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# Difference Calogero–Moser systems and finite Toda chains

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Limits of a recently introduced  $n$ -particle difference Calogero–Moser system with elliptic potentials are studied. We obtain hyperbolic and rational difference Calogero–Moser systems with an eight-parameter external field and (finite) difference Toda chains with four-parameter potentials acting on the boundary particles. Hamiltonians for a number of known integrable  $n$ -particle systems, such as Ruijsenaars' relativistic Calogero–Moser and Toda models and their generalizations associated with classical root systems, can be seen as special cases of the Hamiltonians considered in this paper. © 1995 American Institute of Physics.

## I. INTRODUCTION

The finite Toda chain and the Calogero–Moser system (CM) are nowadays classic examples of integrable  $n$ -particle models in dimension one.<sup>1–4</sup> In the CM system all particles interact pairwise by means of an inverse-square potential [or a (doubly)periodic generalization thereof], whereas in the Toda chain a particle interacts only with its nearest neighbors and the potential is of exponential type. Some years ago, Inozemtsev constructed a Lax pair representation for the classical  $n$ -particle dynamics generated by the generalized CM Hamiltonian<sup>5</sup>

$$H = 1/2 \sum_{1 \leq j \leq n} \theta_j^2 + g \sum_{1 \leq j < k \leq n} (\wp(x_j - x_k) + \wp(x_j + x_k)) + \sum_{1 \leq j \leq n} (g_0 \wp(x_j) + g_1 \wp(\omega_1 + x_j) + g_2 \wp(\omega_2 + x_j) + g_3 \wp(\omega_1 + \omega_2 + x_j)), \quad (1.1)$$

where  $\wp(\cdot)$  denotes the Weierstrasse  $\wp$ -function<sup>6</sup> with primitive periods  $2\omega_1, 2\omega_2$ . A lot of interesting Calogero–Moser and Toda type Hamiltonians, for which Lax pairs have been reported in the literature,<sup>7–10</sup> can be seen as limits of  $H$  (1.1). These limiting cases are characterized by Hamiltonians of the form:

*Calogero–Moser case*

$$H_{CM} = 1/2 \sum_{1 \leq j \leq n} \theta_j^2 + \sum_{1 \leq j < k \leq n} (v(x_j - x_k) + \varepsilon v(x_j + x_k)) + \sum_{1 \leq j \leq n} w(x_j); \quad (1.2)$$

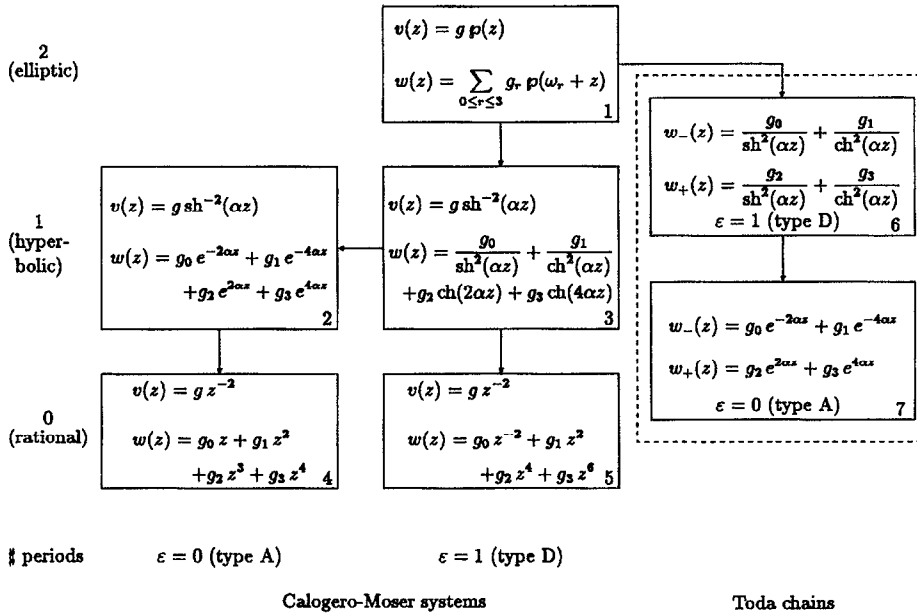


FIG. 1. Potentials of generalized Calogero–Moser systems and finite Toda chains.

Toda case

$$\begin{aligned}
 H_T &= (\theta_1^2 + \theta_2^2)/2 + e^{2\alpha(x_1 - x_2)} \\
 &+ \varepsilon(e^{-2\alpha(x_1 + x_2)} + e^{2\alpha(x_1 + x_2)} + e^{-2\alpha(x_1 - x_2)}) + w_-(x_1) + w_+(x_2), \quad n = 2, \\
 &= 1/2 \sum_{1 \leq j \leq n} \theta_j^2 + \sum_{1 \leq j < n} e^{2\alpha(x_j - x_{j+1})} \\
 &+ \varepsilon(e^{-2\alpha(x_1 + x_2)} + e^{2\alpha(x_{n-1} + x_n)}) + w_-(x_1) + w_+(x_n), \quad n \geq 3.
 \end{aligned}
 \tag{1.3}$$

The relevant potentials  $v, w$  and  $w_{\pm}$  have been collected in the diagram in Fig. 1. The parameter  $\varepsilon$  takes the value one or zero. If the external field potential  $w$  and the boundary terms  $w_{\pm}$  are zero, then the Hamiltonians  $H_{CM}$  (1.2) and  $H_T$  (1.3) are connected with the simply-laced classical root systems:  $A_{n-1}$  if  $\varepsilon = 0$ , and  $D_n$  if  $\varepsilon = 1$ . In this paper we will refer to the two cases  $\varepsilon = 0$  and  $\varepsilon = 1$  as systems of type A and of type D, respectively, regardless of whether the fields  $w, w_{\pm}$  vanish or not.

The Inozemtsev Hamiltonian  $H$  (1.1) corresponds to a type D Calogero–Moser system with potentials as in Block 1 of the diagram. (In Fig. 1 and in the rest of the paper we have used the convention  $\omega_0 = 0, \omega_3 = -\omega_1 - \omega_2$ .) The arrows in the diagram represent limit transitions between the Hamiltonians. For instance, the transition from Block 1 to Block 3 (or  $1 \rightarrow 3$ , for short) corresponds to sending a period to infinity along the real axis (hyperbolic limit). The transition  $3 \rightarrow 2$  amounts to sending the center of mass to infinity:  $x_j \rightarrow x_j + R$  ( $j = 1, \dots, n$ ),  $R \rightarrow \infty$ . Rational potentials are obtained by scaling  $\alpha$  to zero ( $2 \rightarrow 4$  and  $3 \rightarrow 5$ ). The transitions leading to Toda chains with generalized boundary potentials  $w_{\pm}$  (viz.  $1 \rightarrow 6$  and  $6 \rightarrow 7$ ) are explained in Ref. 10. It is important to remark here that, in the process of computing these limits, one has to rescale and reparametrize the coupling constants  $g_r$  and renormalize the Hamiltonian by the subtraction of possibly divergent constants.

Lax pairs for the dynamics generated by the above Hamiltonians can be found in Refs. 7, 8 (Blocks 2, 4), Refs. 8, 9 (Blocks 3, 5), and Ref. 10 (Blocks 6, 7). The integrability of the Hamiltonian is not immediate from the existence of these Lax pairs ( $n$  independent conserved quantities do exist, but their involutivity remains to be shown). A proof of integrability for the type A Calogero–Moser systems of Block 3 and 4 can be found in Refs. 11 and 12. The (quantum) integrability of the type D Calogero–Moser Hamiltonians associated with the Blocks 1, 3, and 5 was shown recently by means of an explicit construction of the integrals.<sup>13,14</sup> For the Toda chains with boundary conditions corresponding to Block 6 and 7 the (quantum) integrability can be proved using the  $R$ -matrix formalism.<sup>15,16</sup> For special values of the coupling constants  $g_r$ , the Hamiltonians of Fig. 1 can be associated with classical root systems (recall that  $w, w_{\pm} = 0$  (i.e.,  $g_r = 0$ ) corresponds to the simply-laced series  $A_{n-1}$  and  $D_n$ ). In these special cases the existence of Lax pairs, and more information regarding the integrability of the systems, was already known from previous work.<sup>1–4,17</sup>

It is clear from the diagram that the Hamiltonians of type D are more general than the type A Hamiltonians: the latter are limits of the former. Nevertheless, from a physical point of view the type A models are more interesting than their type D counterparts. This is because for type A the interaction between the particles depends only on the differences of the particle positions, whereas for type D also terms depending on the sum of the particle positions appear in the Hamiltonian.

In Refs. 18 and 19 we introduced a deformation of the Inozemtsev Hamiltonian  $H$  (1.1). Similar deformations of the type A versions of  $H_{CM}$  (1.2) with  $w = 0$  and  $H_T$  (1.3) with  $w_{\pm} = 0$ , were already introduced by Ruijsenaars *et al.*<sup>20–23</sup> At the quantum level the Hamiltonian of these deformed Calogero–Moser and Toda systems is given by a difference operator rather than a differential operator. For Ruijsenaars' systems (which may be interpreted as relativistic generalizations of the Calogero–Moser and Toda systems) both classical and quantum integrability was proved,<sup>21,22</sup> whereas for our deformation of the Inozemtsev Hamiltonian to date only partial results have been obtained.<sup>18</sup>

In this paper we study limit transitions similar to those in Fig. 1 for our difference counterpart of the quantum version of  $H$  (1.1). As a result we obtain difference counterparts of the Calogero–Moser and Toda Hamiltonians  $H_{CM}$  (1.2) and  $H_T$  (1.3) for each block of the diagram. The paper is organized as follows.

In Sec. II we begin with an analysis of some special cases for which we have complete results as regards the integrability of the system. The models of interest in this section amount to difference versions of the hyperbolic/rational CM systems with potentials as in Blocks 2–5 and with  $g_2, g_3 = 0$ . Starting point is a difference counterpart of the hyperbolic type D Calogero–Moser Hamiltonian corresponding to Block 3 (with  $g_2, g_3 = 0$ , this specialization is associated with the root system  $BC_n$ ). Explicit formulas representing a complete set of quantum integrals for this difference CM system were introduced in Ref. 18. Via limit transitions we arrive at the quantum integrals for difference versions of the quantum models corresponding to Blocks 2, 4 and 5 (again with  $g_2, g_3 = 0$ ).

In Sec. III we consider more general difference CM systems starting from the difference version of the Inozemtsev Hamiltonian  $H$  (1.1). This leads to difference versions of the quantum models corresponding to Blocks 2–5 for arbitrary values of  $g_2, g_3$ . As regards the integrability of the models in this section, only partial results have been obtained: apart from the Hamiltonian we found (to date) only one independent integral (which proves the integrability in the case of two particles).

The transition to difference Toda chains with boundary potentials is discussed in Sec. IV. If the potentials acting on the boundary particles are set to zero, then our chain reduces to the nonperiodic relativistic Toda system (type A) or to a  $D_n^{(1)}$ -type counterpart of this system (type D).

*Remarks:* i. Special cases of the systems under consideration were introduced at the level of classical mechanics by Schneider<sup>24</sup> (external field couplings to the relativistic CM system), Inozemtsev<sup>25</sup> (external field couplings to the relativistic CM system and its type D counterpart in

the case of two particles), and Suris<sup>26</sup> (boundary potentials for the relativistic Toda chain). It turns out that all models in Refs. 24–26 can be seen as limits of the models studied here.

ii. Two important models have not been included in the diagram: the  $A_{n-1}$ -type CM system with elliptic potentials (no external field) and the periodic Toda chain (type  $A_{n-1}^{(1)}$ ). It does not seem possible to view these two models as special (limiting) cases of the Inozemtsev system associated with Block 1 of the diagram. It is, therefore, quite remarkable that for their difference versions such a relation *does* exist: the quantum relativistic CM system with elliptic potentials (the difference version of the elliptic CM system related to the root system  $A_{n-1}$ ) is a limit of our difference version of the Inozemtsev system with Hamiltonian  $H$  (1.1);<sup>18</sup> furthermore, the periodic relativistic Toda chain, in turn, is a limit of the relativistic CM system with elliptic potentials.<sup>23</sup> (The second relation also holds at the nonrelativistic level.<sup>10</sup>)

## II. INTEGRABLE DIFFERENCE CALOGERO-MOSER SYSTEMS WITH HYPERBOLIC OR RATIONAL POTENTIALS

We start with the commuting quantum integrals for a difference version of the (type D) hyperbolic CM system associated with the root system  $BC_n$ . By sending the center of mass to infinity we arrive at the quantum integrals for a (type A) model consisting of Ruijsenaars’ quantum relativistic CM system coupled to an external field. Rational potentials are obtained by scaling the (imaginary) period of the hyperbolic potentials to infinity.

