

Differences of composition operators between weighted Banach spaces of holomorphic functions

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Abstract

We consider differences of composition operators between given weighted Banach spaces H_v^∞ or H_v^0 of analytic functions with weighted sup-norms and give estimates for the distance of these differences to the space of compact operators. We also study boundedness and compactness of the operators. Some examples illustrate our results.

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Introduction

Let v and w be strictly positive bounded continuous functions (*weights*) on the open unit disk D in the complex plane. In this note we are interested in operators defined on Banach spaces of analytic functions of the following form:

$$\begin{aligned} H_v^\infty &:= \{f \in H(D); \|f\|_v = \sup_{z \in D} v(z)|f(z)| < \infty\}, \\ H_v^0 &:= \{f \in H(D); \lim_{|z| \rightarrow 1^-} v(z)|f(z)| = 0\}, \end{aligned}$$

endowed with the norm $\|\cdot\|_v$. Here $H(D)$ denotes the space of all analytic functions. These spaces appear in the study of growth conditions of analytic functions and have been studied in various articles, see e.g. [22], [23], [16], [1], [17], [18], [2].

Let $\phi, \psi : D \rightarrow D$ be analytic mappings. Each such map induces through composition a linear composition operator $C_\phi(f) = f \circ \phi$ resp. $C_\psi(f) = f \circ \psi$ between spaces of holomorphic functions of the type defined above. We will consider differences of composition operators $(C_\phi - C_\psi)(f) = f \circ \phi - f \circ \psi$ acting on these spaces of holomorphic functions.

Composition operators have been studied on various spaces of analytic functions. We refer the reader to the excellent monographs [7] and [21], and the article [15]. The case of operators defined on weighted Banach spaces of the type defined above was treated e.g. in [5], [4] and [6]. Differences of composition operators have been investigated more recently; see [19], [13], [14], [10] and [20]. In this article we are mainly interested in finding an expression for the essential norm $\|C_\phi - C_\psi\|_e$, i.e. the distance of $C_\phi - C_\psi$ to the space of compact operators, when $C_\phi - C_\psi$ is a bounded operator from H_v^∞ into H_w^∞ ; compare with [10] and [12] for the case of H^∞ . It is known that if $\|\varphi\|_\infty < 1$, then C_φ is a compact operator from H_v^∞ into H_w^∞ . Therefore we are interested in the case $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$. In our investigation we also study boundedness and compactness of $C_\phi - C_\psi$. It turns out that we obtain similar conditions to those obtained in [5] and [4], at least when the weight v is radial and satisfies certain natural conditions; see the details below.

Notations and definitions

We refer the reader to [7], [9], [11] and [21] for notation on composition operators and spaces of analytic functions on the unit disc. The compact open topology on the space $H(D)$ will be

denoted by co . The closed unit ball of H_v^∞ resp. H_v^0 is denoted by B_v^∞ resp. B_v^0 . The formulation of many results on weighted spaces of analytic functions and on operators between them requires the so-called *associated weights* (see [3]). For a weight v the associated weight \tilde{v} is defined as follows

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|; f \in H_v^\infty, \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty}}, \quad z \in D,$$

where δ_z denotes the point evaluation of z . The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [3]). Furthermore, for each $z \in D$ there is $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. A weight is called *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in D.$$

For examples of essential weights and conditions when weights are essential see [3], [5] and [4]. Especially interesting are *radial weights* v , i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Every radial weight which is non-increasing with respect to $|z|$ and such that $\lim_{|z| \rightarrow 1} v(z) = 0$ is called a *typical weight*. If the weight v is typical, then the unit ball B_v^∞ coincides with the closure of B_v^0 for the compact open topology. *In the sequel every radial weight is assumed to be non-increasing.*

In order to handle differences of composition operators we need the so called *pseudohyperbolic metric*. Recall that for any $z \in D$, φ_z is the Möbius transformation of D which interchanges the origin and z , namely,

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in D.$$

The pseudohyperbolic distance $\rho(z, w)$ for $z, w \in D$ is defined by $\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \bar{z}w} \right|$. We refer the reader to [9] for more details. According to [8] we define $\rho_v(z, p) := \sup\{|f(z)|\tilde{v}(z); f \in B_v^\infty, f(p) = 0\}$. Note that for any $z, p \in D$,

$$\rho(z, p) \leq \rho_v(z, p).$$

Indeed, let $f(p) = 0$, $f \in H^\infty$, $\|f\|_\infty \leq 1$. For each $z \in D$ there is $g_z \in H_v^\infty$, $\|g_z\|_v \leq 1$, such that $|g_z(z)|\tilde{v}(z) = 1$. Hence $|f(z)| = |f(z)g_z(z)|\tilde{v}(z) \leq \rho_v(z, p)$.

