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DIFFERENCE SPECTRUM AND SPECTRAL SYNTHESIS

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Abstract. As an aid in understanding sets of synthesis for the Fourier algebra $A(G)$ of a locally compact abelian group G, the difference spectrum $\Delta(E)$ for a closed set E in G is studied. Numerous relations involving difference spectra of unions, intersections and cartesian products are obtained and their implications on unions, intersections and cartesian products of sets of spectral synthesis are deduced. The set $\Lambda(E)$ of locally nonsynthesizable points of E is introduced and its relation with $\Delta(E)$ is discussed. The concept of n-difference spectrum is introduced and is used to study weak spectral synthesis. Local methods are employed throughout.

Introduction. The concept of a function belonging locally to an ideal at a point has been studied and exploited in spectral synthesis for a long time. If I, J are two closed ideals of $A(G)$ (the Fourier algebra of a locally compact abelian group G), the set $\Delta(I, J)$ of points where neither I nor J is locally contained in the other has also been utilized in the study of spectral synthesis (e.g., Katznelson's proof [5, p. 36] and Stegeman's proof [8] of extensions of Helson's result and Saeki [6]) under different notations. For a closed subset E of G let $\Delta(E)=\Delta(I(E), J(E))$, where $I(E)$ and $J(E)$ are respectively the largest and the smallest closed ideals of $A(G)$ with cospectrum E. $\Delta(E)$, the difference spectrum of E, has been studied and systematically exploited in problems on spectral synthesis by Saeki [6], Stegeman [9] and Salinger and Stegeman [7]. Using local techniques and difference spectrum some results on unions and intersections of sets of synthesis have been recently given by the authors in [2].

In this paper, besides giving some examples of difference spectrum and further results on difference spectra and spectral synthesis, we introduce the concept of n-difference spectrum and use it to study weak synthesis. We also introduce $\Lambda(E)$, the set of 'locally non synthesizable points' of E, which is closely related to $\Delta(E)$.

1. The difference spectrum. For spectral synthesis we generally follow the notations of [2], [7] and [9]. As general references for harmonic analysis, we cite [1] and [5]. We recall the definition of $\Delta(E)$, the difference spectrum of a closed set E: $\Delta(E)=\{x : I(E)\neq_x J(E)\}$ so that $\Delta(E)$ is a union of perfect subsets of ∂E, the boundary of E. E is a set of synthesis (an S-set) if and only if $\Delta(E)=\varnothing$. No computation of $\Delta(E)$ has been given in the literature. So we start with some examples. We make use of some results of the next section.

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EXAMPLE 1.1. (i) It is a well-known result of Laurent Schwartz that S^n, the unit sphere, is not of synthesis in $\boldsymbol{R}^{n+1}$, $n \geq 2$. Thus $\Delta(S^n) \neq \varnothing$. By rotation invariance of $J(S^n)$ and of $I(S^n)$, it follows that $\Delta(S^n) = S^n$.

(ii) We can get an example with $\varnothing = \Delta(E) \neq E$ as follows. Let E be the union of S^n and a line segment, for instance. Then $\Delta(E) = S^n$ (e.g., use Lemma 2.2 below).

(iii) $\Delta(S^n \times S^m) = S^n \times S^m (n, m \geq 2)$.

(iv) If E is a set of synthesis in G, then $\Delta(S^n \times E) = S^n \times E$. (For (iii) and (iv), use Lemma 2.7 below and (i) above.)

(v) If $S \subset S^n$ is a spherical cap, $n \geq 2$, then $\Delta(S) = S$. (Use Lemma 1(ii) of [2].)

Reiter [5, p. 40] has proved the following local characterization of an S-set (Wiener set in Reiter's terminology): If E is a closed set in G such that each point of E has a closed relative neighbourhood in E which is an S-set for $A(G)$, then E itself is an S-set. So it seems natural to introduce the following set, which also gives some measure of nonsynthesis for E. We define

$$\Lambda(E) = \{x \in E : x \text{ has no closed relative neighbourhood in } E \text{ which is an } S\text{-set}\} .$$

Points of $\Lambda(E)$ may be called locally nonsynthesizable points of E and, by Reiter's result, E is an S-set if and only if $\Lambda(E) = \varnothing$.

