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Difference and Differential Riccati Equations: A Note on the Convergence to the Strong Solution

Giuseppe De Nicolao and Michel Gevers

***Abstract*—This note deals with the convergence of the solutions of the differential and difference Riccati equations to the strong solution of the corresponding ARE. Detectability only is required in the analysis and no assumption is made on the eigenvalues on the real imaginary axis (on the unit circle, in the discrete-time case). In particular, from our result, it follows that, under the sole assumption of detectability, a positive definite initial condition guarantees convergence to the strong solution, even in the presence of unreachable eigenvalues on the imaginary axis or on the unit circle.**

I. Introduction

This note is devoted to the analysis of some convergence properties of the solutions of the following equations of optimal filtering: the continuous-time differential Riccati equation (RE)

$$\dot{P}(t) = AP(t) + P(t)A' + BB' - P(t)C'CP(t), \qquad P(0) = \Pi_0 \quad (1.a)$$

and the discrete-time difference Riccati equation (RE)

$$P(t+1) = AP(t)A' + BB' - AP(t)C'\left[CP(t)C' + I\right]^{-1} \cdot CP(t)A', \qquad P(0) = \Pi_0 \quad (1.b)$$

where A, B, C are constant real matrices and Π_0 is symmetric nonnegative definite. Any constant solution $P(t) = P$, $\forall t$, of the RE satisfies the corresponding (continuous-time or discrete-time) algebraic Riccati equation (ARE)

$$0 = AP + PA' + BB' - PC'CP \quad (2.a)$$

$$P = APA' + BB' - APC'\left[CPC' + I\right]^{-1}CPA'. \quad (2.b)$$

In correspondence of a solution of the ARE, one can define the continuous-time closed-loop state-transition matrix

$$F = A - PC'C$$

and its discrete-time counterpart

$$F = A - APC'\left[CP(t)C' + I\right]^{-1}C.$$

In order to use the same statements in continuous- and discrete-time, the term *stable (boundary) eigenvalues* will be used to denote the eigenvalues in the open left-half plane (on the imaginary axis), in continuous-time, and the eigenvalues in the open unit disk (on the unit circle), in discrete-time.

One of the main topics in the analysis of the Riccati equation is the study of the attractiveness properties of the solutions of the ARE: under which conditions does a solution of the RE asymptotically converge to a solution of the ARE? The classical results, which require reachability and observability [1] or stabilizability and detectability [2], were extended [3], [4], to include the nonstabilizable case. In particular, in [4] a necessary and sufficient condition for convergence was established. However, such a condition was stated under two basic assumptions: the detectability of (A, C) and the nonexistence of (A, B)-unreachable boundary eigenvalues. This last hypothesis was partly relaxed in [5]-[7]. The convergence analysis in these latter works addresses the convergence to the strong solution, when the initial condition is positive semidefinite. In the presence of (A, B)-unreachable boundary eigenvalues, however, there is only one sufficient condition available for the convergence to the strong solution. Such a condition is stated in Theorem 1 below. The purpose of this note is to provide a more general convergence condition (Theorem 2). Besides being of independent interest, this result could prove useful in extending the thorough analysis of [4] to the case with (A, B)-unreachable boundary eigenvalues. As a significant corollary, detectability and a positive definite initial condition always guarantee the convergence to the strong solution, without any further assumption on the reachability of the boundary eigenvalues.

The proofs are rather simple, being based on matrix manipulations and basic notions of linear algebra. The results are worked out for both the continuous- and discrete-time case. After the introduction of the basic tools, Lemma 2 clarifies the effect of the presence of (A, B)-unreachable eigenvalues on the structure of any solution of the ARE. Then, Lemma 2 together with some known convergence results is used to derive the main result of the note (Theorem 3). Needless to say, the results extend by duality to the Riccati equations for the optimal control problem.

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II. Preliminaries

In this section, some preliminary definitions and a lemma of [6], [8] are concisely recalled.