In case v is a radial weight such that the following condition (which is due to Lusky [17]) holds

$$(L1) \quad \inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0,$$

then it is proved in [8] that ρ is equivalent to ρ_v . Several conditions equivalent to (L1) can be seen in [8]. In particular it is equivalent to a condition considered in [23].

An operator $T \in L(E, F)$ from the Banach space E to the Banach space F is called *compact* if it maps the closed unit ball of E onto a relatively compact set in F . We recall that operators $T : E \rightarrow F$ which take weakly null sequences in E to norm null sequences in F are said to be *completely continuous*. The essential norm of a continuous linear operator T is defined by $\|T\|_e := \inf\{\|T - K\| : K \text{ is compact}\}$. Since $\|T\|_e = 0$ if and only if T is compact, the estimates on $\|T\|_e$ lead to conditions for T to be compact.

Results

We start with an auxiliary result.

Lemma 1 *Let v be a radial weight satisfying condition (L1) and let $f \in H_v^\infty$. Then there exists a constant C_v (depending only on the weight v) such that*

$$|f(z) - f(p)| \leq C_v \|f\|_v \max \left\{ \frac{\rho(z, p)}{v(z)}, \frac{\rho(z, p)}{v(p)} \right\}$$

for all $z, p \in D$.

Proof. By Lemma 1 (a) in [8], there are $0 < s < 1$ and constant $0 < C < \infty$ such that $v(z)/v(p) \leq C$ for all $z, p \in D$ with $\rho(z, p) \leq s$. Hence it follows by Lemma 14 in [8] that

$$|f(z) - f(p)|v(z) \leq \frac{4C}{s} \|f\|_v \rho(z, p)$$

for all $z, p \in D$ with $\rho(z, p) \leq s/2$. If $\rho(z, p) > s/2$, then

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq 2\|f\|_v \leq \frac{4\|f\|_v}{s} \rho(z, p).$$

Therefore we conclude

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq C_v \|f\|_v \rho(z, p)$$

for all $z, p \in D$, from which the assertion follows. \square

Now we characterize bounded operators $C_\phi - C_\psi$. Recall that not every composition operator C_φ is bounded on H_v^∞ ; see [5].

Proposition 2 *Let v and w be weights. If $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, then*

$$\max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} < \infty.$$

If v also is radial and satisfies condition (L1), then

$$\max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} < \infty$$

implies the boundedness of $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$.

Proof. Assume that $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Hence we obtain

$$\begin{aligned} \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho_v(\phi(z), \psi(z)) \leq \\ &\leq \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \tilde{v}(\phi(z)) \sup\{|f(\phi(z)) - f(\psi(z))|; f \in B_v^\infty\} = \|C_\phi - C_\psi\| < \infty. \end{aligned}$$

Similarly, $\sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) < \infty$.

For the converse implication we first notice that v is essential by Proposition 2 (b) in [8]. Now we apply Lemma 1, so

$$\begin{aligned} \|C_\phi - C_\psi\| &= \sup_{z \in D} w(z) \sup\{|f(\phi(z)) - f(\psi(z))|; f \in B_v^\infty\} \\ &\leq \sup_{z \in D} w(z) C_v \max \left\{ \frac{\rho(\phi(z), \psi(z))}{v(\phi(z))}, \frac{\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\} < \infty, \end{aligned}$$

and $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded. \square

Since $C_\phi - C_\psi : (H(D), co) \rightarrow (H(D), co)$ is continuous, we immediately get the following result:

Proposition 3 *Let v be a weight such that $\overline{B_v^{co}} = B_v^\infty$. If $C_\phi - C_\psi : H_v^0 \rightarrow H_w^0$ is bounded, then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded.*

Example 4 We give an example of non-bounded composition operators such that their difference is bounded.

Choose $w(z) = 1$ and $v(z) = 1 - |z| = \tilde{v}(z)$ which are radial weights on D . Obviously, v satisfies condition (L1). Moreover, select, $\phi(z) = \frac{z+1}{2}$ and $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, $z \in D$, such that t is real and $|t|$ so small that ψ maps D into D . By [5] Proposition 2.1, $C_\phi : H_v^\infty \rightarrow H_w^\infty$ is not bounded, because for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = \frac{2}{1-r} \rightarrow \infty$ if $r \rightarrow 1$. The fact that $C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded follows in an analogous way: For $z = r \in \mathbb{R}$ we obtain $\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1}{1 - \frac{r+1}{2} - t(r-1)^3} \rightarrow \infty$ if $r \rightarrow 1$. By [19] Example 1 we know $\rho(\phi(z), \psi(z)) \leq \frac{|t|}{\delta} |z - 1|$, where δ is a constant. This yields

$$\begin{aligned} \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \frac{1}{1 - |\frac{z+1}{2}|} \frac{|t|}{\delta} |z - 1| < \infty \text{ and} \\ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) &\leq \sup_{z \in D} \frac{1}{1 - |\frac{z+1}{2} + t(z-1)^3|} \frac{|t|}{\delta} |z - 1| < \infty \end{aligned}$$

Hence, $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded.

