87. *Difference Approximation of Evolution Equations and Generation of Nonlinear Semigroups*

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We consider the following nonlinear evolution equation

(DE) $\qquad (d/dt)u(t) \in Au(t), \qquad 0<t<T,$

where A is a (multi-valued) quasi-dissipative operator. In this note, we construct the solution of the evolution equation (DE) by the method of difference approximation. In addition, we give a generation theorem of nonlinear semigroups through the difference approximation. We sketch here our results. The details will be treated in [6].

1. Preliminaries. Let X be a real Banach space. For the multi-valued operator A, we use the following notations:

$$D(A)=\{x \in X\,;\, Ax \neq \phi\}, \qquad R(A)=\bigcup_{x \in D(A)} \{y\,;\, y \in Ax\},$$
$$\text{and} \quad |||Ax|||=\inf \{\|y\|\,;\, y \in Ax\} \qquad \text{for } x \in D(A).$$

We identify the multi-valued operator A with its graph, so that we write $[x, y] \in A$ if $y \in Ax$.

Let F be the duality map from X into X^*. Then we set

$$\langle y, x\rangle_i=\inf \{\langle y, f\rangle\,;\, f \in F(x)\} \qquad \text{for } x, y \in X.$$

Let $A \subset X \times X$. A is said to be *dissipative* if for any $[x_i, y_i] \in A$ $(i=1,2)$,

$$\langle y_i-y_2, x_1-x_2\rangle_i \leq 0.$$

According to Takahashi [9], we introduce the following notion as a generalization of that of dissipative operators.

Definition 1. Let $A \subset X \times X$. A is said to be *quasi-dissipative* if for any $[x_i, y_i] \in A$ $(i=1,2)$,

$$\langle y_1, x_1-x_2\rangle_i+\langle y_2, x_2-x_1\rangle_i \leq 0.$$

The following example shows that quasi-dissipative operators are not always dissipative.

Example (I. Miyadera). Let $X=R^2$ with the maximum norm. Let $x_1=(1,1)$ and $x_2=(0,0)$. We set $D(A)=\{x_1, x_2\}$, $Ax_1=\{(\alpha, \beta)\,;\, \alpha \leq 0$ or $\beta \leq 0\}$ and $Ax_2=\{(\alpha, \beta)\,;\, \alpha \geq 0$ or $\beta \geq 0\}$. Then A is quasi-dissipative in X but $A-\omega$ is not dissipative in X for any real ω. In addition, $R(I-\lambda A) \supset D(A)$ for any $\lambda>0$.

The following plays a central role in our argument.

Lemma 1. *Let $A \subset X \times X$. Then the following are equivalent:*

(i) *A is quasi-dissipative;*

(ii) *for any* $[x_i, y_i] \in A$ $(i=1,2)$ *and* $\lambda, \mu>0$,

$$(\lambda+\mu)\|x_1-x_2\|\leq\lambda\|x_1-x_2-\mu y_1\|+\mu\|x_2-x_1-\lambda y_2\|;$$

(iii) *for any* $[x_i, y_i] \in A$ $(i=1,2)$ *and* $\lambda>0$,

$$2\|x_1-x_2\|\leq\|x_1-x_2-\lambda y_1\|+\|x_2-x_1-\lambda y_2\|.$$

We can verify Lemma 1 similarly as Lemma 1.1 in Kato [4].

Let $X_0\subset X$. A one parameter family $\{T(t); t\geq 0\}$ of operators from X_0 into itself is called (nonlinear) *contraction semigroup* on X_0 if it has the following properties:

(i) $\|T(t)x-T(t)y\|\leq\|x-y\|$ for $x, y \in X_0$ and $t\geq 0$;

(ii) $T(0)x=x$ for $x\in X_0$ and $T(t+s)=T(t)T(s)$ for $t, s\geq 0$:

(iii) for each $x\in X_0$, $T(t)x$ is strongly continuous in $t\geq 0$.