### A. Type D

In Ref. 18 we studied the quantum integrability of a one-dimensional  $n$ -particle Hamiltonian of the form

$$\hat{H}_1 = \sum_{\substack{1 \leq j \leq n \\ \epsilon = \pm 1}} (V_{\epsilon j}^{1/2} e^{-\epsilon \beta \hat{\theta}_j} V_{-\epsilon j}^{1/2} - V_{\epsilon j}), \tag{2.1}$$

with

$$\hat{\theta}_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j}, \quad V_{\epsilon j} = w(\epsilon x_j) \prod_{k \neq j} v(\epsilon x_j + x_k) v(\epsilon x_j - x_k). \tag{2.2}$$

One has

$$(e^{-\epsilon \beta \hat{\theta}_j f})(x_1, \dots, x_n) = f(x_1, \dots, x_{j-1}, x_j + i \epsilon \beta \hbar, x_{j+1}, \dots, x_n), \quad \epsilon = \pm 1,$$

so  $\hat{H}_1$  is a second order (analytic) difference operator. It turns out that for particular potentials  $v$ ,  $w$  the above Hamiltonian has  $n$  independent commuting quantum integrals given by higher-order difference operators of the form

$$\hat{H}_l = \sum_{\substack{J \subset \{1, \dots, n\}, |J| \leq l \\ \epsilon_j = \pm 1, j \in J}} U_{J^c, l - |J|} V_{\epsilon J, J^c}^{1/2} e^{-\beta \hat{\theta}_{\epsilon J}} V_{-\epsilon J, J^c}^{1/2}, \quad l = 1, \dots, n, \tag{2.3}$$

with

$$\hat{\theta}_{\epsilon J} = \sum_{j \in J} \epsilon_j \hat{\theta}_j, \tag{2.4}$$

$$\begin{aligned}
 V_{\varepsilon J;K} = & \prod_{j \in J} w(\varepsilon_j x_j) \prod_{\substack{j, j' \in J \\ j < j'}} v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + i\beta\hbar) \\
 & \times \prod_{\substack{j \in J \\ k \in J}} v(\varepsilon_j x_j + x_k) v(\varepsilon_j x_j - x_k), \tag{2.5}
 \end{aligned}$$

$$U_{K,p} = \sum_{\substack{\emptyset \subsetneq I_1 \subsetneq \dots \subsetneq I_q \subset K, 1 \leq q \leq p \\ |I_q| = p, \varepsilon_i = \pm 1, i \in I_q}} (-1)^q V_{\varepsilon I_1;K \setminus I_1} V_{\varepsilon(I_2 \setminus I_1);K \setminus I_2} \cdots V_{\varepsilon(I_q \setminus I_{q-1});K \setminus I_q} \tag{2.6}$$

( $U_{I,0} = 1$ ). For  $l = 1$ , the operator  $\hat{H}_l$  (2.3)-(2.5) reduces to  $\hat{H}_1$  (2.1), (2.2).

The difference operators  $\hat{H}_1, \dots, \hat{H}_n$  do not commute for arbitrary functions  $v$  and  $w$ . In fact, requiring  $[\hat{H}_l, \hat{H}_{l'}] = 0$  for  $l \neq l'$  leads to functional equations for these potentials. Some solutions of the functional equations were obtained indirectly, using previous work related to multi-variable  $q$ -polynomials:<sup>27</sup>

*Hyperbolic potentials (type D)*

$$\begin{aligned}
 v(z) &= \frac{\text{sh } \alpha(\mu + z)}{\text{sh}(\alpha z)}, \\
 w(z) &= \frac{\text{sh } \alpha(\mu_0 + z)}{\text{sh}(\alpha z)} \frac{\text{ch } \alpha(\mu_1 + z)}{\text{ch}(\alpha z)} \frac{\text{sh } \alpha(\mu'_0 + \gamma + z)}{\text{sh } \alpha(\gamma + z)} \frac{\text{ch } \alpha(\mu'_1 + \gamma + z)}{\text{ch } \alpha(\gamma + z)}, \tag{2.7}
 \end{aligned}$$

*Rational potentials (type D)*

$$v(z) = \frac{\mu + z}{z}, \quad w(z) = \left( \frac{\mu_0 + z}{z} \right) \left( \frac{\mu'_0 + \gamma + z}{\gamma + z} \right), \tag{2.8}$$

where we have introduced the (dependent) parameter

$$\gamma = i\beta\hbar/2. \tag{2.9}$$

Very recently we realized that for a potential  $v$  as in (2.7), (2.8), the function  $U_{K,p}$  (2.6) can be rewritten in a much simpler form:

$$\begin{aligned}
 U_{K,p} = & (-1)^p \sum_{\substack{I \subset K, |I| = p \\ \varepsilon_i = \pm 1, i \in I}} \left( \prod_{i \in I} w(\varepsilon_i x_i) \prod_{\substack{i, i' \in I \\ i < i'}} v(\varepsilon_i x_i + \varepsilon_{i'} x_{i'}) v(-\varepsilon_i x_i - \varepsilon_{i'} x_{i'} - 2\gamma) \right. \\
 & \left. \times \prod_{\substack{i \in I \\ k \in K \setminus I}} v(\varepsilon_i x_i + x_k) v(\varepsilon_i x_i - x_k) \right). \tag{2.10}
 \end{aligned}$$

The equivalence of (2.6) and (2.10) hinges on functional equations for the potential  $v$  that are stated and proved in an appendix. In the next subsection, when we compute the limit resulting in our type A difference CM systems with external fields, it will be convenient to use (2.10) rather than (2.6) for  $U_{K,p}$ .

To see that the Hamiltonian  $\hat{H}_1$  (2.1), (2.2), with potentials given by (2.7) or (2.8), constitutes a difference counterpart of a type D Calogero–Moser Hamiltonian one substitutes

$$\mu = i\beta g, \quad \mu_r = i\beta g_r, \quad \mu'_r = i\beta g'_r \tag{2.11}$$

( $r=0,1$ ), and expands in the step size parameter  $\beta$ :

$$\hat{H}_1(\beta) = \hat{H}_{1,0}\beta^2 + o(\beta^2), \tag{2.12}$$

with

$$\hat{H}_{1,0} = \sum_{1 \leq j \leq n} \hat{\theta}_j^2 + \sum_{1 \leq j \neq k \leq n} (\nu(x_j + x_k) + \nu(x_j - x_k)) + \sum_{1 \leq j \leq n} \omega(x_j) + \text{constant}, \tag{2.13}$$

and

$$\nu(z) = \frac{g(g - \hbar)\alpha^2}{\text{sh}^2(\alpha z)}, \quad \omega(z) = \frac{\tilde{g}_0(\tilde{g}_0 - \hbar)\alpha^2}{\text{sh}^2(\alpha z)} - \frac{\tilde{g}_1(\tilde{g}_1 - \hbar)\alpha^2}{\text{ch}^2(\alpha z)} \quad (\text{Hyperbolic case}),$$

$$\nu(z) = \frac{g(g - \hbar)}{z^2}, \quad \omega(z) = \frac{\tilde{g}_0(\tilde{g}_0 - \hbar)}{z^2} \quad (\text{Rational case}),$$

where  $\tilde{g}_0 = g_0 + g'_0$ ,  $\tilde{g}_1 = g_1 + g'_1$ . It follows that the differential operator  $\hat{H}_{1,0}$  (2.13), which amounts to a quantization of the type D Calogero–Moser Hamiltonian  $H_{CM}$  (1.2) with potentials taken from Block 3 (and  $g_2, g_3 = 0$ ) or Block 5 (and  $g_1, g_2, g_3 = 0$ ) of Fig. 1, can be obtained as a limit of  $\hat{H}_1$  by sending the step size to zero:  $\hat{H}_{1,0} = \lim_{\beta \rightarrow 0} \beta^{-2l} \hat{H}_1$ . More generally, one has for the higher-order integrals<sup>18</sup>

$$\hat{H}_l(\beta) = \hat{H}_{l,0}\beta^{2l} + o(\beta^{2l}), \quad \hat{H}_{l,0} = \sum_{J \subset \{1, \dots, n\}} \prod_{j \in J} \hat{\theta}_j^2 + \text{l.o.}, \tag{2.14}$$

where l.o. stands for terms of lower order in the partials  $\hat{\theta}_j$ .

*Remarks:* i. For every difference system in this paper there exists an associated classical  $n$ -particle system (see also Refs. 18 and 19). To pass from the quantum to the classical system one substitutes real variables  $\theta_j$  for the partials  $\hat{\theta}_j$  and sets  $\hbar$  equal zero (so  $\gamma = 0$ , cf. Eq. (2.9)). The commutativity of our difference operators then implies the Poisson commutativity of the corresponding classical quantities.<sup>18</sup>

ii. The rational potentials are a limit of the hyperbolic potentials: by sending  $\alpha$  to zero Eq. (2.7) goes over in Eq. (2.8). A more general external field is obtained if, before scaling  $\alpha$  to zero, the coupling constants  $\mu_1, \mu'_1$  are shifted over a half-period (turning  $\text{ch } \alpha(\mu_1^{(r)} + z)$  into  $\text{sh } \alpha(\mu_1^{(r)} + z)$ ). Then  $\lim_{\alpha \rightarrow 0} \alpha^{-2l} \hat{H}_l$  leads to operators with rational potentials given by

$$\nu(z) = \frac{\mu + z}{z}, \quad w(z) = \frac{(\mu_0 + z)(\mu_1 + z)(\mu'_0 + \gamma + z)(\mu'_1 + \gamma + z)}{z(\gamma + z)}. \tag{2.15}$$

The potentials (2.8) are recovered after multiplication of  $\hat{H}_l$  by  $(\mu_1 \mu'_1)^{-l}$  and sending  $\mu_1, \mu'_1$  to infinity. If one substitutes  $\mu = i\beta g$ ,  $\mu_0^{(r)} = i\beta g_0^{(r)}$  and  $\mu_1^{(r)} = 1/(i\beta g_1^{(r)})$  in the Hamiltonian  $\hat{H}_1$  (2.1), (2.2) with  $\nu, w$  taken from (2.15), then expansion of  $-\beta^2 g_1 g'_1 \hat{H}_1$  in  $\beta$  leads to a Hamiltonian that consists of the rational version of  $\hat{H}_{1,0}$  (2.13) coupled to a harmonic external field:

$$\nu(z) = \frac{g(g - \hbar)}{z^2}, \quad \omega(z) = \frac{\tilde{g}_0(\tilde{g}_0 - \hbar)}{z^2} + \tilde{g}_1^2 z^2 \tag{2.16}$$

( $\tilde{g}_r = g_r + g'_r$ ). Thus, the model with rational potentials as in (2.15) amounts to a difference version of the type D Calogero–Moser system with potentials taken from Block 5 of the diagram in the introduction and  $g_2, g_3 = 0$ .