It would be natural to investigate the relation between $\Delta(E)$ and $\Lambda(E)$. Here is a first step. We recall that $j(E) = \{f \in A(G) : f$ vanishes in some neighbourhood of $E\}$ so that $J(E) = \overline{j(E)}$.

LEMMA 1.2. $\Delta(E) \subset \Lambda(E)$.

PROOF. If $x \notin \Lambda(E)$, then there is a closed relative neighbourhood V of x in E which is an S-set. Choose a neighbourhood W of x with $E \cap \bar{W} \subset V$. Choose $k \in A(G)$ such that $k = 1$ near x and $\operatorname{supp} k \subset W$. Let $f \in I(E)$. Since V is an S-set, for each positive integer n, there is a $g_n \in j(V)$ with $\|f - g_n\| < 1/n\|k\|$. Then $g_n k \in j(E)$ and $fk = \lim g_n k \in J(E)$. Thus $f \in_x J(E)$, so $x \notin \Delta(E)$.

COROLLARY 1.3. *If E is a closed set in G and if each point of ∂E has a closed relative neighbourhood in E which is an S-set, then E is an S-set.*

PROOF. $\Delta(E) \subset \Lambda(E) \cap \partial E$.

Observe that $E \setminus \Lambda(E)$ is relatively open in E, so $\Lambda(E)$ is relatively closed in E. Since E itself is closed, this implies that $\Lambda(E)$ is always closed (in contrast to $\Delta(E)$). Moreover, from the definition, it does not seem to follow that $\Lambda(E) \subset \partial E$. (However, in the case of $\boldsymbol{R}^n$ this inclusion does hold.) When is $\Delta(E) = \Lambda(E)$? We begin by proving the following result.

THEOREM 1.4. *If E is a closed subset of $\boldsymbol{R}$ (or of $\boldsymbol{T}$, the unit circle), then $\Delta(E) = \Lambda(E)$.*

PROOF. In view of Lemma 1.2 we have to prove that $\Lambda(E) \subset \Delta(E)$. Let $x \in \Lambda(E)$.

Let $V=[x-\varepsilon, x+\varepsilon]\cap E$ be a closed relative neighbourhood of x. Then $\Delta(V)\neq\emptyset$. If $y\in\Delta(V)$ then $y\in\partial E$. Indeed, suppose $y\in\Delta(V)$ and y lies in the interior of E. Let I be a closed interval around y with $I\subset E$. Then $V\cap I=[x-\varepsilon, x+\varepsilon]\cap I$ is a closed interval and is a relative neighbourhood of y in V, so $y\notin\Delta(V)$ and hence $y\notin\Delta(V)$. Now $\Delta(V)$, being a union of perfect sets (actually a perfect set in this case), is uncountable and $E\setminus V$ has only two possible limit points in V. Hence we can choose $y\in\Delta(V)$ such that y is not a limit point of $E\setminus V$. Then there is a neighbourhood W of y with $W\cap(E\setminus V)=\emptyset$. Choose $k\in A(G)$ such that $k=1$ near y and $\operatorname{supp}k\subset W$. Since $y\in\Delta(V)$, there is an $f\in I(V)$ with $f\notin_y J(V)$. But $fk\in I(E)$, $fk=_y f$ and $f\notin_y J(V)\supset J(E)$. Hence $y\in\Delta(E)$.

Thus, for each positive integer n, we get a $y_n\in[x-1/n, x+1/n]\cap E$ with $y_n\in\Delta(E)$. Since $\Delta(E)$ is closed, $x=\lim y_n\in\Delta(E)$.

Essentially the same proof holds for $\boldsymbol{T}$.

Remark 1.5. It is likely that the same result holds for $\boldsymbol{R}^n$ (and for $\boldsymbol{T}^n$) as well, but we are unable to settle this. (A more general conjecture would be: $\Delta(E)=\Lambda(E)\cap\partial E$ when G is metrizable.)