2. Cauchy problems and difference approximation. Let A be a quasi-dissipative operator in X. Let $x_0\in X$ and $T>0$. Then we treat the following Cauchy problem for the evolution equation (DE):

$$(\mathrm{CP}; x_0)\qquad \begin{cases}(d/dt)u(t)\in Au(t) & \text{for } t\in(0,T),\\ u(0)=x_0.\end{cases}$$

For the Cauchy problem $(\mathrm{CP}; x_0)$, we consider the following type of difference approximation:

$$(\mathrm{DS}; x_0)\qquad \begin{cases}\dfrac{x_k^n-x_{k-1}^n}{t_k^n-t_{k-1}^n}-\varepsilon_k^n\in Ax_k^n, & k=1,2,\cdots,N_n;\ n\geq 1,\\ x_0^n=x_0,\end{cases}$$

where for each n, $\{t_k^n\}$ represents the partition of $[0, T]$ such that $0=t_0^n<t_1^n<\cdots<t_{N_n-1}^n<T\leq t_{N_n}^n$ and $\delta_n=\max_{1\leq k\leq N_n}(t_k^n-t_{k-1}^n)\to 0$ as $n\to\infty$. The ε_k^n may be referred as an error which occurs at the k-th step of the n-th approximation of the difference approximation. In this sense $(\mathrm{DS}; x_0)$ can be regarded as an approximating difference scheme for $(\mathrm{CP}; x_0)$ which permits errors.

Definition 2. Let $u_n(t)$ be a sequence in $L^\infty(0, T; X)$. We say that $u_n(t)$ is a (backward) DS-*approximate solution* of the Cauchy problem $(\mathrm{CP}; x_0)$ if there exists a difference approximation $(\mathrm{DS}; x_0)$ satisfying the followings:

(i) $u_n(0)=x_0^n=x_0$, $n\geq 1$;

(ii) $u_n(t)=x_k^n$ for $t\in(t_{k-1}^n, t_k^n]\cap(0,T]$, $k=1,2,\cdots,N_n$; $n\geq 1$,

(iii) $\sum_{k=1}^{N_n}\|\varepsilon_k^n\|(t_k^n-t_{k-1}^n)\to 0$ as $n\to\infty$.

Then we have

Theorem 1. *Let* $x_0\in\overline{D(A)}$ *and* $u_n(t)$ *be a DS-approximate solution of* $(\mathrm{CP}; x_0)$ *on* $[0, T]$. *Then there exists a* $u(t)\in C([0,T]; X)$ *satisfying the followings*:

(i) $u(t)=\lim_{n\to\infty}u_n(t)$ *for* $t\in[0, T]$, *and the convergence is uniform on* $[0, T]$;

(ii) $u(t)\in\overline{D(A)}$ *for* $t\in[0, T]$ *and* $u(0)=x_0$;

(iii) *for any DS-approximate solution* $\hat{u}_n(t)$ *of* $(\mathrm{CP}; x_0)$,

$$u(t)=\lim_{n\to\infty} \hat{u}_n(t) \qquad \text{for } t \in [0, T].$$

Remarks. 1) Kenmochi-Oharu [5] and Takahashi [9], [10] studied the convergence (i) under the additional condition, which is called the stability condition by them. Our result is an extension of their results.

2) By Bénilan's method [2], we find that the limiting function $u(t)$ is the unique integral solution of the Cauchy problem (CP; x_0).

The proof of Theorem 1 is based on the following.

Lemma 2. *Let* (DS; x_0) *and* (DS; $\hat{x}_0$) *be two difference approximations as above of the Cauchy problems* (CP; x_0) *and* (CP; $\hat{x}_0$) *on* $[0, T]$, *respectively. Let the notations with the symbol "$\hat{\ }$" represent the difference approximation* (DS; $\hat{x}_0$). *Then we have*

$$\begin{aligned} \|x_i^m-\hat{x}_j^n\| \le & \|x_0-u\|+\|\hat{x}_0-u\| \\ & +\{(t_i^m-\hat{t}_j^n)^2+\delta_m t_i^m+\hat{\delta}_n \hat{t}_j^n\}^{1/2}|||Au||| \\ & +\sum_{k=1}^{i}\|\varepsilon_k^m\|(t_k^m-t_{k-1}^m)+\sum_{k=1}^{j}\|\hat{\varepsilon}_k^n\|(\hat{t}_k^n-\hat{t}_{k-1}^n), \end{aligned} \tag{1}$$

for $0 \le i \le N_m$, $0 \le j \le \hat{N}_n$ *and* $u \in D(A)$.

Lemma 2 is proved by the method of Crandall-Liggett [3], modified by Rasmussen [8] (see also Yosida [13]), by using Lemma 1.

Remark. Let A be a dissipative operator in X such that $R(I-\lambda A) \supset D(A)$ for $\lambda>0$. Then the estimate (1) gives

$$\|(I-\lambda A)^{-n}x-(I-\mu A)^{-m}x\| \le \{(n\lambda-m\mu)^2+n\lambda^2+m\mu^2\}^{1/2}|||Ax|||$$

for $n, m \ge 1$, $\lambda, \mu>0$ and $x \in D(A)$. This estimate is similar to but different from that of Crandall-Liggett [3].