Example 5 We give a non-trivial example of a non-bounded difference of composition operators.

Choose $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, where t is real and $|t|$ is so small that ψ maps D into D . Select now $w(z) = v(z) = e^{-\frac{1}{1-|z|}} = \tilde{v}(z)$, which are radial weights not satisfying (L1). By [5] Proposition 2.1, $C_\phi : H_v^\infty \rightarrow H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = e^{-\frac{1}{1-r} + \frac{1}{1-\frac{r+1}{2}}} = e^{-\frac{1}{1-r} + \frac{2}{1+r}} = e^{\frac{1}{1-r}} \rightarrow \infty$ if $r \rightarrow 1$. Analogously $C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\psi(r))} = e^{-\frac{1}{1-r} + \frac{1}{1 - \frac{r+1}{2} - t(r-1)^3}} = e^{-\frac{1}{1-r} + \frac{2}{1-r-2t(r-1)^3}} \rightarrow \infty$ if $r \rightarrow 1$. By [19] Example 1 we know $\rho(\phi(z), \psi(z)) \leq \frac{|t|}{\delta} |z - 1|$, where $\delta > 0$ is a constant. Since $|\phi(z)| \rightarrow 1$ or $|\psi(z)| \rightarrow 1$ is equivalent to $z \rightarrow 1$ and $\lim_{z \rightarrow 1} \frac{|t|}{\delta} |z - 1| = 0$, we get

$$\lim_{|\phi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = 0.$$

Now, for $z = r \in \mathbb{R}$, we have

$$\begin{aligned} \frac{w(r)}{\tilde{v}(\phi(r))} \rho(\phi(r), \psi(r)) &= e^{\frac{1}{1-r}} \left| \frac{t(r-1)^3}{1 - (\frac{r+1}{2})(\frac{r+1}{2} + t(r-1)^3)} \right| \\ &= e^{\frac{1}{1-r}} |t| \left| \frac{(r-1)^3}{1 - (\frac{(r+1)^2}{4}) - (\frac{r+1}{2}t(r-1)^3)} \right| \end{aligned}$$

and $\frac{w(r)}{\tilde{v}(\phi(r))} \rho(\phi(r), \psi(r)) \rightarrow \infty$ for $r \rightarrow 1$. Hence $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded.

The proof of our next result exploits a method presented in [4].

Theorem 6 Let v and w be radial weights such that v is typical and satisfies condition (L1). There is a constant $C_v > 0$ such that, if $\phi, \psi : D \rightarrow D$ are analytic maps such that $\max\{||\phi||_\infty, ||\psi||_\infty\} = 1$ and $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, then

$$\begin{aligned} \max \left\{ \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} &\leq ||C_\phi - C_\psi||_e \\ &\leq C_v \max \left\{ \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\}. \end{aligned}$$

Proof. We first prove the lower estimate of the essential norm by contradiction. Assume we can find $b > c > d > 0$, a compact operator $K : H_v^\infty \rightarrow H_w^\infty$ and a sequence $(z_n) \in D$ with $|\phi(z_n)| \rightarrow 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\phi(z_n), \psi(z_n)) \geq b > c > d > \|C_\phi - C_\psi - K\| \quad \text{for all } n.$$

Now we select an increasing sequence $(\alpha(n))_n$ of natural numbers going to infinity such that $|\phi(z_n)|^{\alpha(n)} \geq c/b$ for all n . Since v is typical, it follows that for every n we can find $f_n \in B_v^0$ such that $|f_n(\phi(z_n))| \geq \frac{1}{\tilde{v}(\phi(z_n))} \frac{d}{c}$.