2. Spectral synthesis. Here we discuss some results on unions, intersections and cartesian products of sets of synthesis using the difference spectrum. Some of the results given below on difference spectra have already been made use of in the previous section to compute some examples on difference spectrum.

The union problem for sets of spectral synthesis is a central unsolved problem in the subject. There have been several attempts, giving rise to partial results. As our contribution in this direction, we obtain some relations between the difference spectra of two closed sets E_1, E_2 and their union $E_1\cup E_2$. We make use of local equality of sets. Recall that $E=_x F$ means $E\cap V=F\cap V$ for some neighbourhood V of x.

Lemma 2.1. *Let E, F be closed sets in G and let $x\in G$. If $E=_x F$, then* (i) $J(E)=_x J(F)$ *and* (ii) $\Delta(E)=_x\Delta(F)$.

Proof. Suppose V is a neighbourhood of x such that $E\cap V=F\cap V$. (i) Choose a neighbourhood W of x with $\bar{W}\subset V$ and a $k\in A(G)$ such that $k=1$ near x and $\operatorname{supp}k\subset W$. Let $f\in J(E)$, so $f=\lim f_n$, $f_n\in j(E)$. Then $kf_n\in j(F)$, $kf=\lim kf_n\in J(F)$ and $f=_x kf$. Thus $J(E)\subset_x J(F)$ and the result follows by interchanging E and F. (ii) We prove that $\Delta(E)\cap V=\Delta(F)\cap V$. Let $y\in\Delta(E)\cap V$. Now choose a neighbourhood W of y with $\bar{W}\subset V$ and choose a $k\in A(G)$ supported in W with $k=1$ near y. Since $y\in\Delta(E)$ there is an $f\in I(E)$ such that $f\notin_y J(E)$. But $E=_y F$ and so $kf=_y f\notin_y J(F)$ by (i), whereas $kf\in I(F)$. Thus $y\in\Delta(F)$. This proves that $\Delta(E)\cap V\subset\Delta(F)\cap V$ and, by symmetry, (ii) follows.

We shall also make use of the observations that the boundary of a closed set E in G is $\partial E=\{x\in E\colon \emptyset\neq E\neq_x G\}$ and that the inclusion $A\subset B\cup C$ holds if $A\subset_x B$ for each $x\notin C$ for subsets A, B, C of G.

LEMMA 2.2. *Let E_1, E_2 be closed sets in G and let*

$$E_{12} = \{x \in \partial E_1 \cap \partial E_2 \cap \partial(E_1 \cup E_2) : x \text{ is a limit point of } E_1 \Delta E_2\},$$

where $E_1 \Delta E_2$ denotes the symmetric difference. Then

$$\Delta(E_1 \cup E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup E_{12}.$$

PROOF. Let $x \notin E_{12}$. Then $x \notin \partial E_1$ or $x \notin \partial E_2$ or $x \notin \partial(E_1 \cup E_2)$ or x is not a limit point of $E_1 \Delta E_2$. If $x \notin \partial E_1$, either $E_1 =_x \varnothing$ or $E_1 =_x G$, that is, $E_1 \cup E_2 =_x E_2$ or $E_1 \cup E_2 =_x G$ and so $\Delta(E_1 \cup E_2) =_x \Delta(E_2)$ or $\Delta(E_1 \cup E_2) =_x \varnothing$. The cases $x \notin \partial E_2$, $x \notin \partial(E_1 \cup E_2)$ are similar. If x is not a limit point of $E_1 \Delta E_2$, then $E_1 \Delta E_2 =_x \varnothing$, so $E_1 \cup E_2 =_x E_1$ and $\Delta(E_1 \cup E_2) =_x \Delta(E_1)$. Thus in any case $x \notin E_{12}$ implies $\Delta(E_1 \cup E_2) \subset_x \Delta(E_1) \cup \Delta(E_2)$. Hence the result follows.