**B. Type A: The relativistic CM system in an external field**

Let us now discuss how one arrives at the quantum integrals of a relativistic CM system in an external field, by sending the center of mass to infinity in the hyperbolic type D system of the previous subsection. If we substitute

$$\begin{aligned} x_j \rightarrow x_j + R, \quad & \mu_0 \rightarrow \mu_0 - R, \quad \mu_1 \rightarrow \mu_1 + i\pi/(2\alpha) - R, \\ & \mu'_0 \rightarrow \mu'_0 - \gamma + R, \quad \mu'_1 \rightarrow \mu'_1 - i\pi/(2\alpha) - \gamma + R, \end{aligned} \tag{2.17}$$

in the difference operators  $\hat{H}_l$  (2.3) with hyperbolic potentials (2.7) (and  $U_{K,p}$  taken from Eq. (2.10)), then for  $R \rightarrow \infty$  we have (it is assumed throughout that  $\alpha > 0$ ):

$$\begin{aligned} v(\varepsilon_j x_j + \varepsilon_k x_k) &\rightarrow \frac{\text{sh } \alpha(\mu + \varepsilon(x_j - x_k))}{\text{sh } \alpha\varepsilon(x_j - x_k)}, \quad \text{if } \varepsilon_j = -\varepsilon_k = \varepsilon, \\ &\rightarrow (\lambda_v)^\varepsilon, \quad \text{if } \varepsilon_j = \varepsilon_k = \varepsilon, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} w(\varepsilon_j x_j) &\rightarrow \lambda_w (1 - e^{-2\alpha(\mu_0 + x_j)})(1 - e^{-2\alpha(\mu_1 + x_j)}), \quad \text{if } \varepsilon_j = +1, \\ &\rightarrow \lambda_w^{-1} (1 - e^{2\alpha(\mu'_0 - x_j)})(1 - e^{2\alpha(\mu'_1 - x_j)}), \quad \text{if } \varepsilon_j = -1, \end{aligned} \tag{2.19}$$

where

$$\lambda_v = e^{\alpha\mu}, \quad \lambda_w = e^{\alpha(\mu_0 + \mu'_0 + \mu_1 + \mu'_1 - 2\gamma)}.$$

It thus follows that our commuting difference operators turn into operators of the form

$$\hat{H}_l = \sum_{\substack{|J_+| + |J_-| \leq l \\ J_+ \cap J_- = \emptyset}} U_{J_+^c \cap J_-^c, l - |J_+| - |J_-|} V_{J_+, J_-, J_+^c \cap J_-^c}^{1/2} e^{-\beta(\hat{\theta}_{J_+} - \hat{\theta}_{J_-})} V_{J_-, J_+, J_+^c \cap J_-^c}^{1/2}, \tag{2.20}$$

$l = 1, \dots, n$  (and  $J_+, J_- \subset \{1, \dots, n\}$ ), where

$$\begin{aligned} V_{J_+, J_-; K} &= \prod_{j \in J_+} w_+(x_j) \prod_{j \in J_-} w_-(-x_j) \prod_{\substack{j \in J_+ \\ j' \in J_-}} v(x_j - x_{j'}) v(x_j - x_{j'} + 2\gamma) \\ &\times \prod_{\substack{j \in J_+ \\ k \in K}} v(x_j - x_k) \prod_{\substack{j \in J_- \\ k \in K}} v(x_k - x_j), \end{aligned} \tag{2.21}$$



$$\begin{aligned}
 U_{K,p} = & (-1)^p \sum_{\substack{I_+, I_- \subset K, I_+ \cap I_- = \emptyset \\ |I_+| + |I_-| = p}} \left( \lambda_{|K|, |I_+|, |I_-|} \prod_{i \in I_+} w_+(x_i) \prod_{i \in I_-} w_-(-x_i) \right. \\
 & \times \prod_{\substack{i \in I_+ \\ i' \in I_-}} v(x_i - x_{i'}) v(x_{i'} - x_i - 2\gamma) \prod_{k \in K \setminus (I_+ \cup I_-)} v(x_i - x_k) \prod_{k \in K \setminus (I_+ \cup I_-)} v(x_k - x_i) \Big),
 \end{aligned} \tag{2.22}$$

with

*Hyperbolic potentials (type A)*

$$v(z) = \text{sh } \alpha(\mu + z) / \text{sh } (\alpha z), \tag{2.23}$$

$$w_+(z) = (1 - e^{-2\alpha(\mu_0 + z)})(1 - e^{-2\alpha(\mu_1 + z)}), \quad w_-(z) = (1 - e^{2\alpha(\mu'_0 + z)})(1 - e^{2\alpha(\mu'_1 + z)}),$$

and

$$\lambda_{|K|, |I_+|, |I_-|} = \lambda_w^{|I_+| - |I_-|} \lambda_v^{(|K| - |I_+| - |I_-|)(|I_+| - |I_-|)}.$$

If we divide  $\hat{H}_l$  by  $(2\alpha)^{2l}$  and send  $\alpha$  to zero, then we obtain operators of the same form as in Eqs. (2.20)-(2.22), but now with

*Rational potentials (type A)*

$$v(z) = (\mu + z)/z, \quad w_+(z) = (\mu_0 + z)(\mu_1 + z), \quad w_-(z) = (\mu'_0 + z)(\mu'_1 + z), \tag{2.24}$$

and  $\lambda_v, \lambda_w, \lambda_{|K|, |I_+|, |I_-|} = 1$ .

Both in the hyperbolic and the rational case the operator  $\hat{H}_l$  (2.20)-(2.22) reduces for  $l = 1$  to

$$\begin{aligned}
 \hat{H}_1 = & \sum_{1 \leq j \leq n} \left( w_+^{1/2}(x_j) \prod_{k \neq j} v^{1/2}(x_j - x_k) \exp(-\beta \hat{\theta}_j) \prod_{k \neq j} v^{1/2}(x_k - x_j) w_-^{1/2}(-x_j) \right. \\
 & \left. + w_-^{1/2}(-x_j) \prod_{k \neq j} v^{1/2}(x_k - x_j) \exp(\beta \hat{\theta}_j) \prod_{k \neq j} v^{1/2}(x_j - x_k) w_+^{1/2}(x_j) \right) \\
 & + U(x_1, \dots, x_n),
 \end{aligned} \tag{2.25}$$

with

$$U = - \sum_{1 \leq j \leq n} \left( \lambda_w \lambda_v^{n-1} w_+(x_j) \prod_{k \neq j} v(x_j - x_k) + \lambda_w^{-1} \lambda_v^{1-n} w_-(-x_j) \prod_{k \neq j} v(x_k - x_j) \right). \tag{2.26}$$

This is the Hamiltonian of a type A difference Calogero–Moser system consisting of the relativistic CM system coupled to an external field. To switch off the external field we set  $\mu_0, \mu_1 = R$ ,  $\mu'_0, \mu'_1 = -R$ , and send  $R$  to infinity (in the rational case we first renormalize  $\hat{H}_l$  by multiplying it by  $R^{-2l}$ ). For the operator  $\hat{H}_l$  this limit amounts to setting  $w_+, w_- = 1$ . Specifically, the Hamiltonian  $\hat{H}_1$  (2.25), (2.26) then reduces, apart from an additive constant, to the Hamiltonian of the relativistic CM system introduced by Ruijsenaars.<sup>21</sup> To compare our expression for the Hamiltonian with that of Ref. 21, one should notice that for  $w_+, w_- = 1$  the function  $U$  (2.26) (with  $v$

given by (2.23) or (2.24)) is constant in  $x_j, j=1, \dots, n$ . The proof hinges on Liouville’s theorem:  $U$  is both entire in  $x_j$  (generically simple poles congruent to  $x_j=x_k, k \neq j$ , caused by the zeros in the denominators of  $v$  cancel each other as a consequence of the permutation symmetry) and bounded in  $x_j$  ( $U$  has bounded asymptotics for  $x_j \rightarrow \infty$ ). A more detailed version of this type of reasoning can be found in the appendix below, where it was used to demonstrate the equality of (2.6) and (2.10).

*Remarks:* i. In general the difference operators  $\hat{H}_l$  are *not* formally self-adjoint (and the corresponding classical integrals are not real-valued). If we assume  $\alpha$  and  $\beta$  to be real, then the type D difference operators (2.3)–(2.8) become formally self-adjoint by picking  $\mu, \mu_0^{(')}, \mu_1^{(')} \in i\mathbb{R}$ , whereas for the type A difference operators (2.20)–(2.24) one is led to the constraint  $\mu \in i\mathbb{R}$  and  $\mu_0' = -\bar{\mu}_0, \mu_1' = -\bar{\mu}_1$ .

ii. Sending the step size parameter  $\beta$  to zero in our type A difference model corresponds to the nonrelativistic limit: the relativistic Ruijsenaars system with external field goes over in the nonrelativistic Calogero–Moser system with external field. Specifically, if we substitute  $\mu = i\beta g, \mu_0 = (i\beta g_0 - \log(\beta k_0) + i\pi/2)/2\alpha, \mu_0' = (i\beta g_0 + \log(\beta k_0) + i\pi/2)/2\alpha, \mu_1 = (i\beta g_1 - \log(-\beta k_1) - i\pi/2)/2\alpha,$  and  $\mu_1' = (i\beta g_1 + \log(-\beta k_1) - i\pi/2)/2\alpha$  in the hyperbolic version of the Hamiltonian  $\hat{H}_1$  (2.25), (2.26), then the expansion in  $\beta$  is of the form (2.12) with

$$\hat{H}_{1,0} = \sum_{1 \leq j \leq n} \hat{\theta}_j^2 + \sum_{1 \leq j \neq k \leq n} v(x_j - x_k) + \sum_{1 \leq j \leq n} \omega(x_j) + \text{constant} \tag{2.27}$$

and

$$v(z) = \frac{g(g - \hbar)\alpha^2}{\text{sh}^2(\alpha z)}, \quad \omega(z) = \tilde{g}_0 \exp(-2\alpha z) + \tilde{g}_1 \exp(-4\alpha z), \tag{2.28}$$

where  $\tilde{g}_0 = 2(k_0 + k_1)(g_0 + g_1 - 2\alpha\hbar)$  and  $\tilde{g}_1 = (k_0 + k_1)^2$ . In the rational case we substitute  $\mu = i\beta g$  and  $\mu_r' = \mu_r = 1/(i\beta g_r), r=0,1$ . After multiplying the Hamiltonian by  $-\beta^2 g_0 g_1$  (cf. Remark ii of Sec. II A), expansion in  $\beta$  leads to  $\hat{H}_{1,0}$  (2.27) with potentials given by

$$v(z) = \frac{g(g - \hbar)}{z^2}, \quad \omega(z) = (g_0 + g_1)^2 z^2. \tag{2.29}$$

Thus, we recover the quantum versions of the nonrelativistic Calogero–Moser Hamiltonians of Block 2 and 4 in the introduction with  $g_2, g_3 = 0$ . (The linear term in the rational external potential of Block 4 may be obtained from the harmonic part by a translation of the center of mass.)

### III. MORE GENERAL DIFFERENCE CALOGERO-MOSER SYSTEMS

In this section generalizations of the systems in Section II are studied. We start with an elliptic generalization of the type D operators  $\hat{H}_1, \hat{H}_n$  from Section II A. This elliptic system was introduced in Ref. 18. (The Hamiltonian amounts to a difference version of  $H$  (1.1).) It will be explained how limit transitions similar to those considered above lead to hyperbolic and rational difference CM systems of type A and D with a more general external field potential than the systems in the previous section. All systems below are given by a Hamiltonian  $\hat{H}_1$  (a difference operator of order 2) and an additional independent quantum integral  $\hat{H}_n$  (a difference operator of order  $2n$ ). So, the quantum integrability of these generalized external field models follows for  $n=2$ . To keep our treatment self-contained, some preliminaries regarding sigma functions are recalled in Sec. III A.

