ideas of Bessaga [1] to the more general setting of a differentiable C^p norm $\|\cdot\|$ ($p \in \mathbb{N} \cup \{\infty\}$) whose sphere might contain segments and consequently

the usual stereographic projection might not be well defined for the whole sphere.

Let us choose a point $x_0 \in S_X$ and see first that $S_X \setminus \{x_0\}$ is diffeomorphic to any hyperplane H in X . Put $x^* = d\|\cdot\|(x_0)$, $Z = \ker x^*$, and consider the decomposition $X = [x_0] \oplus Z = \mathbb{R} \times Z$. Take a C^∞ convex body U on the plane \mathbb{R}^2 such that the set $\{(t, s) : t^2 + s^2 = 1, t \geq 0\} \cup \{(-1, s) : |s| \leq 1/2\}$ is contained in ∂U , the boundary of U . Consider the Minkowski functional of U , $q_U(t, s) = \inf\{\lambda > 0 : (t, s) \in \lambda U\}$, which is C^∞ smooth away from $(0, 0)$. Define $Q(t, z) = q_U(t, \|z\|)$ for every $(t, z) \in \mathbb{R} \times Z$. It is quite clear that Q is a C^p function away from the ray $\{\lambda x_0 : \lambda > 0\}$ (and Q is C^1 smooth on $X \setminus \{0\}$). Now consider the convex body $V = \{(t, z) \in X : Q(t, z) \leq 1\}$ and its boundary ∂V . The proof of Lemma 4 (see [2] or [7]) shows that the sets $\partial V \setminus \{x_0\}$ and $S_X \setminus \{x_0\}$ are C^p diffeomorphic (whereas ∂V and S_X are C^1 diffeomorphic). Note that for every $z \in Z$ the ray joining z to x_0 intersects the set ∂V at a unique point. This means that the stereographic projection $\pi : \partial V \setminus \{x_0\} \rightarrow Z_{-1}$ (where $Z_{-1} = \{x \in X : x^*(x) = -1\}$ is the tangent hyperplane to ∂V at $-x_0$), defined by means of the rays emanating from x_0 , is a well defined one-to-one mapping from $\partial V \setminus \{x_0\}$ onto Z_{-1} , and it is easy to check that π is a C^p diffeomorphism between $\partial V \setminus \{x_0\}$ and Z_{-1} . Since any two closed hyperplanes in X are isomorphic this proves that $\partial V \setminus \{x_0\}$ is C^p diffeomorphic to each hyperplane H in X , and hence so is $S_X \setminus \{x_0\}$.

Thus, to complete the proof of Theorem 1 it only remains to show that $S_X \setminus \{x_0\}$ and S_X are C^p diffeomorphic, which we can do by choosing a suitable atlas for S_X and using Theorem 6. Recall that $x^* = d\|\cdot\|(x_0)$ and $Z = \ker x^*$. Define $D_1 = \{x \in S_X : x^*(x) > -1/2\}$ and $D_2 = \{x \in S_X : x^*(x) < 1/2\}$, and let $\pi_1 : D_1 \rightarrow Z$ be the stereographic projection defined by means of the rays coming from $-x_0$, and $\pi_2 : D_2 \rightarrow Z$ the stereographic projection defined by means of the rays emanating from x_0 . Note that, although the sphere S_X might contain segments, these stereographic projections are well defined because they have been restricted to D_1 and D_2 , sets which cannot contain a segment passing through $-x_0$ and x_0 respectively. Let $G_1 = \{x \in D_1 : x^*(x) > 1/2\}$ and consider $\pi_1(G_1) \subseteq Z$. Since $\pi_1(G_1)$ is an open set in Z containing 0, there exists $\varepsilon > 0$ such that $\{z \in Z : \|z\| \leq \varepsilon\} \subseteq \pi_1(G_1)$. Now, from Theorem 6 we get a diffeomorphism $\varphi : Z \rightarrow Z \setminus \{0\}$ such that $\varphi(z) = z$ whenever $\|z\| \geq 1$. Let $h(z) = \varepsilon\varphi(\varepsilon^{-1}z)$ for each $z \in Z$. It is clear that h is a C^p diffeomorphism between Z and $Z \setminus \{0\}$ such that $h(z) = z$ whenever $\|z\| \geq \varepsilon$. Finally, define $g : S_X \rightarrow S_X \setminus \{x_0\}$ by

$$g(x) = \begin{cases} x & \text{if } x \in D_2, \\ \pi_1^{-1}(h(\pi_1(x))) & \text{if } x \in D_1. \end{cases}$$

It is easy to check that g is a C^p diffeomorphism from S_X onto $S_X \setminus \{x_0\}$. This concludes the proof of Theorem 1.

FINAL REMARKS. 1. It is worth noting that Theorem 6 above enlarges the class of spaces for which some results of B. M. Garay [8, 9] concerning the existence of solutions to ordinary differential equations and cross-sections of solution funnels in infinite-dimensional Banach spaces are valid.

2. Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space having a (not necessarily complete) Fréchet differentiable norm ϱ . It is natural to consider the unit sphere $S_\varrho = \{x \in X : \varrho(x) = 1\}$ and ask whether S_ϱ is diffeomorphic to each closed hyperplane H in X . One can show that this is the case, using Theorems 6 or 7 as in the proof of Theorem 1.

3. Let X be the reflexive Banach space constructed by W. T. Gowers and B. Maurey in [12] which is not isomorphic and therefore is not diffeomorphic to its closed hyperplanes. Being reflexive, X has an equivalent Fréchet differentiable norm $\|\cdot\|$ (see, e.g., [15] or [4]). By Theorem 1, the unit sphere S_X is diffeomorphic to a hyperplane of X and hence S_X is not diffeomorphic to the whole of X .

4. The following problem concerning negligibility of points in infinite-dimensional Banach spaces remains unsolved: let X be an infinite-dimensional Banach space having a C^p smooth bump function. Is there a C^p diffeomorphism φ between X and $X \setminus \{0\}$ such that $\varphi(x) = x$ whenever $\|x\| \geq 1$?

Acknowledgements. I am very grateful to J. A. Jaramillo and T. Dobrowolski for several valuable conversations and for encouraging me to study these problems.

References

- [1] C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 27–31.
- [2] —, *Interplay between infinite-dimensional topology and functional analysis. Mappings defined by explicit formulas and their applications*, Topology Proc. 19 (1994), 15–35.
- [3] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Monograf. Mat. 58, PWN, Warszawa, 1975.
- [4] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs Surveys Pure and Appl. Math. 64, Longman, 1993.
- [5] T. Dobrowolski, *Smooth and \mathbb{R} -analytic negligibility of subsets and extension of homeomorphisms in Banach spaces*, Studia Math. 65 (1979), 115–139.
- [6] —, *Every infinite-dimensional Hilbert space is real-analytically isomorphic with its unit sphere*, J. Funct. Anal. 134 (1995), 350–362.
- [7] —, *Relative classification of smooth convex bodies*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), 309–312.

- [8] B. M. Garay, *Cross-sections of solution funnels in Banach spaces*, Studia Math. 97 (1990), 13–26.
- [9] —, *Deleting homeomorphisms and the failure of Peano's existence theorem in infinite-dimensional Banach spaces*, Funkcial. Ekvac. 34 (1991), 85–93.
- [10] K. Goebel and J. Wośko, *Making a hole in the space*, Proc. Amer. Math. Soc. 114 (1992), 475–476.
- [11] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. 26 (1994), 523–530.
- [12] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.
- [13] R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. 113 (1964), 129–140.
- [14] V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, *ibid.* 74 (1953), 10–43.
- [15] S. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. 37 (1971), 173–180.

Departamento de Análisis Matemático
 Facultad de Ciencias Matemáticas
 Universidad Complutense
 Madrid, 28040, Spain
 E-mail: daniel@sunam1.mat.ucm.es

Received December 29, 1996
 Revised version March 11, 1997

(3811)

Non-reflexive pentagon subspace lattices

by

M. S. LAMBROU (Iraklion) and
 W. E. LONGSTAFF (Nedlands, Western Australia)

*Dedicated to Paul R. Halmos
 in celebration of his 80th birthday*

Abstract. On a complex separable (necessarily infinite-dimensional) Hilbert space H any three subspaces K , L and M satisfying $K \cap M = (0)$, $K \vee L = H$ and $L \subset M$ give rise to what has been called by Halmos [4, 5] a *pentagon subspace lattice* $\mathcal{P} = \{(0), K, L, M, H\}$. Then $n = \dim M \ominus L$ is called the *gap-dimension* of \mathcal{P} . Examples are given to show that, if $n < \infty$, the order-interval $[L, M]_{\text{Lat Alg } \mathcal{P}} = \{N \in \text{Lat Alg } \mathcal{P} : L \subseteq N \subseteq M\}$ in $\text{Lat Alg } \mathcal{P}$ can be either (i) a nest with $n + 1$ elements, or (ii) an atomic Boolean algebra with n atoms, or (iii) the set of all subspaces of H between L and M . For $n > 1$, since $\text{Lat Alg } \mathcal{P} = \mathcal{P} \cup [L, M]_{\text{Lat Alg } \mathcal{P}}$, all such examples of pentagons are non-reflexive, the examples in case (iii) extremely so.

1. Introduction. On a complex separable Hilbert space H any three (closed) subspaces K , L and M satisfying $K \cap M = (0)$, $K \vee L = H$ and $L \subset M$ give rise to what has been called by Halmos [4, 5] a *pentagon subspace lattice* $\mathcal{P} = \{(0), K, L, M, H\}$. Here inclusion is the partial order and a labelled Hasse diagram of \mathcal{P} is given in Figure 1.

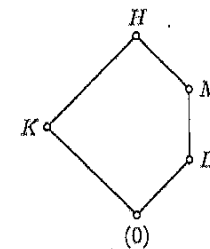


Fig. 1