A relation in the reverse direction is given in [2]. A consequence of Lemma 2.2 is the following result (a slight improvement of a result of Saeki [6]).

THEOREM 2.3. *Let E_1, E_2 be S-sets. If there exists a C-set C such that $E_{12} \subset C \subset E_1 \cup E_2$, then $E_1 \cup E_2$ is an S-set.*

PROOF. This is immediate from Lemma 2.2 and the result of Stegeman [9] that E is an S-set if there is a C-set C with $\Delta(E) \subset C \subset E$.

COROLLARY 2.4. *Let E_1, E_2 be two S-sets and suppose $E_1 \cap E_2$ is relatively open in $E_1 \cup E_2$, then $E_1 \cup E_2$ is an S-set.*

PROOF. In this case $E_{12} = \varnothing$.

It is well-known that the intersection of two S-sets need not be an S-set; indeed this phenomenon occurs in any nondiscrete G (see [3]). The following lemma and its corollary, giving a sufficient condition for the intersection of two S-sets to be an S-set, are improvements of corresponding results in [2].

LEMMA 2.5. *Let E_1, E_2 be closed sets in G and let $E^{12} = \{x \in \partial E_1 \cap \partial E_2$: x is a limit point of $E_1 \Delta E_2\}$. Then $\Delta(E_1 \cap E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup E^{12}$.*

COROLLARY 2.6. *If E_1, E_2 are S-sets in G and if there is a C-set C with $E^{12} \subset C \subset E_1 \cap E_2$, then $E_1 \cap E_2$ is an S-set.*

An outstanding open problem is the cartesian product problem for S-sets: Is the cartesian product of two S-sets an S-set? The next lemma on the difference spectrum of a cartesian product gives a result in the converse direction to this problem.

LEMMA 2.7. *If E_1 and E_2 are nonempty closed sets in G, then*

$$E_1 \times \Delta(E_2) \cup \Delta(E_1) \times E_2 \subset \Delta(E_1 \times E_2).$$

PROOF. Let $(x, y) \in E_1 \times \Delta(E_2)$. Then $y \in \Delta(E_2)$, so there is a function $f_2 \in I(E_2)$ such that $f_2 \notin_y J(E_2)$. Choose $f_1 \in A(G)$ with $f_1 = 1$ near x. Then $f_1 \otimes f_2 \in I(E_1 \times E_2)$, but we prove that $f_1 \otimes f_2 \notin_{(x,y)} J(E_1 \times E_2)$. Here $f_1 \otimes f_2(x_1, x_2) = f_1(x_1) f_2(x_2)$.

Suppose $f_1 \otimes f_2 \in_{(x,y)} J(E_1 \times E_2)$, so that there is a $g \in J(E_1 \times E_2)$ with $f_1 \otimes f_2 =_{(x,y)} g$. Let $\{g_n\}$ be a sequence in $j(E_1 \times E_2)$ such that g_n converges to g. Then $g_n(x, \cdot) \in j(E_2)$ and $g_n(x, \cdot)$ converges to $g(x, \cdot)$. This leads to the contradiction $f_2 =_y g(x, \cdot) \in J(E_2)$.

By symmetry we also get $\Delta(E_1) \times E_2 \subset \Delta(E_1 \times E_2)$.

An immediate consequence is the following converse of the cartesian product problem first proved in [4] by different methods.

THEOREM 2.8. *Let E_1, E_2 be two nonempty closed subsets of G. If $E_1 \times E_2$ is an S-set (in $G \times G$), then so are E_1, E_2 (in G).*

Both Lemma 2.7 and Theorem 2.8 remain valid even if E_1, E_2 are subsets of two different groups G_1, G_2.

In view of Lemma 2.7 we ask the following question:

Question: Is $E_1 \times \Delta(E_2) \cup \Delta(E_1) \times E_2 = \Delta(E_1 \times E_2)$?

3. The n-difference spectrum and weak synthesis. The concept of difference spectrum is generalized here to define n-difference spectrum, which is then applied to the study of weak spectral synthesis defined in [11] and studied also in [4].