### A. Sigma functions

This subsection summarizes some useful properties of the Weierstrass  $\sigma$ -function. For a more detailed treatment the reader is referred to, e.g., Whittaker and Watson.<sup>6</sup>

For our purposes it is convenient to introduce the  $\sigma$ -function as a function of the form

$$\begin{aligned}\sigma(z) &= e^{\kappa z^2} \alpha^{-1} \operatorname{sh}(\alpha z) \prod_{m=1}^{\infty} \left( 1 - \frac{\operatorname{sh}^2(\alpha z)}{\operatorname{sh}^2(2m\alpha\omega_1)} \right), \\ &= e^{\kappa z^2} \alpha^{-1} \operatorname{sh}(\alpha z) \prod_{m=1}^{\infty} \frac{(1 - e^{2\alpha z} e^{-4m\alpha\omega_1})(1 - e^{-2\alpha z} e^{-4m\alpha\omega_1})}{(1 - e^{-4m\alpha\omega_1})^2},\end{aligned}\quad (3.1)$$

with  $\alpha = i\pi/(2\omega_2)$  and  $\operatorname{Im}(\omega_2/\omega_1) > 0$  (so  $\operatorname{Re}(\alpha\omega_1) > 0$ ). In this paper the actual value of constant  $\kappa$  is not very important; however, to keep agreement with the literature one should take  $\kappa = \zeta(\omega_2)/(2\omega_2)$ , where  $\zeta(z)$  denotes the Weierstrass  $\zeta$ -function.<sup>6</sup>

The  $\sigma$ -function is entire and odd in  $z$ , and it has (simple) zeros in the points of the lattice

$$\Gamma = 2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}. \quad (3.2)$$

It is not difficult to verify from Eq. (3.1) that  $\sigma(z)$  is quasi-periodic in  $z$  with primitive quasi-periods  $2\omega_1$  and  $2\omega_2$ :

$$\sigma(z + 2\omega_r) = -e^{2\eta_r(\omega_r + z)} \sigma(z), \quad (3.3)$$

where  $\eta_1 = 2\kappa\omega_1 + i\pi/(2\omega_2)$  and  $\eta_2 = 2\kappa\omega_2$ . The constants  $\eta_1, \eta_2$  satisfy Legendre's relation

$$\eta_1\omega_2 - \eta_2\omega_1 = i\pi/2. \quad (3.4)$$

It is convenient to distinguish a third (dependent) half-period  $\omega_3 = -\omega_1 - \omega_2$ . Equation (3.3) then holds for  $r = 1, 2, 3$ , with  $\eta_3 = -\eta_1 - \eta_2$ . By shifting the argument over the half-periods  $\omega_r$  three associated sigma functions are introduced:

$$\sigma_r(z) = e^{-\eta_r z} \sigma(\omega_r + z) / \sigma(\omega_r), \quad r = 1, 2, 3. \quad (3.5)$$

The sigma functions are related to the Weierstrass  $\wp$ -function via

$$\frac{\sigma_r(\mu + z)\sigma_r(\mu - z)}{\sigma_r(z)\sigma_r(-z)} = \sigma^2(\mu)(\wp(\mu) - \wp(\omega_r + z)), \quad -\frac{d^2}{dz^2} \ln \sigma_r(z) = \wp(z + \omega_r), \quad r = 0, 1, 2, 3 \quad (3.6)$$

(with the convention  $\sigma_0(z) = \sigma(z)$  and  $\omega_0 = 0$ ). Another useful relation is the duplication formula

$$\sigma(2z) = 2\sigma(z)\sigma_1(z)\sigma_2(z)\sigma_3(z). \quad (3.7)$$

In the second identity of Formula (3.6) the above choice of  $\kappa$  is important: for other values than  $\kappa = \zeta(\omega_2)/(2\omega_2)$  the two sides of the equation differ by a constant.

It is immediate from Eqs. (3.1), (3.5) that the sigma functions degenerate into hyperbolic functions when  $\omega_1$  becomes infinite. One has the following asymptotics for  $\omega_1 \rightarrow \infty$ :

$$\sigma(z) \sim e^{\kappa z^2} \alpha^{-1} \operatorname{sh}(\alpha z) (1 - 4 \operatorname{sh}^2(\alpha z) e^{-4\alpha\omega_1} + O(e^{-8\alpha\omega_1})), \quad (3.8)$$

$$\sigma_1(z) \sim e^{\kappa z^2} (1 - 4 \operatorname{sh}^2(\alpha z) e^{-2\alpha\omega_1} - 8 \operatorname{sh}^2(\alpha z) e^{-4\alpha\omega_1} + O(e^{-6\alpha\omega_1})), \quad (3.9)$$

$$\sigma_2(z) \sim e^{\kappa z^2} \operatorname{ch}(\alpha z) (1 + 4 \operatorname{sh}^2(\alpha z) e^{-4\alpha\omega_1} + O(e^{-8\alpha\omega_1})), \tag{3.10}$$

$$\sigma_3(z) \sim e^{\kappa z^2} (1 + 4 \operatorname{sh}^2(\alpha z) e^{-2\alpha\omega_1} - 8 \operatorname{sh}^2(\alpha z) e^{-4\alpha\omega_1} + O(e^{-6\alpha\omega_1})). \tag{3.11}$$

**B. Elliptic potentials (type D)**

The elliptic generalization of the type D difference CM Hamiltonian from Sec. II A is given by<sup>18</sup>

$$\hat{H}_1 = \sum_{\substack{1 \leq j < n \\ \epsilon = \pm 1}} V_{\epsilon j}^{1/2} e^{-\epsilon \beta \hat{\theta}_j} V_{-\epsilon j}^{1/2} + U(x_1, \dots, x_n), \tag{3.12}$$

$$V_{\epsilon j} = w(\epsilon x_j) \prod_{k \neq j} v(\epsilon x_j + x_k) v(\epsilon x_j - x_k),$$

with

*Elliptic potentials I*

$$v(z) = \frac{\sigma(\mu + z)}{\sigma(z)}, \quad w(z) = \prod_{0 \leq r \leq 3} \frac{\sigma_r(\mu_r + z) \sigma_r(\mu'_r + \gamma + z)}{\sigma_r(z) \sigma_r(\gamma + z)}, \tag{3.13}$$

and

$$U = \sum_{0 \leq r \leq 3} c_r \prod_{1 \leq j \leq n} \frac{\sigma_r(\mu - \gamma + x_j) \sigma_r(\mu - \gamma - x_j)}{\sigma_r(-\gamma + x_j) \sigma_r(-\gamma - x_j)}, \tag{3.14}$$

$$c_r = 2(\sigma(\mu) \sigma(\mu - 2\gamma))^{-1} \prod_{0 \leq s \leq 3} \sigma_s(\mu_{\pi_r(s)} - \gamma) \sigma_s(\mu'_{\pi_r(s)}), \quad r = 0, \dots, 3,$$

where we have introduced permutations  $\pi_0 = id$ ,  $\pi_1 = (01)(23)$ ,  $\pi_2 = (02)(13)$ , and  $\pi_3 = (03)(12)$ . Comparison of Eq. (3.12) with (2.1) reveals that, in passing from hyperbolic/rational potentials to elliptic potentials, the structure of the terms in the Hamiltonian of degree  $\pm 1$  in the differences  $\exp(\beta \hat{\theta}_j)$  has remained the same, whereas the structure of the part of degree zero (i.e., the function  $U$ ) has changed considerably.

In Ref. 18 we found an independent difference operator of order  $2n$  that commutes with our elliptic difference CM Hamiltonian if

$$\sum_{0 \leq r \leq 3} (\mu_r + \mu'_r) = 0. \tag{3.15}$$

This quantum integral is given by an elliptic generalization of the operator  $\hat{H}_n$  that has the same form as in Sec. II A (but now with  $v$  and  $w$  taken from (3.13)):

$$\hat{H}_n = \sum_{\substack{J \subset \{1, \dots, n\} \\ \epsilon_j = \pm 1, j \in J}} U_{J^c} V_{\epsilon J, J^c}^{1/2} e^{-\beta \hat{\theta}_{\epsilon J}} V_{-\epsilon J, J^c}^{1/2}, \tag{3.16}$$

with

$$\begin{aligned}
 V_{\varepsilon J;K} = & \prod_{j \in J} w(\varepsilon_j x_j) \prod_{\substack{j, j' \in J \\ j < j'}} v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + 2\gamma) \\
 & \times \prod_{\substack{j \in J \\ k \in K}} v(\varepsilon_j x_j + x_k) v(\varepsilon_j x_j - x_k), \tag{3.17}
 \end{aligned}$$

$$U_K = \sum_{\varepsilon_k = \pm 1, k \in K} (-1)^{|K|} \prod_{k \in K} w(\varepsilon_k x_k) \prod_{\substack{k, k' \in K \\ k < k'}} v(\varepsilon_k x_k + \varepsilon_{k'} x_{k'}) v(-\varepsilon_k x_k - \varepsilon_{k'} x_{k'} - 2\gamma). \tag{3.18}$$

To see that  $\hat{H}_1$  (3.12)-(3.14) is indeed a difference version of the Inozemtsev Hamiltonian  $H$  (1.1), one reparametrizes the coupling constants as in Eq. (2.11) and expands in the step size parameter  $\beta$  (using the second identity in Eq. (3.6)):

$$\hat{H}_1 = \text{constant} + \hat{H}_{1,0} \beta^2 + o(\beta^2), \tag{3.19}$$

with

$$\begin{aligned}
 \hat{H}_{1,0} = & \sum_{1 \leq j \leq n} \hat{\theta}_j^2 + g(g - \hbar) \sum_{1 \leq j \neq k \leq n} (\wp(x_j + x_k) + \wp(x_j - x_k)) \\
 & + \sum_{\substack{0 \leq r \leq 3 \\ 1 \leq j \leq n}} \tilde{g}_r (\tilde{g}_r - \hbar) \wp(\omega_r + x_j), \tag{3.20}
 \end{aligned}$$

and  $\tilde{g}_r = g_r + g'_r$ . Thus, for  $\beta \rightarrow 0$  a quantization of the Hamiltonian  $H$  (1.1) arises.

We will see in the next subsection that for  $\omega_1 \rightarrow \infty$  the operators  $\hat{H}_1$  and  $\hat{H}_n$  reduce to operators with hyperbolic potentials. These are essentially the same as the corresponding operators in Sec. II A. Hyperbolic (and rational) difference CM systems with more general external field potentials are obtained when, before sending periods to infinity, the coupling constants  $\mu_r$  and  $\mu'_r$  are shifted over the half-periods  $\omega_r$  (cf. Remark ii. in Sec. II A for a similar state of affairs in the transition from hyperbolic to rational potentials). The relevant parameter shift

$$\mu_r \rightarrow \mu_r - \omega_r, \quad \mu'_r \rightarrow \mu'_r - \omega_r, \tag{3.21}$$

leads, after rewriting and apart from multiplicative constants, to operators  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)-(3.18) with

*Elliptic potentials II*

$$v(z) = \sigma(\mu + z) / \sigma(z), \quad w(z) = (\sigma(2z) \sigma(2\gamma + 2z))^{-1} \prod_{0 \leq r \leq 3} \sigma(\mu_r + z) \sigma(\mu'_r + \gamma + z), \tag{3.22}$$

and

$$U = \sum_{0 \leq r \leq 3} c_r \prod_{1 \leq j \leq n} \frac{\sigma_r(\mu - \gamma + x_j)}{\sigma_r(-\gamma + x_j)} \frac{\sigma_r(\mu - \gamma - x_j)}{\sigma_r(-\gamma - x_j)}, \tag{3.23}$$

$$c_r = (2\sigma(\mu)\sigma(\mu - 2\gamma))^{-1} \prod_{0 \leq s \leq 3} \sigma(\mu_s - \gamma)\sigma(\mu'_s), \quad r=0,$$

$$= \frac{\exp(-4\eta_r\omega_r)\sigma^8(\omega_r)}{2\sigma(\mu)\sigma(\mu - 2\gamma)} \prod_{0 \leq s \leq 3} \sigma_r(\mu_s - \gamma)\sigma_r(\mu'_s), \quad r=1,2,3.$$

To arrive at the potentials (3.22), (3.23) we used the duplication-formula (3.7) to rewrite the denominator of  $w$ , and we used Eqs. (3.3), (3.5) (taking into account also Legendre’s relation (3.4)) to rewrite expressions of the form  $\sigma_s(z - \omega_r)$  in terms of  $\sigma_{\pi_r(s)}(z)$ .

**C. Generalized external fields for the hyperbolic and rational systems of type D**

The sigma functions degenerate into elementary functions when periods are sent to infinity. For  $\omega_1 = \infty$  one has (cf. Expansions (3.8)-(3.11)):

$$\sigma(z) = e^{\kappa z^2} \text{sh}(\alpha z) / \alpha, \quad \sigma_1(z) = e^{\kappa z^2}, \quad \sigma_2(z) = e^{\kappa z^2} \text{ch}(\alpha z), \quad \sigma_3(z) = e^{\kappa z^2}, \quad (3.24)$$

with  $\alpha = i\pi / (2\omega_2)$  and  $\kappa = \zeta(\omega_2) / (2\omega_2) (= -\alpha^2/6)$ . If one substitutes (3.24) for the sigma functions, then in  $U$  (3.14) the exponentials  $\exp(\kappa z^2)$  give rise to an overall multiplicative constant, whereas in  $v, w$  (3.13) factors of the form  $\exp(az + b)$  emerge. After commuting these factors to one side of the differences  $\exp(\varepsilon\beta\theta_j)$ , the same multiplicative constant appearing in  $U$  also arises in front of the terms  $V_{\varepsilon j}^{1/2} \exp(-\varepsilon\beta\theta_j) V_{-\varepsilon j}^{1/2}$ . It is not difficult to verify that in  $\hat{H}_n$  (3.16)-(3.18) the factors  $\exp(\kappa z^2)$  also give rise to an overall multiplicative constant. In the latter case, however, one needs to invoke Condition (3.15) to get rid of the  $x_j$ -dependence of factors induced by  $w$ .