Definition 1: An eigenvalue of A is said to be unobservable (of rank p) if and only if there exist n-dimensional vectors $y_i \neq 0$, $i = 1, \cdots, p$, $y_0 = 0$ such that

$$Ay_i = \lambda y_i + y_{i-1}$$
$$Cy_i = 0. \quad ■$$

As is well known, (A, C) is detectable iff all the nonstable eigenvalues of A are (A, C)-observable.

Definition 2: An eigenvalue of A is said to be (A, B)-unreachable (of rank p) if and only if there exist n-dimensional vectors $v_i \neq 0$, $i = 1, \cdots, p$, $v_0 = 0$ such that

$$A' v_i = \lambda v_i + v_{i-1}$$
$$B' v_i = 0. \quad ■$$

The subspace spanned by the vectors v_i will be termed (A, B)-unreachable eigenspace associated with λ. The sum of all the (A, B)-unreachable eigenspaces of A associated with boundary eigenvalues will be denoted by $\bar{E}(A, B)$.

Definition 3: A real symmetric positive semidefinite solution P of the ARE is called *strong* if the eigenvalues of the corresponding closed-loop state-transition matrix are only stable or boundary eigenvalues.

Lemma 1: Consider two RE's with the same A, B, C matrices but possibly different initial conditions Π_1 and Π_2. Then, $\Pi_1 \geq \Pi_2$ implies $P_1(t) \geq P_2(t)$, $t \geq 0$, where $P_i(t)$ denotes the solution of the RE with initial condition $P_i(0) = \Pi_i$.

III. Convergence Analysis

Let P_S denote the strong solution of the ARE and $P(\cdot)$ the solution of the RE with initial condition $P(0) = \Pi_0$. The following two sufficient conditions for the convergence to the strong solution were proven in [5]–[7].

Theorem 1: Subject to

i) (A, C) is detectable,

ii) $\Pi_0 \geq P_S$,

then $\lim_{t\to\infty} P(t) = P_S$.

Theorem 2: Subject to

i) (A, C) is detectable,

ii) A has no (A, B)-unreachable boundary eigenvalue,

iii) $\Pi_0 > 0$,

then $\lim_{t\to\infty} P(t) = P_S$. ■

In order to extend these convergence results, we first have to prove a lemma that relates the structure of any solution of the ARE to the (A, B)-unreachable eigenspaces associated with boundary eigenvalues.

Lemma 2: Assume that (A, C) is detectable and let P be any symmetric solution of the ARE. Then, $N[P] \supseteq \bar{E}(A, B)$.

Proof: Let λ be an (A, B)-unreachable boundary eigenvalue of A and consider the vectors v_i in Definition 2. We proceed by induction. Suppose that $Pv_s = 0$ hold for $s = i - 1$. Now the proof divides into a continuous-time and a discrete-time branch.

In continuous-time

$$v_i^*(AP + PA')v_i = (\lambda + \lambda^*)v_i^* P v_i + v_i^* P v_{i-1} + v_{i-1}^* P v_i = 0$$

where v_i^* denotes the transpose of the complex conjugate of v_i. Then, the ARE (2.a) and $v_i^* BB' v_i = 0$ imply $CPv_i = 0$. In view of (2.a), $APv_i + PA'v_i = 0$. Letting $y = Pv_i$, it follows that $Ay = -\lambda y$ and $Cy = 0$. Since $-\lambda$ is a boundary eigenvalue, in order not to contradict the detectability of (A, C), we have that $y = 0$.

In discrete-time

$$v_i^*(P - APA')v_i = v_i^*(1 - |\lambda|^2)Pv_i + v_{i-1}^* P v_{i-1} + v_i^* P v_{i-1} + v_{i-1}^* P v_i = 0.$$

In view of [9, theorem 2.5], $CPA + I > 0$.