DEFINITION 3.1. Let n be a positive integer. For a closed subset E of G we define

$$\Delta_n(E) = \bigcup_{f \in I(E)} \{x : f^n \notin_x J(E)\}$$

and we call it the n-difference spectrum of E.

REMARKS. (i) Weak synthesis can also be studied using the sets

$$\tilde{\Delta}_n(E) = \{x \in G : \text{there is an } f \in I^n(E) \text{ with } f \notin_x J(E)\}.$$

(Here I^n denotes the closed ideal generated by $\{f_1 \cdots f_n : f_j \in I\}$.) Then $\tilde{\Delta}_1(E) = \Delta_1(E) = \Delta(E)$ and $\tilde{\Delta}_n(E) = \varnothing$ if and only if $\Delta_n(E) = \varnothing$. Thus Δ_n and $\tilde{\Delta}_n$ give rise to the same notion of weak synthesis. (ii) Since the difference spectrum of two closed ideals is sequentially closed ([7]), $\tilde{\Delta}_n(E)$ is sequencially closed. Salinger and Stegeman ([7]) showed that every nonmetrizable G contains a closed set E for which $\Delta(E)$ is not closed. Adapting their proof, we have proved the following result (the details can be found elsewhere): *Every nonmetrizable locally compact abelian group G contains a closed set E for which $\tilde{\Delta}_n(E)$ is nonclosed for each n.*

DEFINITION 3.2. We say that a closed set E in G is a set of weak synthesis (or a weak S-set) of characteristic n if $\Delta_n(E) = \varnothing$ and $\Delta_k(E) \neq \varnothing$ for $k < n$.

We have the following inclusions for the n-difference spectra of unions and intersections.

LEMMA 3.3. (a) *Let E_1, E_2 be closed sets in G. Then*

(i) $\Delta_n(E_1 \cap E_2) \subset \Delta_n(E_1) \cup \Delta_n(E_2) \cup E^{12}$,
(ii) $\Delta_n(E_1 \cup E_2) \subset \Delta_n(E_1) \cup \Delta_n(E_2) \cup E_{12}$,
(iii) $\Delta_n(E_1) \cup \Delta_n(E_2) \subset \Delta_n(E_1 \cup E_2) \cup (E_1 \cap E_2)$,
(iv) $E_1 \times \Delta_n(E_2) \cup \Delta_n(E_1) \times E_2 \subset \Delta_n(E_1 \times E_2)$.

(b) *Suppose* $\{E_i\}$ *is a collection of mutually disjoint closed sets in G such that, for each j,* $\bigcup_{i \neq j} E_i$ *is closed. Then* $\Delta_n(\bigcup E_i) = \bigcup \Delta_n(E_i)$.

PROOF. Lemma 2.1 holds with Δ_n in place of Δ, so the proofs of (i) and (ii) are the same as for Lemmas 2.5 and 2.2, respectively. Similarly, the proofs of the case $n=1$ of (iii) and (b) given in [2] and that of (iv) given in Lemma 2.7 carry over for any n.

EXAMPLE 3.4. (i) S^n in $\boldsymbol{R}^{n+1}$ has characteristic $k=[n/2]+1$ for $n \geq 2$ (Varopoulos [10]). If $m<k$, then $\Delta_m(S^n)=S^n$. (ii) If E is the union of S^n and a line segment, then $\Delta_m(E)=\Delta_m(S^n)$ for any m. The other examples discussed in 1.1 can also be treated easily.

COROLLARY 3.5 (Warner). *There is a pairwise disjoint sequence* $\{E_k\}$ *of weak S-sets in* $\boldsymbol{T}^\infty$ *whose union is closed but is not of weak synthesis.*

PROOF. $\boldsymbol{T}^\infty$ contains mutually disjoint weak S-sets E_k of characteristic k such that $\bigcup_{j \neq k} E_j$ is closed for each k (see proof of Theorem 2.6 of [11]). We can now apply Lemma 3.3(b).

To deduce results on weak synthesis from Lemma 3.3, we need a Stegeman-type result for weak synthesis. To prove such a result, we make use of the following simple lemma proved by Stegeman [9].