It thus follows that for  $\omega_1 = \infty$  the operators  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)-(3.18) with potentials (3.13), (3.14) reduce, apart from multiplicative (in  $\hat{H}_1$  also additive) constants, to operators with potentials given by

$$v(z) = \frac{\text{sh}\alpha(\mu + z)}{\text{sh}(\alpha z)},$$

$$w(z) = \frac{\text{sh}\alpha(\mu_0 + z)}{\text{sh}(\alpha z)} \frac{\text{ch}\alpha(\mu_2 + z)}{\text{ch}(\alpha z)} \frac{\text{sh}\alpha(\mu'_0 + \gamma + z)}{\text{sh}\alpha(\gamma + z)} \frac{\text{ch}\alpha(\mu'_2 + \gamma + z)}{\text{ch}\alpha(\gamma + z)}, \quad (3.25)$$

and

$$U = c_0 \prod_{1 \leq j \leq n} \frac{\text{sh}\alpha(\mu - \gamma + x_j)}{\text{sh}\alpha(-\gamma + x_j)} \frac{\text{sh}\alpha(\mu - \gamma - x_j)}{\text{sh}\alpha(-\gamma - x_j)}$$

$$+ c_2 \prod_{1 \leq j \leq n} \frac{\text{ch}\alpha(\mu - \gamma + x_j)}{\text{ch}\alpha(-\gamma + x_j)} \frac{\text{ch}\alpha(\mu - \gamma - x_j)}{\text{ch}\alpha(-\gamma - x_j)}, \quad (3.26)$$

$$c_0 = 2\text{sh}\alpha(\mu_0 - \gamma)\text{ch}\alpha(\mu_2 - \gamma)\text{sh}(\alpha\mu'_0)\text{ch}(\alpha\mu'_2) / (\text{sh}(\alpha\mu)\text{sh}\alpha(\mu - 2\gamma)),$$

$$c_2 = 2\text{ch}\alpha(\mu_0 - \gamma)\text{sh}\alpha(\mu_2 - \gamma)\text{ch}(\alpha\mu'_0)\text{sh}(\alpha\mu'_2) / (\text{sh}(\alpha\mu)\text{sh}\alpha(\mu - 2\gamma)).$$

For these potentials the operators  $\hat{H}_1$  and  $\hat{H}_n$  coincide, apart from a constant term in  $\hat{H}_1$  and an interchange of parameters  $\mu_1^{(j)} \leftrightarrow \mu_2^{(j)}$ , with the corresponding operators from Sec. II A. For  $\hat{H}_n$  this is immediate, whereas for  $\hat{H}_1$  this follows because  $U$  (3.26) differs from the function

−∑<sub>j,ε</sub> V<sub>εj</sub> (with V<sub>εj</sub> as in (3.12)) only by a constant. (This is again seen by invoking Liouville’s theorem after having verified that the difference of the two functions is free of poles and bounded in x<sub>j</sub>, j=1, …, n).

Notice that in the hyperbolic limit Condition (3.15) is no longer needed to ensure the commutativity of  $\hat{H}_1$  and  $\hat{H}_n$  because the dependence on  $\mu_r, \mu'_r$ , r=1,3 has dropped out.

Next, we turn to the more general difference CM system that one obtains as hyperbolic limit of the system with elliptic potentials (3.22), (3.23) (which arose from the parameter shift  $\mu_r^{(r)} \rightarrow \mu_r^{(r)} - \omega_r$ ). The calculation is again based on the sigma function asymptotics (3.8)-(3.11). In the case of v and w we just substitute (as before) the leading part of the asymptotics:  $\sigma(z) = \exp(\kappa z^2) \text{sh}(\alpha z) / \alpha$ . For the function U, however, the computation of the hyperbolic limit is now more cumbersome because higher order terms in the asymptotics contribute (cf. Remark i. below for further details). After subtracting (divergent) additive constants emerging in U, and dividing by overall multiplicative constants caused by the factors  $\exp(\kappa z^2)$ , the limit  $\omega_1 \rightarrow \infty$  leads to operators of the form  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)-(3.18) with

*Hyperbolic potentials with generalized external field (type D)*

$$v(z) = \text{sh}\alpha(\mu + z) / \text{sh}(\alpha z), \tag{3.27}$$

$$w(z) = (\alpha^6 \text{sh}(2\alpha z) \text{sh}2\alpha(\gamma + z))^{-1} \prod_{0 \leq r \leq 3} \text{sh}\alpha(\mu_r + z) \text{sh}\alpha(\mu'_r + \gamma + z),$$

and

$$\begin{aligned} U = & c_0 \prod_{1 \leq j \leq n} \frac{\text{sh}\alpha(\mu - \gamma + x_j)}{\text{sh}\alpha(-\gamma + x_j)} \frac{\text{sh}\alpha(\mu - \gamma - x_j)}{\text{sh}\alpha(-\gamma - x_j)} \\ & + c_1 \prod_{1 \leq j \leq n} \frac{\text{ch}\alpha(\mu - \gamma + x_j)}{\text{ch}\alpha(-\gamma + x_j)} \frac{\text{ch}\alpha(\mu - \gamma - x_j)}{\text{ch}\alpha(-\gamma - x_j)} \\ & + \sum_{1 \leq j \leq n} (c_2 \text{ch}(2\alpha x_j) + c_3 \text{ch}(4\alpha x_j)) \\ & + c_4 \sum_{1 \leq j < k \leq n} \text{ch}(2\alpha x_j) \text{ch}(2\alpha x_k), \end{aligned} \tag{3.28}$$

where

$$c_0 = (2\alpha^6 \text{sh}(\alpha\mu) \text{sh}\alpha(\mu - 2\gamma))^{-1} \prod_{0 \leq s \leq 3} \text{sh}\alpha(\mu_s - \gamma) \text{sh}(\alpha\mu'_s),$$

$$c_1 = (2\alpha^6 \text{sh}(\alpha\mu) \text{sh}\alpha(\mu - 2\gamma))^{-1} \prod_{0 \leq s \leq 3} \text{ch}\alpha(\mu_s - \gamma) \text{ch}(\alpha\mu'_s),$$

$$c_2 = 2^{-4} \alpha^{-6} \sum_{0 \leq s \leq 3} (\text{ch}2\alpha(\mu_s - \gamma) + \text{ch}(2\alpha\mu'_s)),$$

$$c_3 = -2^{-4} \alpha^{-6} \text{ch}(2\alpha\gamma), \quad c_4 = 2^{-2} \alpha^{-6} \text{sh}(\alpha\mu) \text{sh}\alpha(\mu - 2\gamma).$$

In the present case, sending the period  $\omega_1$  to infinity has not diminished the number of coupling constants parametrizing the external field. Thus, we need Condition (3.15) to ensure the commutativity of  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)–(3.18).

In order to demonstrate that the potentials (3.27), (3.28) generalize those in Eqs. (3.25), (3.26), we substitute

$$\mu_1, \mu'_1 = R, \quad \mu_2 \rightarrow \mu_2 + i\pi/(2\alpha), \quad \mu'_2 \rightarrow \mu'_2 - i\pi/(2\alpha), \quad \mu_3, \mu'_3 = -R.$$

The limits

$$\lim_{R \rightarrow \infty} (2\alpha)^6 e^{-4\alpha R} \hat{H}_1, \quad \lim_{R \rightarrow \infty} (2\alpha)^{6n} e^{-4n\alpha R} \hat{H}_n,$$

recover the operators with potentials (3.25), (3.26).

Sending  $\alpha$  to zero leads to a rational degeneration of the system with potentials (3.27), (3.28). For  $v, w$  we obtain (after division of  $w$  by a factor 4)

*Rational potentials with generalized external field (type D)*

$$v(z) = (\mu + z)/z, \quad w(z) = z^{-1}(\gamma + z)^{-1} \prod_{0 \leq r \leq 3} (\mu_r + z) (\mu'_r + \gamma + z). \quad (3.29)$$

It is clear that the resulting operator  $\hat{H}_n$  (3.16)–(3.18) generalizes the one in Remark ii. of Sec. II A. To determine the corresponding Hamiltonian  $\hat{H}_1$  we must also compute the rational version of  $U$  (3.28). The calculation, which consists of expanding (3.28) in  $\alpha$  and subtracting all divergent (constant) terms, is rather cumbersome and has only been done for  $n=2, \dots, 5$  (by computer). As a result we obtained a function of the form

$$U = c_0 \prod_{1 \leq j \leq n} \left( \frac{\mu - \gamma + x_j}{-\gamma + x_j} \right) \left( \frac{\mu - \gamma - x_j}{-\gamma - x_j} \right) + \sum_{1 \leq j \leq n} (c_1 x_j^2 + c_2 x_j^4 + c_3 x_j^6) + c_4 \sum_{1 \leq j < k \leq n} x_j^2 x_k^2, \quad (3.30)$$

with constants  $c_0, \dots, c_4$  that depend in a rather complicated way (especially  $c_1$ ) on the coupling constants  $\mu, \mu_r, \mu'_r$ :

$$c_0 = (2\mu)^{-1} (\mu - 2\gamma)^{-1} \prod_{0 \leq s \leq 3} (\mu_s - \gamma) \mu'_s,$$

$$c_1 = a_0 + a_1 \gamma + a_2 \gamma^2 + a_3 \gamma^3,$$

$$c_2 = 1/4 \sum_{0 \leq r \leq 3} ((\mu_r - \gamma)^2 + \mu_r'^2) + (n-1)\mu(\mu/2 - \gamma) - 5\gamma^2/2,$$

$$c_3 = -1/2, \quad c_4 = 3\mu(\mu/2 - \gamma). \quad (3.31)$$

with



$$\begin{aligned}
a_0 &= 2^{-4} \sum_{0 \leq r \leq 3} (\mu_r^2 - \mu_r'^2)^2 - 2^{-3} \sum_{0 \leq r < s \leq 3} (\mu_r \mu_s + \mu_r^2 \mu_s'^2 + \mu_r'^2 \mu_s'^2) \\
&\quad - 2^{-2}(n-1)\mu^2 \sum_{0 \leq r \leq 3} (\mu_r^2 + \mu_r'^2) - 2^{-2}(n-1)(n-2)\mu^4, \\
a_1 &= -2^{-1} \sum_{0 \leq r \leq 3} (\mu_r^3 + \mu_r'^3) + 2^{-2} \sum_{0 \leq r, s \leq 3} \mu_r (\mu_r^2 + \mu_r'^2) + 2^{-1}(n-1)\mu \sum_{0 \leq r \leq 3} (\mu_r^2 + \mu_r'^2) \\
&\quad + 2^{-1}(n-1)\mu^2 \sum_{0 \leq r \leq 3} (\mu_r + \mu_r'), \\
a_2 &= 2^{-1} \sum_{0 \leq r \leq 3} \mu_r^2 - 2^{-1} \sum_{0 \leq r < s \leq 3} \mu_r \mu_s - (n-1)\mu \sum_{0 \leq r \leq 3} \mu_r - 2^{-1}(n-1)(2n-5)\mu^2, \\
a_3 &= -2^{-1}(n-1)\mu - \sum_{0 \leq r \leq 3} \mu_r.
\end{aligned}$$

Undoubtedly this formula for  $U$  also holds for  $n > 5$ .