The ARE (2.b) and $v_i^* BB' v_i = 0$ imply $CPA' v_i = 0$. Since $A' v_i = \lambda v_i + v_{i-1}$, it turns out that $CPv_i = 0$. In view of (2.b), $Pv_i = APA'v_i$, from which $Pv_i = \lambda APv_i$. Letting $y = Pv_i$, it follows that $Ay = \lambda^{-1} y$ and $Cy = 0$. Then, since λ^{-1} is a boundary eigenvalue, the detectability of (A, C) entails $y = 0$. ■

Without any loss of generality, one can always choose a basis such that the triple (A, B, C) takes the following form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where the eigenvalues of the square matrix A_{22} are all and only the (A, B)-unreachable boundary eigenvalues of A. Such a basis will be called standard basis. In the standard basis, by Lemma 2, any solution P of the ARE takes the form

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Matrix P_{11} turns out to be a solution of the reduced-order ARE characterized by the triple (A_{11}, B_1, C_1) in place of the triple (A, B, C). Moreover, by exploiting the block-partitioned structure, it can be verified that, if P is a strong solution of the ARE, P_{11} is a strong solution of the reduced-order ARE. Finally, it is easy to see that, if $N[\Pi_0] \supseteq \bar{E}(A, B)$, i.e.,

$$\Pi_0 = \begin{bmatrix} \Pi_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

in the standard basis, then $N[P(t)] \supseteq \bar{E}(A, B)$, $t \geq 0$, where $P(\cdot)$ denotes the solution of the RE with initial condition $P(0) = \Pi_0$. This means that

$$P(t) = \begin{bmatrix} P_{11}(t) & 0 \\ 0 & 0 \end{bmatrix}$$

$P_{11}(\cdot)$ being the solution of the reduced-order RE with initial condition $P_{11}(0) = \Pi_{11}$.

Now, after an auxiliary lemma, the main convergence result is provided.

Lemma 3: Let $\Pi_0 \geq 0$ and $\bar{E}(A, B) \supseteq N[\Pi_0]$, and let

$$\Pi_0 = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}' & \Pi_{22} \end{bmatrix}$$

be the representation of Π_0 in the standard basis with $\Pi_{11} > 0$ and of the same dimension as A_{11}. Then, there exists $\tilde{\Pi}_{11} > 0$, of the same dimension as Π_{11}, such that

$$\Pi_0 \geq \begin{bmatrix} \tilde{\Pi}_{11} & 0 \\ 0 & 0 \end{bmatrix} \triangleq \tilde{\Pi}_0.$$

Proof: Let the vector $[x_1' \;\; x_2']'$ be partitioned similarly to Π_0 and denote

$$\lambda_{\min} = \min_{[x:\, x_1 \neq 0]} \frac{x' \Pi_0 x}{x_1' x_1}.$$

Since $\bar{E}(A, B) \supseteq N[\Pi_0]$, it follows that $\Pi_0 x = 0$ only if $x_1 = 0$. Therefore, and since $\Pi_0 \geq 0$, we have $\lambda_{\min} > 0$. Define $\tilde{\Pi}_{11} = \epsilon I$ with $0 < \epsilon < \lambda_{\min}$. Now consider

$$x'(\Pi_0 - \tilde{\Pi}_0)x = x_1'(\Pi_{11} - \tilde{\Pi}_{11})x_1 + 2x_1'\Pi_{12}x_2 + x_2'\Pi_{22}x_2$$

for some nonzero x.

Case i): If $x_1 = 0$, then $x'(\Pi_0 - \tilde{\Pi}_0)x = x_2'\Pi_{22}x_2 \geq 0$ because $\Pi_0 \geq 0$.

Case ii): If $x_1 \neq 0$, then

$$\begin{aligned} x'(\Pi_0 - \tilde{\Pi}_0)x &= x'\Pi_0 x - \epsilon x_1' x_1 \\ &\geq \lambda_{\min} x_1' x_1 - \epsilon x_1' x_1 \\ &= (\lambda_{\min} - \epsilon) x_1' x_1 > 0. \end{aligned}$$

Theorem 3: Subject to

i) (A, C) is detectable,

ii) $\Pi_0 \geq 0$,

iii) $\bar{E}(A, B) \supseteq N[\Pi_0]$,

then $\lim_{t\to\infty} P(t) = P_S$, where $P(\cdot)$ is the solution of the RE with initial condition $P(0) = \Pi_0$ and P_S is the strong solution of the ARE.