LEMMA 3.6 (Stegeman). *Let* $f \in A(G)$, *I a closed ideal in* $A(G)$ *and E a closed set in G. If* $f \in_x I$ *for every* $x \notin E$, *then* $f\overline{j(E)} \subset I$.

THEOREM 3.7. *Let E be a closed subset of G. If there is a weak S-set W with characteristic n such that* $\Delta_m(E) \subset W \subset E$, *then E is of weak synthesis with characteristic* $\leq m+n$.

PROOF. Let $f \in I(E)$. If $x \notin W$, then $x \notin \Delta_m(E)$ and so $f^m \in_x J(E)$. Thus, by Lemma 3.6, $f^m\overline{j(W)} \subset J(E)$. Now $I(E) \subset I(W)$ as $W \subset E$, so $f \in I(W)$. Hence $f^n \in J(W) = \overline{j(W)}$ and $f^{m+n} \in f^m\overline{j(W)} \subset J(E)$. The theorem is proved. (Note that if W is a C-set, the proof shows that E is of characteristic $\leq m$).

We can now give some consequences of Lemma 3.3 and Theorem 3.7. Some of these are new results, some others are improvements of known ones and the rest give simplified proofs of existing results.

COROLLARY 3.8. *If* $\overline{\Delta_m(E)}$ *is of weak synthesis* (*of characteristic n*), *then E is of weak synthesis* (*of characteristic* $\leq m+n$).

One of the problems posed by Stegeman [9, Problem (iv)] is whether $\overline{\Delta(E)}$ is of synthesis implies E is of synthesis. This problem appears to be rather difficult. Corollary

3.8 gives an easy answer to the corresponding problem for weak synthesis. The next corollary gives a partial answer to Stegeman's question (and also to the more general question asking for the actual value of the characteristic of E in Corollary 3.8).

COROLLARY 3.9. *Suppose E is a Helson set. If $\overline{\Delta(E)}$ (or even $\overline{\Delta_m(E)}$, for some m) is of synthesis, then E is of synthesis.*

PROOF. This is a consequence of Corollary 3.8 and the easily proved fact (see [12]) that a Helson set of weak synthesis is automatically a set of synthesis.

COROLLARY 3.10. *If ∂E is a weak S-set (of characteristic n), then E is a weak S-set (of characteristic $\leq n+1$).*

COROLLARY 3.11. *If E_1, E_2 are weak S-sets of characteristic m, n respectively, then $E_1 \cup E_2$ is a weak S-set of characteristic $\leq m+n$.*

These two corollaries are due to Warner [11]. The next one is an improvement of a result of Warner [11] who got $mn+m$ in place of $m+n$. His methods are entirely different.

COROLLARY 3.12. *Let E_1, E_2 be closed sets in G. Suppose $E_1 \cup E_2$ and $E_1 \cap E_2$ are weak S-sets of characteristic m and n respectively. Then E_1, E_2 are weak S-sets of characteristic $\leq m+n$.*

It is not known whether the intersection of two weak S-sets is a weak S-set. Here is a sufficient condition for this to hold.

COROLLARY 3.13. *If E_1, E_2 are weak S-sets in G and if there is a weak S-set W such that $\partial E_1 \cap \partial E_2 \subset W \subset E_1 \cap E_2$ then $E_1 \cap E_2$ is a weak S-set.*

Our last result in this list of corollaries is due to Parthasarathy and Varma [4].

COROLLARY 3.14. *Let E_1, E_2 be nonempty closed sets in G. If $E_1 \times E_2$ is a weak S-set of characteristic n in $G \times G$, then E_1, E_2 are weak S-sets with characteristic $\leq n$.*

REMARKS 3.15. (i) Warner's union result (Corollary 3.11) can be given a direct one line proof: $f \in I(E_1 \cup E_2) \Rightarrow f^m =_x g \in J(E_1)$ and $f^n =_x h \in J(E_2)$ for all $x \Rightarrow f^{m+n} =_x gh \in J(E_1)J(E_2) \subset J(E_1 \cup E_2)$ for all x.