*Remarks:* i. To determine the hyperbolic degeneration of  $U$  (3.23), we used Eqs. (3.8)–(3.11) to derive the following asymptotics for  $\omega_1 \rightarrow \infty$ :

$$\begin{aligned}
c_r &\sim \kappa_0 \alpha^{-6} \prod_{0 \leq s \leq 3} \operatorname{sh} \alpha(\mu_s - \gamma) \operatorname{sh}(\alpha \mu_s'), \quad r=0, \\
&\sim \kappa_0 \alpha^{-6} 2^{-8} \exp(4\alpha\omega_1) (1 - 2\kappa_1 e^{-2\alpha\omega_1} + O(e^{-4\alpha\omega_1})), \quad r=1, \\
&\sim \kappa_0 \alpha^{-6} \prod_{0 \leq s \leq 3} \operatorname{ch} \alpha(\mu_s - \gamma) \operatorname{ch}(\alpha \mu_s'), \quad r=2, \\
&\sim \kappa_0 \alpha^{-6} 2^{-8} \exp(4\alpha\omega_1) (1 + 2\kappa_1 e^{-2\alpha\omega_1} + O(e^{-4\alpha\omega_1})), \quad r=3,
\end{aligned} \tag{3.32}$$

with  $\kappa_0 = (2 \operatorname{sh}(\alpha\mu) \operatorname{sh} \alpha(\mu - 2\gamma))^{-1} \exp(\kappa \sum_{0 \leq s \leq 3} (\mu_s^2 + \mu_s'^2 - 2\gamma\mu_s) - 2\kappa\mu(\mu - 2\gamma))$ ,  $\kappa_1 = \sum_{0 \leq s \leq 3} (\operatorname{ch} 2\alpha(\mu_s - \gamma) + \operatorname{ch}(2\alpha\mu_s'))$ ; and

$$\frac{\sigma_r(\mu - \gamma + x_j) \sigma_r(\mu - \gamma - x_j)}{\sigma_r(-\gamma + x_j) \sigma_r(-\gamma - x_j)} \tag{3.33}$$

$$\begin{aligned}
&\sim \lambda_0 \frac{\operatorname{sh} \alpha(\mu - \gamma + x_j) \operatorname{sh} \alpha(\mu - \gamma - x_j)}{\operatorname{sh} \alpha(-\gamma + x_j) \operatorname{sh} \alpha(-\gamma - x_j)}, \quad r=0, \\
&\sim \lambda_0 (1 - \lambda_1 \operatorname{ch}(2\alpha x_j) e^{-2\alpha\omega_1} + (\text{constant} - \lambda_2 \operatorname{ch}(4\alpha x_j)) e^{-4\alpha\omega_1} \\
&\quad + O(e^{-6\alpha\omega_1})), \quad r=1, \\
&\sim \lambda_0 \frac{\operatorname{ch} \alpha(\mu - \gamma + x_j) \operatorname{ch} \alpha(\mu - \gamma - x_j)}{\operatorname{ch} \alpha(-\gamma + x_j) \operatorname{ch} \alpha(-\gamma - x_j)}, \quad r=2,
\end{aligned}$$

$$\begin{aligned} &\sim \lambda_0(1 + \lambda_1 \operatorname{ch}(2\alpha x_j) e^{-2\alpha\omega_1} + (\text{constant} - \lambda_2 \operatorname{ch}(4\alpha x_j)) e^{-4\alpha\omega_1} \\ &\quad + O(e^{-6\alpha\omega_1})), \quad r=3, \end{aligned}$$

with  $\lambda_0 = \exp(2\kappa\mu(\mu - 2\gamma))$ ,  $\lambda_1 = 8 \operatorname{sh}(\alpha\mu) \operatorname{sh}\alpha(\mu - 2\gamma)$ ,  $\lambda_2 = 16 \operatorname{ch}(2\alpha\gamma) \operatorname{sh}(\alpha\mu) \operatorname{sh}\alpha(\mu - 2\gamma)$ .

Substituting Eqs. (3.32), (3.33) into  $U$  (3.23) leads, after subtracting a divergent constant term of order  $O(e^{4\alpha\omega_1})$  and omitting of multiplicative constants stemming from the exponentials  $\exp(\kappa z^2)$ , to the hyperbolic limit given in Eq. (3.28).

ii. A further limit of the system with hyperbolic potentials (3.27), (3.28) leads to the  $n$ -particle version of a two-particle system studied in Ref. 25. If we set  $\mu_1 = -(\mu_0 + \mu'_0)/2 + i\pi/(2\alpha) + R$ ,  $\mu'_1 = -(\mu_0 + \mu'_0)/2 - i\pi/(2\alpha) - R$ ,  $\mu_2 = \mu'_2 = 0$  and  $\mu_3 = -\mu'_3 = i\pi/(2\alpha)$  in the operators with potentials (3.27), (3.28), then the limits  $\lim_{R \rightarrow \infty} 2^4 \alpha^6 e^{\alpha(\gamma - 2R)} \hat{H}_1$  and  $\lim_{R \rightarrow \infty} 2^{(4n)} \alpha^{(6n)} e^{n\alpha(\gamma - 2R)} \hat{H}_n$  result in operators of the form  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)-(3.18) with potentials given by

$$v(z) = \operatorname{sh}\alpha(\mu + z) / \operatorname{sh}(\alpha z), \quad w(z) = \operatorname{sh}\alpha(\mu_0 + z) \operatorname{sh}\alpha(\mu'_0 + \gamma + z),$$

$$U = -\operatorname{ch}\alpha(\mu_0 + \mu'_0 + \gamma) \sum_{1 \leq j \leq n} \operatorname{ch}(2\alpha x_j). \tag{3.34}$$

For  $n=2$  the Liouville integrability of the corresponding classical system was proved by Inozemtsev.<sup>25</sup>

### D. Generalized external fields for the hyperbolic and rational systems of type A

We have seen that the type D difference CM systems in the previous subsection form a generalization of the systems in Sec. II A. It should, therefore, not come as a surprise that sending the center of mass to infinity leads to more general external fields for the relativistic CM system than the ones given in Sec. II B. If we substitute

$$x_j \rightarrow x_j + R, \quad \mu_r \rightarrow \mu_r - R, \quad \mu'_r \rightarrow \mu'_r - \gamma + R, \tag{3.35}$$

in operators  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)-(3.18) with hyperbolic potentials (3.27), (3.28), then for  $R \rightarrow \infty$  we have:

$$\begin{aligned} v(\varepsilon_j x_j + \varepsilon_k x_k) &\sim \frac{\operatorname{sh}\alpha(\mu + \varepsilon(x_j - x_k))}{\operatorname{sh}\alpha\varepsilon(x_j - x_k)}, \quad \text{if } \varepsilon_j = -\varepsilon_k = \varepsilon, \\ &\sim \exp(\alpha\varepsilon\mu), \quad \text{if } \varepsilon_j = \varepsilon_k = \varepsilon, \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} w(\varepsilon_j x_j) &\sim 2^{-2} \alpha^{-6} e^\alpha \sum_{0 \leq r \leq 3} (\mu'_r - \gamma/2) e^{4\alpha R} \prod_{0 \leq r \leq 3} \operatorname{sh}\alpha(\mu_r + x_j), \quad \text{if } \varepsilon_j = +1, \\ &\sim 2^{-2} \alpha^{-6} e^{-\alpha} \sum_{0 \leq r \leq 3} (\mu_r - \gamma/2) e^{4\alpha R} \prod_{0 \leq r \leq 3} \operatorname{sh}\alpha(\mu'_r - x_j), \quad \text{if } \varepsilon_j = -1. \end{aligned} \tag{3.37}$$

It is not difficult to derive the corresponding asymptotics for  $U$  (3.28) (cf. Remark i. for further details). It then follows that for  $R \rightarrow \infty$  we obtain, after the usual renormalizations (i.e., the division by and subtraction of divergent constants), operators of the form

$$\hat{H}_1 = \sum_{1 \leq j \leq n} \left( w_+^{1/2}(x_j) \prod_{k \neq j} v^{1/2}(x_j - x_k) \exp(-\beta \hat{\theta}_j) \prod_{k \neq j} v^{1/2}(x_k - x_j) w_-^{1/2}(-x_j) \right. \\ \left. + w_-^{1/2}(-x_j) \prod_{k \neq j} v^{1/2}(x_k - x_j) \exp(\beta \hat{\theta}_j) \prod_{k \neq j} v^{1/2}(x_j - x_k) w_+^{1/2}(x_j) \right) + U(x_1, \dots, x_n), \tag{3.38}$$

and

$$\hat{H}_n = \sum_{\substack{J_+, J_- \subset \{1, \dots, n\} \\ J_+ \cap J_- = \emptyset}} U_{J_+^c \cap J_-^c} V_{J_+, J_-}^{1/2} e^{-\beta(\hat{\theta}_{J_+} - \hat{\theta}_{J_-})} V_{J_-, J_+}^{1/2} U_{J_+ \cap J_-^c} \tag{3.39}$$

with

$$V_{J_+, J_-; K} = \prod_{j \in J_+} w_+(x_j) \prod_{j \in J_-} w_-(-x_j) \prod_{\substack{j \in J_+ \\ j' \in J_-}} v(x_j - x_{j'}) v(x_j - x_{j'} + 2\gamma) \\ \times \prod_{\substack{j \in J_+ \\ k \in K}} v(x_j - x_k) \prod_{\substack{j \in J_- \\ k \in K}} v(x_k - x_j), \\ U_K = \sum_{\substack{K_+ \cup K_- = K \\ K_+ \cap K_- = \emptyset}} \left( \lambda_w^{|K_+| - |K_-|} \prod_{k \in K_+} w_+(x_k) \prod_{k \in K_-} w_-(-x_k) \right. \\ \left. \times \prod_{\substack{k \in K_+ \\ k' \in K_-}} v(x_k - x_{k'}) v(x_{k'} - x_k - 2\gamma) \right),$$

where

*Hyperbolic potentials with generalized external field (type A)*

$$v(z) = \text{sh}\alpha(\mu + z) / \text{sh}(\alpha z),$$

$$w_+(z) = \alpha^{-4} \prod_{0 \leq r \leq 3} \text{sh}\alpha(\mu_r + z), \quad w_-(z) = \alpha^{-4} \prod_{0 \leq r \leq 3} \text{sh}\alpha(\mu'_r + z), \tag{3.40}$$

$\lambda_w = \exp\alpha(\sum_{0 \leq r \leq 3} (\mu_r + \mu'_r - \gamma)/2)$ , and

$$U = 2^{-2} \alpha^{-4} \sum_{\substack{0 \leq r \leq 3 \\ 1 \leq j \leq n}} (\text{sh}^2 \alpha(\mu_r - \gamma - x_j - \lambda) + \text{sh}^2 \alpha(\mu'_r - \gamma + x_j + \lambda)) \\ - 2^{-1} \alpha^{-4} \text{ch}(2\alpha\gamma) \sum_{1 \leq j \leq n} \text{sh}^2 \alpha(2x_j + \lambda) \\ + \alpha^{-4} \text{sh}(\alpha\mu) \text{sh}\alpha(\mu - 2\gamma) \sum_{1 \leq j < k \leq n} \text{sh}^2 \alpha(x_j + x_k + \lambda), \tag{3.41}$$

$$\lambda = \sum_{0 \leq r \leq 3} (\mu_r - \mu'_r) / 4.$$

For  $\alpha \rightarrow 0$  we obtain the corresponding rational system. The renormalized limit leads to operators  $\hat{H}_1$  (3.38) and  $\hat{H}_n$  (3.39) with

*Rational potentials with generalized external field (type A)*

$$v(z) = (\mu + z) / z, \quad w_+(z) = \prod_{0 \leq r \leq 3} (\mu_r + z), \quad w_-(z) = \prod_{0 \leq r \leq 3} (\mu'_r + z), \quad (3.42)$$

$\lambda_w = 1$ , and

$$U = 12^{-1} \sum_{\substack{0 \leq r \leq 3 \\ 1 \leq j \leq n}} ((\mu_r - \gamma - x_j - \lambda)^4 + (\mu'_r - \gamma + x_j + \lambda)^4) - \sum_{1 \leq j \leq n} (\gamma^2 (2x_j + \lambda)^2 + (2x_j + \lambda)^4 / 6) + \mu(\mu - 2\gamma) \sum_{1 \leq j < k \leq n} (x_j + x_k + \lambda)^2, \quad (3.43)$$

$$\lambda = \sum_{0 \leq r \leq 3} (\mu_r - \mu'_r) / 4.$$

Notice that the condition on the coupling constants of the external field ensuring the commutativity of  $\hat{H}_1$  and  $\hat{H}_n$  has changed as a consequence of the reparametrization (3.35). This condition now reads  $\sum_{0 \leq r \leq 3} (\mu_r + \mu'_r) - 4\gamma = 0$ .