By virtue of Theorem 1, we define the following.

Definition 3. Let $u(t) \in C([0, T]; X)$ and $x_0 \in \overline{D(A)}$. We say that $u(t)$ is a (backward) DS-*limit solution* of the Cauchy problem (CP; x_0) on $[0, T]$ if there exists a (backward) DS-approximate solution $u_n(t)$ of (CP; x_0) on $[0, T]$, such that $u_n(t)$ converges to $u(t)$, uniformly for $t \in [0, T]$.

By Lemma 2, we have also

Corollary. *Let* $u(t), \hat{u}(t)$ *be two DS-limit solutions of* (CP) *on* $[0, T]$. *Then*

$$\|u(t)-\hat{u}(t)\| \le \|u(0)-\hat{u}(0)\| \qquad \text{for } t \in [0, T].$$

3. Generation of semigroups. By Theorem 1 and Corollary, we have a generation theorem of semigroups.

Definition 4. Let A be a quasi-dissipative operator in X. We say that A has the *property* ($\mathfrak{D}$) if for any $x \in \overline{D(A)}$ and $T>0$, there exists a DS-approximate solution of the Cauchy problem (CP; x) on $[0, T]$.

Theorem 2. *Let A be a quasi-dissipative operator in X, having the property* ($\mathfrak{D}$). *Then there exists a contraction semigroup* $\{T(t); t \ge 0\}$ *on* $\overline{D(A)}$ *such that for each* $x \in \overline{D(A)}$ *and* $T>0, u(t)=T(t)x$ *is the unique DS-limit solution of the Cauchy problem* (CP; x) *on* $[0, T]$.

We give a sufficient condition that a quasi-dissipative operator has the property ($\mathfrak{D}$). Let A be a quasi-dissipative operator in X. We add the following condition on A:

(R_t) for any $x \in \overline{D(A)}$, there exist a sequence $\delta_n \downarrow 0$ and $[x_n, y_n] \in A$ $(n \geq 1)$ such that

$$\lim_{n\to\infty} \delta_n^{-1} \| x_n - x - \delta_n y_n \| = 0.$$

Then we have

Theorem 3. *Let A be a quasi-dissipative operator in X, satisfying the condition (R_t). Then A has the property ($\mathfrak{D}$). Thus A generates a contraction semigroup on $\overline{D(A)}$, in the sense of Theorem* 2.

Remarks. 1) This theorem implies the fundamental result of Crandall-Liggett [3]; a part of the results of Martin [7] on ordinary differential equations; and the results of Webb [11] and Barbu [1] on the continuous perturbations of m-dissipative operators.

2) Yorke announces in [12] that he obtained a similar result.

Sketch of the proof of Theorem 3. Let $x_0 \in \overline{D(A)}$ and $\varepsilon_n \downarrow 0$. Let n be fixed. Then for each $x \in \overline{D(A)}$, we define

$$\delta_n(x) = \sup \{\delta\,;\, 0 < \delta \leq \varepsilon_n \text{ and there exists } [x_\delta, y_\delta] \in A \text{ such that } \| x_\delta - x - \delta y_\delta \| \leq \delta \varepsilon_n \}.$$

Then each $\delta_n(x)$ is positive by the assumption. Therefore, inductively, we can choose $h_k^n > 0$ and $[x_k^n, y_k^n] \in A$, for $k = 1, 2, \cdots$, so that they satisfy the followings:

(i) $x_0^n = x_0$;

(ii) $(1/2)\delta_n(x_{k-1}^n) < h_k^n \leq \varepsilon_n$, for $k = 1, 2, \cdots$,

(iii) $\| x_k^n - x_{k-1}^n - h_k^n y_k^n \| \leq h_k^n \varepsilon_n$, for $k = 1, 2, \cdots$.

Then we set $t_i^n = \sum_{k=1}^{i} h_k^n$. We may show that $t_i^n \to \infty$ as $i \to \infty$. For the purpose, we establish the following estimate:

$$\| x_i^n - x_j^n \| \leq (t_i^n - t_j^n) \| y_k^n \| + \varepsilon_n (t_i^n - t_k^n) + \varepsilon_n (t_j^n - t_k^n) \tag{2}$$

for any $i \geq j \geq k \geq 1$. This estimate may be verified by the induction for (i, j) with $i \geq j \geq k$ for each fixed $k \geq 1$, by using Lemma 1.

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