(ii) It follows from Corollary 3.12, for example, that a spherical cap is of weak synthesis in $\boldsymbol{R}^{n+1}$.

Finally, we have a look at the difference spectrum in restriction algebras. For closed sets $E \subset F \subset G$, let $J_F(E)$, $I_F(E)$ and $\Delta_n^F(E)$ denote the analogues of $J(E)$, $I(E)$ and $\Delta_n(E)$ in the restriction algebra $A(F)$.

LEMMA 3.16. (i) *If $E \subset F_1 \subset F_2 \subset G$ are closed sets, then $\Delta_n^{F_1}(E) \subset \Delta_n^{F_2}(E)$.*

(ii) *If E, E_1, E_2 are closed sets in G with $E \subset E_1 \cup E_2$, then*

$$\Delta_n^{E_1 \cup E_2}(E) \subset \Delta_n^{E_1}(E \cap E_1) \cup \Delta_n^{E_2}(E \cap E_2) \cup (E_1 \cap E_2).$$

PROOF. We omit the simple proof of (i). To prove (ii), let $x \in \Delta_n^{E_1 \cup E_2}(E)$. Suppose $x \notin E_1 \cap E_2$. Then $x \in E \cap E_1$ or $x \in E \cap E_2$, but not both. Assume $x \in E \cap E_1$. Choose a neighbourhood V of x such that $\bar{V} \cap E_1 \cap E_2 = \varnothing$ and $\bar{V} \cap E \cap E_2 = \varnothing$. There is an $f \in I_{E_1 \cup E_2}(E)$ with $f^n \notin_x J_{E_1 \cup E_2}(E)$. If $x \notin \Delta_n^{E_1}(E \cap E_1)$, then $f^n|_{E_1} =_x h \in J_{E_1}(E \cap E_1)$. Choose $k \in A(G)$ such that $k = 1$ near x and $\operatorname{supp} k \subset V$. If $g \in A(G)$ satisfies $h = g|_{E_1}$, then $(kg)|_{E_1 \cup E_2} \in J_{E_1 \cup E_2}(E)$. So $f^n =_x g|_{E_1 \cup E_2} = (kg)|_{E_1 \cup E_2} \in J_{E_1 \cup E_2}(E)$, a contradiction. Thus $x \in \Delta_n^{E_1}(E \cap E_1)$. In the same way we get $x \in \Delta_n^{E_2}(E \cap E_2)$ if $x \in E \cap E_2$.

We need a definition to state our next theorem.

DEFINITION 3.17. A closed set E is called a set of weak spectral resolution if every closed subset of E is a weak S-set.

THEOREM 3.18. (i) *If weak synthesis fails for $A(F_1)$, then it fails for $A(F_2)$. More precisely, if $E \subset F_1 \subset F_2 \subset G$ and if E has characteristic n in $A(F_2)$, then in $A(F_1)$ it has characteristic $\leq n$.*

(ii) *Let E_1, E_2 be closed, non-empty sets in G. Assume that $E_1 \cap E_2$ is a set of weak spectral resolution for $A(E_1 \cup E_2)$. Then weak synthesis holds for $A(E_1 \cup E_2)$ if and only if it holds for both $A(E_1)$ and (E_2).*

PROOF. (i) is immediate from (i) of Lemma 3.16, while (ii) follows from Lemma 3.16 and the analogue of Theorem 3.7 for $A(E_1 \cup E_2)$. (Note that the assumption that $E_1 \cap E_2$ is a set of weak spectral resolution is needed in one implication only.)

REMARKS 3.19. The qualitative and the quantitative versions of Theorem 3.18(i) coincide for $n=1$. When $F_2 = G$, this is [1, 39.19]. Following the proof given there (which is entirely different from the present one), the qualitative part of (i) has been obtained in [4].

We are indebted to J. D. Stegeman for suggesting the use of local equality of sets in the proofs of Lemmas 2.2 and 2.5 in place of our earlier arguments.

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