*Remarks:* i. The behavior for  $R \rightarrow \infty$  of the function  $U$  (3.28) (after Substitution (3.35)) can be derived with the aid of the asymptotics

$$\begin{aligned} c_0 &\sim \kappa_0 e^{8\alpha R} (1 - \kappa_1 e^{-2\alpha R} + O(e^{-4\alpha R})), \\ c_1 &\sim \kappa_0 e^{8\alpha R} (1 + \kappa_1 e^{-2\alpha R} + O(e^{-4\alpha R})), \\ c_2 &\sim \kappa_2 e^{2\alpha R} + O(e^{-2\alpha R}), \end{aligned} \quad (3.44)$$

where

$$\kappa_0 = 2^{-9} \alpha^{-6} (\text{sh}(\alpha\mu) \text{sh}(\alpha(\mu - 2\gamma)))^{-1} \exp \alpha \left( \sum_{0 \leq r \leq 3} (\mu'_r - \mu_r) \right),$$

$$\kappa_1 = \sum_{0 \leq r \leq 3} (e^{2\alpha(\mu_r - \gamma)} + e^{-2\alpha(\mu'_r - \gamma)}), \quad \kappa_2 = 2^{-5} \alpha^{-6} \sum_{0 \leq r \leq 3} (e^{-2\alpha(\mu_r - \gamma)} + e^{2\alpha(\mu'_r - \gamma)}), \quad (3.45)$$

and

$$\begin{aligned} &\frac{\text{sh}(\alpha(\mu - \gamma + x_j + R))}{\text{sh}(\alpha(-\gamma + x_j + R))} \frac{\text{sh}(\alpha(\mu - \gamma - x_j - R))}{\text{sh}(\alpha(-\gamma - x_j - R))} \\ &\quad \sim (1 - \lambda_1 e^{-2\alpha x_j} e^{-2\alpha R} - \lambda_2 e^{-4\alpha x_j} e^{-4\alpha R} + O(e^{-6\alpha R})), \\ &\frac{\text{ch}(\alpha(\mu - \gamma + x_j + R))}{\text{ch}(\alpha(-\gamma + x_j + R))} \frac{\text{ch}(\alpha(\mu - \gamma - x_j - R))}{\text{ch}(\alpha(-\gamma - x_j - R))} \\ &\quad \sim (1 + \lambda_1 e^{-2\alpha x_j} e^{-2\alpha R} - \lambda_2 e^{-4\alpha x_j} e^{-4\alpha R} + O(e^{-6\alpha R})), \end{aligned} \quad (3.46)$$

with

$$\lambda_1 = 4 \operatorname{sh}(\alpha \mu) \operatorname{sh} \alpha (\mu - 2 \gamma), \quad \lambda_2 = 8 \operatorname{ch}(2 \alpha \gamma) \operatorname{sh}(\alpha \mu) \operatorname{sh} \alpha (\mu - 2 \gamma). \quad (3.47)$$

ii. The external field couplings to the relativistic Calogero–Moser system introduced in Refs. 24 and 25 can be seen as special (limiting) cases of the fields considered here.

#### IV. DIFFERENCE TODA CHAINS

In this section it is explained how difference Toda chains with boundary conditions arise as limits of the type D difference CM system with elliptic potentials of Sec. III B.

We start with the type D difference CM operators  $\hat{H}_1$  (3.12) and  $\hat{H}_n$  (3.16)–(3.18) with potentials given by Eqs. (3.13), (3.14). To arrive at Toda type difference operators we substitute

$$x_j \rightarrow x_j + \omega_1(j-1)/(n-1), \quad \mu \rightarrow \mu + \omega_2 - \omega_1/(n-1), \quad (4.1)$$

and send  $\omega_1$  to infinity. The Hamiltonian then becomes

$$\begin{aligned} \hat{H}_1 = & w_-^{1/2}(x_1)v^{1/2}(x_1+x_2)v^{1/2}(x_1-x_2)e^{-\beta\hat{\theta}_1}v^{1/2}(x_2-x_1)v^{1/2}(-x_1-x_2)w_-^{1/2}(-x_1) \\ & + w_-^{1/2}(-x_1)v^{1/2}(-x_1-x_2)v^{1/2}(x_2-x_1)e^{\beta\hat{\theta}_1}v^{1/2}(x_1-x_2)v^{1/2}(x_1+x_2)w_-^{1/2}(x_1) \\ & + w_+^{1/2}(x_2)v^{1/2}(x_1+x_2)v^{1/2}(x_2-x_1)e^{-\beta\hat{\theta}_2}v^{1/2}(x_1-x_2)v^{1/2}(-x_1-x_2)w_+^{1/2}(-x_2) \\ & + w_+^{1/2}(-x_2)v^{1/2}(-x_1-x_2)v^{1/2}(x_1-x_2)e^{\beta\hat{\theta}_2}v^{1/2}(x_2-x_1)v^{1/2}(x_1+x_2)w_+^{1/2}(x_2) \\ & + U(x_1, x_2), \end{aligned} \quad (4.2)$$

if  $n=2$ ,

$$\begin{aligned} \hat{H}_1 = & w_-^{1/2}(x_1)v^{1/2}(x_1+x_2)e^{-\beta\hat{\theta}_1}v^{1/2}(x_2-x_1)w_-^{1/2}(-x_1) \\ & + w_-^{1/2}(-x_1)v^{1/2}(x_2-x_1)e^{\beta\hat{\theta}_1}v^{1/2}(x_1+x_2)w_-^{1/2}(x_1) \\ & + v^{1/2}(x_1+x_2)v^{1/2}(x_2-x_1)e^{-\beta\hat{\theta}_2}v^{1/2}(x_3-x_2)v^{1/2}(-x_2-x_3) \\ & + v^{1/2}(-x_2-x_3)v^{1/2}(x_3-x_2)e^{\beta\hat{\theta}_2}v^{1/2}(x_2-x_1)v^{1/2}(x_1+x_2) \\ & + w_+^{1/2}(x_3)v^{1/2}(x_3-x_2)e^{-\beta\hat{\theta}_3}v^{1/2}(-x_2-x_3)w_+^{1/2}(-x_3) \\ & + w_+^{1/2}(-x_3)v^{1/2}(-x_2-x_3)e^{\beta\hat{\theta}_3}v^{1/2}(x_3-x_2)w_+^{1/2}(x_3) \\ & + U(x_1, x_2, x_3), \end{aligned} \quad (4.3)$$

if  $n=3$ , and

$$\begin{aligned} \hat{H}_1 = & w_-^{1/2}(x_1)v^{1/2}(x_1+x_2)e^{-\beta\hat{\theta}_1}v^{1/2}(x_2-x_1)w_-^{1/2}(-x_1) \\ & + w_-^{1/2}(-x_1)v^{1/2}(x_2-x_1)e^{\beta\hat{\theta}_1}v^{1/2}(x_1+x_2)w_-^{1/2}(x_1) \\ & + v^{1/2}(x_1+x_2)v^{1/2}(x_2-x_1)e^{-\beta\hat{\theta}_2}v^{1/2}(x_3-x_2) \\ & + v^{1/2}(x_3-x_2)e^{\beta\hat{\theta}_2}v^{1/2}(x_2-x_1)v^{1/2}(x_1+x_2) \\ & + \sum_{3 \leq j \leq n-2} (v^{1/2}(x_j-x_{j-1})e^{-\beta\hat{\theta}_j}v^{1/2}(x_{j+1}-x_j)) \end{aligned}$$

$$\begin{aligned}
 &+ v^{1/2}(x_{j+1}-x_j)e^{\beta\hat{\theta}_j v^{1/2}(x_j-x_{j-1})} \\
 &+ v^{1/2}(x_{n-1}-x_{n-2})e^{-\beta\hat{\theta}_{n-1}v^{1/2}(x_n-x_{n-1})}v^{1/2}(-x_{n-1}-x_n) \\
 &+ v^{1/2}(-x_{n-1}-x_n)v^{1/2}(x_n-x_{n-1})e^{\beta\hat{\theta}_{n-1}v^{1/2}(x_{n-1}-x_{n-2})} \\
 &+ w_+^{1/2}(x_n)v^{1/2}(x_n-x_{n-1})e^{-\beta\hat{\theta}_n v^{1/2}(-x_{n-1}-x_n)}w_+^{1/2}(-x_n) \\
 &+ w_+^{1/2}(-x_n)v^{1/2}(-x_{n-1}-x_n)e^{\beta\hat{\theta}_n v^{1/2}(x_n-x_{n-1})}w_+^{1/2}(x_n) \\
 &+ U(x_1, \dots, x_n),
 \end{aligned} \tag{4.4}$$

if  $n > 3$ , with

$$\begin{aligned}
 v(z) &= (1 + e^{-2\alpha(\mu+z)}), \\
 w_-(z) &= \frac{\text{sh}\alpha(\mu_0+z)}{\text{sh}(\alpha z)} \frac{\text{ch}\alpha(\mu_2+z)}{\text{ch}(\alpha z)} \frac{\text{sh}\alpha(\mu'_0+\gamma+z)}{\text{sh}\alpha(\gamma+z)} \frac{\text{ch}\alpha(\mu'_2+\gamma+z)}{\text{ch}\alpha(\gamma+z)}, \\
 w_+(z) &= \frac{\text{sh}\alpha(\mu_1+z)}{\text{sh}(\alpha z)} \frac{\text{ch}\alpha(\mu_3+z)}{\text{ch}(\alpha z)} \frac{\text{sh}\alpha(\mu'_1+\gamma+z)}{\text{sh}\alpha(\gamma+z)} \frac{\text{ch}\alpha(\mu'_3+\gamma+z)}{\text{ch}\alpha(\gamma+z)},
 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
 U &= c_0 \frac{(1 + e^{-2\alpha(\mu-\gamma)}e^{-2\alpha x_2})(1 + e^{-2\alpha(\mu-\gamma)}e^{2\alpha x_2})\delta_{n,2}}{\text{sh}\alpha(-\gamma+x_1)\text{sh}\alpha(-\gamma-x_1)} \\
 &+ c_1 \frac{(1 + e^{-2\alpha(\mu-\gamma)}e^{2\alpha x_{n-1}})(1 + e^{-2\alpha(\mu-\gamma)}e^{-2\alpha x_{n-1}})\delta_{n,2}}{\text{sh}\alpha(-\gamma+x_n)\text{sh}\alpha(-\gamma-x_n)} \\
 &+ c_2 \frac{(1 - e^{-2\alpha(\mu-\gamma)}e^{-2\alpha x_2})(1 - e^{-2\alpha(\mu-\gamma)}e^{2\alpha x_2})\delta_{n,2}}{\text{ch}\alpha(-\gamma+x_1)\text{ch}\alpha(-\gamma-x_1)} \\
 &+ c_3 \frac{(1 - e^{-2\alpha(\mu-\gamma)}e^{2\alpha x_{n-1}})(1 - e^{-2\alpha(\mu-\gamma)}e^{-2\alpha x_{n-1}})\delta_{n,2}}{\text{ch}\alpha(-\gamma+x_n)\text{ch}\alpha(-\gamma-x_n)},
 \end{aligned} \tag{4.6}$$

where  $\delta_{i,j}$  denotes the Kronecker delta and

$$\begin{aligned}
 c_0 &= 2\text{sh}\alpha(\mu_0-\gamma)\text{ch}\alpha(\mu_2-\gamma)\text{sh}(\alpha\mu'_0)\text{ch}(\alpha\mu'_2), \\
 c_1 &= 2\text{sh}\alpha(\mu_1-\gamma)\text{ch}\alpha(\mu_3-\gamma)\text{sh}(\alpha\mu'_1)\text{ch}(\alpha\mu'_3), \\
 c_2 &= 2\text{ch}\alpha(\mu_0-\gamma)\text{sh}\alpha(\mu_2-\gamma)\text{ch}(\alpha\mu'_0)\text{sh}(\alpha\mu'_2), \\
 c_3 &= 2\text{ch}\alpha(\mu_1-\gamma)\text{sh}\alpha(\mu_3-\gamma)\text{ch}(\alpha\mu'_1)\text{sh}(\alpha\mu'_3).
 \end{aligned} \tag{4.7}$$