Proof: In view of Lemma 3, it is always possible to find $\tilde{\Pi}_{11} > 0$ such that in the standard basis

$$\tilde{\Pi}_0 = \begin{bmatrix} \tilde{\Pi}_{11} & 0 \\ 0 & 0 \end{bmatrix} \leq \Pi_0.$$

Consider now the reduced-order RE associated with the triple (A_{11}, B_1, C_1). The pair (A_{11}, C_1) is detectable and A_{11} has no (A_{11}, B_1)-unreachable boundary eigenvalue. Then, by Theorem 2, $\tilde{P}_{11}(\cdot)$ converges to P_{11S}, where $\tilde{P}_{11}(t)$ denotes the solution of the reduced-order RE with initial condition $\tilde{P}_{11}(0) = \tilde{\Pi}_{11}$, and P_{11S} is the strong solution of the corresponding reduced-order ARE. Note that

$$P_S = \begin{bmatrix} P_{11S} & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, denoting by $\tilde{P}(\cdot)$ the solution of the RE (1) with initial condition $\tilde{P}(0) = \tilde{\Pi}_0$, $\tilde{P}(\cdot)$ converges to P_S.

It is also always possible to find $\bar{\Pi}_0$ such that $\bar{\Pi}_0 \geq \Pi_0$ and $\bar{\Pi}_0 \geq P_S$. Then, by Theorem 1, letting $\bar{P}(\cdot)$ be the solution of the RE (1), with initial condition $\bar{P}(0) = \bar{\Pi}_0$, $\bar{P}(\cdot)$ converges to P_S.

Finally, Lemma 1 entails that $\tilde{P}(t) \leq P(t) \leq \bar{P}(t)$, $t \geq 0$, so that the thesis follows.

Corollary: If (A, C) is detectable and $\Pi_0 > 0$, then $\lim_{t\to\infty} P(t) = P_S$. ■

Theorem 3 improves on existing convergence results in that it handles systems having possibly unreachable boundary eigenvalues. If we restrict our attention to the class of detectable systems with no unreachable boundary eigenvalues, a necessary and sufficient condition for convergence to the strong solution is already available [4]. A comparison of [4] with our Theorem 3 shows that, for detectable systems with a nonnegative Π_0, condition iii) of Theorem 3 is only sufficient. In conclusion, the search for a necessary and sufficient condition for convergence to the strong solution in the case of detectable systems is still an open question.

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Norm Based Robust Control of State-Constrained Discrete-Time Linear Systems

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Abstract—**Most realistic control problems involve some type of constraint. However, to date, all the algorithms that deal with constrained problems assume that the system is perfectly known. On the other hand, during the last decade a considerable amount of time has been spent in the robust control problem. However, in its present form, the robust control theory can address only the idealized situation of completely unconstrained problems. In this note we present a theoretical framework to analyze the stability properties of constrained discrete-time systems under the presence of uncertainty and we show that this formalism provides a unifying approach, including as a particular case the well-known technique of estimating robustness bounds from the solution of a Lyapunov equation. These results are applied to the problem of designing feedback controllers capable of stabilizing a family of systems, while at the same time satisfying state-space constraints.**

I. Introduction

A large class of problems frequently encountered in practice involves the control of linear systems with states restricted to closed convex regions of space. Several methods have been proposed recently to deal with this class of problems (see [1] for a thorough discussion and several examples), but as a rule, all of these schemas assume exact knowledge of the dynamics involved (i.e., exact knowledge of the model). Such an assumption can be too restrictive, ruling out cases where good qualitative models of the plant are available but the numerical values of various parameters are unknown or even change during operation. On the other hand, during the last decade a considerable amount of time has been spent analyzing the question of whether some relevant quantitative properties of a system (most notably asymptotic stability) are preserved under the presence of unknown perturbations. This research effort has led to procedures for designing controllers, termed "robust controllers," capable of achieving desirable properties under various classes of perturbations. However, these design procedures cannot accommodate directly time domain constraints, although some progress has been made recently in this direction [2]-[4].

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