The corresponding quantum integral  $\hat{H}_n$  is obtained by substituting in Eqs. (3.16)-(3.18):

$$w(\varepsilon_j x_j) \rightarrow \begin{cases} w_-(\varepsilon_1 x_1), & \text{if } j=1 \\ \exp\left(\alpha\varepsilon_j \sum_{0 \leq r \leq 3} (-1)^r (\mu_r + \mu'_r)\right), & \text{if } 1 < j < n, \\ w_+(\varepsilon_n x_n), & \text{if } j=n \end{cases}$$

and

$$\begin{aligned}
 v(x_j - x_k) &\rightarrow \begin{cases} (1 + e^{-2\alpha(\mu+x_j-x_{j-1})}), & \text{if } k=j-1 \\ (1 + e^{-2\alpha(\mu+x_1-x_2)}), & \text{if } n=2 \text{ and } (j,k)=(1,2), \\ 1, & \text{otherwise} \end{cases} \\
 v(x_j + x_k) &\rightarrow \begin{cases} (1 + e^{-2\alpha(\mu+x_1+x_2)}), & \text{if } (j,k) \text{ or } (k,j)=(1,2) \\ 1, & \text{otherwise} \end{cases}, \\
 v(-x_j - x_k) &\rightarrow \begin{cases} (1 + e^{-2\alpha(\mu-x_{n-1}-x_n)}), & \text{if } (j,k) \text{ or } (k,j)=(n-1,n) \\ 1, & \text{otherwise} \end{cases}
 \end{aligned}$$

(together with a similar substitution for the ‘shifted’ potentials  $v(\varepsilon_j x_j + \varepsilon_k x_k + 2\gamma)$ ). Commutativity holds again provided  $\sum_{0 \leq r \leq 3} (\mu_r + \mu'_r) = 0$ .

If we set the coupling constants  $\mu_r, \mu'_r$  equal to zero then the boundary potentials become trivial:  $w_-, w_+ = 1$  and  $U = 0$ . The resulting operators can be associated with the loop algebra  $D_n^{(1)}$ : the classical version of the Hamiltonian  $(\theta_j \rightarrow \theta_j, \gamma \rightarrow 0)$  coincides up to a canonical gauge transformation with the Hamiltonian of the  $D_n^{(1)}$ -type discrete time Toda chain introduced by Suris.<sup>26</sup> For this special case the classical integrability of the model follows for arbitrary particle number from the  $R$ -matrix construction in Ref. 26. Recently, also the quantum integrability of the  $D_n^{(1)}$ -type specialization of our difference Toda chain was shown with the aid of the  $R$ -matrix method.<sup>28</sup> Via limit transitions similar to those between Block 6 and 7 of the diagram in the introduction we arrive at the other types of boundary conditions considered by Refs. 26 and 28.

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**APPENDIX: SOME FUNCTIONAL IDENTITIES**

In this appendix it is shown that the functions  $U_{K,p}$  in (2.6) and (2.10) are equal for potentials  $v$  of the form (2.7), (2.8), without restrictions on the external potential  $w$ . The point is that equality of (2.6) and (2.10) is implied by the following identity for  $v(z)$  ( $p = 1, 2, \dots$ ):

$$\begin{aligned}
 \sum_{\substack{\emptyset \subseteq I_1 \subseteq \dots \subseteq I_q = \{1, \dots, p\} \\ 1 \leq q \leq p}} (-1)^q \prod_{1 \leq q' \leq q} \prod_{\substack{i, i' \in I_{q'} \setminus I_{q'-1} \\ i < i'}} v(y_i + y_{i'}) \prod_{\substack{i \in I_{q'} \setminus I_{q'-1} \\ i' \in \{1, \dots, p\} \setminus I_{q'}}} v(y_i - y_{i'}) \\
 = (-1)^p \prod_{\substack{i, i' \in \{1, \dots, p\} \\ i < i'}} v(-y_i - y_{i'}) \tag{A1}
 \end{aligned}$$

(where  $I_0 = \emptyset$ ). To see that Eq. (A1) is indeed sufficient for equality to hold, one has to compare the terms in (2.6) and (2.10) corresponding to a fixed index set  $I = I_q \subset K$ , with the signs  $\varepsilon_i, i \in I$  chosen in a fixed configuration. One infers that after dividing by a common factor of the form

$$\prod_{i \in I} w(\varepsilon_i x_i) \prod_{\substack{i, i' \in I \\ i < i'}} v(\varepsilon_i x_i + \varepsilon_{i'} x_{i'}) \prod_{\substack{i \in I \\ k \in K \setminus I}} v(\varepsilon_i x_i + x_k) v(\varepsilon_i x_i - x_k)$$

(which eliminates the dependence on  $w$ ), equality of the corresponding terms amounts to Eq. (A1), where we have set  $\varepsilon_i x_i + \gamma = y_i$  and renumbered such that  $I = \{1, \dots, p\}$  (recall  $|I| = p$ ).

It remains to verify that  $v(z)$  (2.7), (2.8) satisfies (A1). For  $p = 1, 2$  this is easy to check: for  $p = 1$  the equation is trivial (as in both sides the product is empty) and for  $p = 2$  it reduces to the identity  $-v(y_1 + y_2) + v(y_1 - y_2) + v(y_2 - y_1) = v(-y_1 - y_2)$ , which hinges on the property of  $v(z)$  (2.7), (2.8) that  $v(z) + v(-z)$  is constant in  $z$ . In order to demonstrate (A1) for general  $p$ , it is convenient to rewrite the equation in a simpler form by performing induction on  $p$ . Notice to this end that for a fixed (non-empty) index set  $I_1 = J \subset \{1, \dots, p\}$  the corresponding terms in the l.h.s. of (A1) split in a product of

$$- \prod_{\substack{j, j' \in J \\ j < j'}} v(y_j + y_{j'}) \prod_{\substack{j \in J \\ k \in \{1, \dots, p\} \setminus J}} v(y_j - y_k)$$

and an expression that has the same form as the l.h.s. but with  $I_q$  taken to be  $\{1, \dots, p\} \setminus J$  (instead of  $\{1, \dots, p\}$ ). If we replace the second part by the corresponding r.h.s. of (A1) using induction on  $p$ , and we take the sum over all possible choices of the index set  $J$ , then we arrive at an equation of the form

$$\sum_{J \subset \{1, \dots, p\}} (-1)^{|J|} \prod_{\substack{j, j' \in J \\ j < j'}} v(y_j + y_{j'}) \prod_{\substack{j \in J \\ k \notin J}} v(y_j - y_k) \prod_{\substack{k, k' \notin J \\ k < k'}} v(-y_k - y_{k'}) = 0 \tag{A2}$$

(where we have brought all terms to one side: the term in (A2) corresponding to  $J = \emptyset$  originates from the r.h.s. of (A1)).

Equations (A1) and (A2) form equivalent systems. For  $p \leq 2$  the equations are exactly the same whereas for  $p > 2$  the latter is combinatorially much simpler than the former. Our proof that the potentials  $v$  (2.7), (2.8) satisfy (A2) (and thus (A1)) relies on Liouville’s theorem. The terms in the l.h.s. of (A2) have (generically) simple poles caused by the zeros in the denominators of  $v(z)$  at (rational case) or congruent mod  $i\pi/\alpha$  to (hyperbolic case) one of

$$\begin{aligned} y_j - y_k = 0 & \quad \text{with } j \in J \text{ and } k \notin J & \quad (\text{type } 1) \\ y_j + y_{j'} = 0 & \quad \text{with } j, j' \in J & \quad (\text{type } 2.a) \\ y_k + y_{k'} = 0 & \quad \text{with } k, k' \notin J & \quad (\text{type } 2.b). \end{aligned} \tag{A3}$$

Residues at poles of type 1 cancel in (A2) because of the permutation symmetry. Similarly, a residue at  $y_i + y_{i'} = 0$  of type 2.a ( $i, i' \in J$ ) or type 2.b ( $i, i' \notin J$ ) cancels pairwise against the residue of the term corresponding to the index set obtained by bringing the pair  $\{i, i'\}$  from  $J$  to  $J^c$  (type 2.a) or vice versa (type 2.b). We conclude that the l.h.s. of (0.2) is an entire function of  $y_1, \dots, y_p$ . Furthermore, from the limits

$$\lim_{z \rightarrow \infty} v(z) = 1 \quad (\text{rational}), \quad \lim_{\text{Re}(\alpha z) \rightarrow \pm \infty} v(z) = \exp(\pm \alpha \mu) \quad (\text{hyperbolic}), \tag{A4}$$

it follows that the l.h.s. is also bounded in these variables. By Liouville’s theorem it must then be a constant function independent of  $y_i$ ,  $i = 1, \dots, p$ . To see that this constant is indeed equal to zero one observes that the l.h.s. of (A2) is odd in  $y_i$ . (A sign flip  $y_i \rightarrow -y_i$  in the terms of (A2) amounts to pulling the index  $i$  from  $J$  to  $J^c$  or vice versa, and multiplying the resulting term by  $-1$  to compensate for the sign change caused by the decrease/increase of the cardinality of  $J$  by one.)

*Remarks:* i. Using the limits (A4) it is also possible to compute directly that the constant in the l.h.s. of (A2) is zero. By setting  $y = R\tilde{y}$  with  $\tilde{y}_1 > \tilde{y}_2 > \dots > \tilde{y}_p > 0$  and sending  $R \rightarrow +\infty$  we obtain for the l.h.s.



$$\sum_{J \subset \{1, \dots, p\}} (-1)^{|J|} = \sum_{q=0}^p (-1)^q \binom{p}{q} = (1-1)^p = 0.$$

in the rational case and

$$\begin{aligned} & \sum_{J \subset \{1, \dots, p\}} (-1)^{|J|} e^{\frac{\alpha\mu}{2}|J|(|J|-1)} e^{\alpha\mu(|\{j \in J, k \notin J, j < k\}| - |\{j \in J, k \notin J, j > k\}|)} e^{-\frac{\alpha\mu}{2}(p-|J|)(p-|J|-1)} \\ &= \sum_{J \subset \{1, \dots, p\}} (-1)^{|J|} e^{\alpha\mu} \left( \sum_{j \in J} \binom{p-j}{j} - \sum_{k \notin J} \binom{p-k}{k} \right) \\ &= (e^{-\alpha\mu(p-1)} - e^{\alpha\mu(p-1)}) \dots (e^{-\alpha\mu} - e^{\alpha\mu})(1-1) = 0. \end{aligned}$$

in the case of hyperbolic potentials (where in the latter situation we assumed  $\alpha$  to be positive).

ii. In Ref. 18 we conjectured the commutativity of the difference operators  $\hat{H}_i$  (2.3)-(2.6) with elliptic potentials of the form

$$v(z) = \frac{\sigma_s(z)}{\sigma(z)}, \quad w(z) = \left( \prod_{0 \leq r \leq 3} \frac{\sigma_r(\mu_r + z)}{\sigma_r(z)} \right) \left( \frac{\sigma(\mu'_0 + \gamma + z)}{\sigma(\gamma + z)} \frac{\sigma_s(\mu'_s + \gamma + z)}{\sigma_s(\gamma + z)} \right), \quad (\text{A5})$$

where  $(\mu'_0 + \mu'_s) + \sum_{0 \leq r \leq 3} \mu_r = 0$  and  $s = 1, 2$ , or  $3$ . (For  $n \leq 4$  the commutativity was checked by computer.) This difference system is a special case of the elliptic system in Sec. III B corresponding to  $\mu = \omega_s$  and  $\mu'_r = 0$  for  $r \neq 0, s$ . It is not hard to check by the above method that also  $v(z)$  (A5) is a solution of Eq. (A2): the l.h.s. is again entire, odd, and bounded (because doubly periodic) in  $y_i$ . This means that also for the potentials (A5) one may replace the functions  $U_{K,p}$  (2.6) by the simpler form (2.10).

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