Set $h_n(z) := z^{\alpha(n)} \varphi_{\psi(z_n)}(z) f_n(z)$. Thus, $h_n \in H_v^0$ with $\|h_n\|_v \leq 1$. Moreover (h_n) converges to zero in the compact open topology, and consequently $h_n \rightarrow 0$ weakly in H_v^0 ; see e.g. [25]. Since the operator K is compact, $\lim_{n \rightarrow \infty} \|Kh_n\|_w = 0$. Thus, for each n ,

$$c > \|C_\phi - C_\psi - K\| \geq \|(C_\phi - C_\psi)h_n\|_w - \|Kh_n\|_w,$$

and we conclude that

$$\begin{aligned} d > \|C_\phi - C_\psi - K\| &\geq \limsup_n \|(C_\phi - C_\psi)h_n\|_w = \limsup_n \|h_n \circ \phi - h_n \circ \psi\|_w \geq \\ &\geq \limsup_n w(z_n) |h_n(\phi(z_n)) - h_n(\psi(z_n))| = \\ &= \limsup_n w(z_n) |\phi(z_n)|^{\alpha(n)} |\varphi_{\psi(z_n)}(\phi(z_n)) f_n(\phi(z_n))| \geq \\ &\geq \frac{d}{c} \limsup_n \frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\psi(z_n), \phi(z_n)) |\phi(z_n)|^{\alpha(n)} \geq c \frac{d}{c}, \end{aligned}$$

which is a contradiction.

We now prove the upper estimate. Take the sequence of linear operators $C_k : H(D) \rightarrow H(D)$, $k \in \mathbb{N}$, defined by $C_k f(z) = f(\frac{k}{k+1}z)$, which are continuous for the compact open topology and $C_k f \rightarrow f$ uniformly on every compact subset of D . Moreover, the operators $C_k : H_v^\infty \rightarrow H_v^\infty$ are well-defined and compact with $\|C_k\| \leq 1$.

For fixed $k \in \mathbb{N}$ we have,

$$\|C_\phi - C_\psi\|_e \leq \|C_\phi - C_\psi - (C_\phi - C_\psi)C_k\| = \|(C_\phi - C_\psi)(Id - C_k)\|.$$

Let $f \in H_v^\infty$ with $\|f\|_v \leq 1$ and fix an arbitrary $r \in (0, 1)$. Set $g_k := (Id - C_k)f$. Then $g_k \in H_v^\infty$ and $\|g_k\|_v \leq 2$. Hence

$$\begin{aligned} &\|(C_\phi - C_\psi)g_k\|_w \leq \\ &\leq \sup_{\{z: |\phi(z)| \leq r \text{ and } |\psi(z)| \leq r\}} |g_k(\phi(z)) - g_k(\psi(z))| w(z) + \\ &+ \sup_{\{z: |\phi(z)| > r \text{ or } |\psi(z)| > r\}} |g_k(\phi(z)) - g_k(\psi(z))| w(z) \leq \\ &\leq \sup_{\{z: |\phi(z)| \leq r\}} |g_k(\phi(z))| w(z) + \sup_{\{z: |\psi(z)| \leq r\}} |g_k(\psi(z))| w(z) \\ &+ \sup_{\{z: |\phi(z)| > r \text{ or } |\psi(z)| > r\}} |g_k(\phi(z)) - g_k(\psi(z))| w(z). \end{aligned}$$

The sequence of operators $(Id - C_k)_k$ satisfies $\lim_k (Id - C_k)g = 0$ for each g in $H(D)$, and the space $H(D)$ endowed with the compact open topology co is a Fréchet space. By the Banach-Steinhaus theorem, $(Id - C_k)_k$ converges to zero uniformly on the compact subsets of $(H(D), co)$. Since the closed unit ball of H_v^∞ is a compact subset of $(H(D), co)$ we obtain that

$$\lim_k \sup_{\|f\|_v \leq 1} \sup_{|\xi| \leq r} |(Id - C_k)f(\xi)| = 0.$$

By Lemma 1,

$$|f(\phi(z)) - f(\psi(z))|w(z) \leq C_v \max \left\{ \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\}$$

for all $z \in D$ and $f \in H_v^\infty$, $\|f\|_v \leq 1$. Since v is non-increasing we conclude from this

$$\begin{aligned} & \lim_k \|(C_\phi - C_\psi)(Id - C_k)\| \leq \\ & \leq 2C_v \max \left\{ \sup_{\{z: |\phi(z)| > r\}} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \sup_{\{z: |\psi(z)| > r\}} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|C_\phi - C_\psi\|_e \leq \\ & 2C_v \max \left\{ \lim_{r \rightarrow 1} \sup_{\{z: |\phi(z)| > r\}} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\phi(z))}, \lim_{r \rightarrow 1} \sup_{\{z: |\psi(z)| > r\}} \frac{w(z)\rho(\phi(z), \psi(z))}{v(\psi(z))} \right\}. \end{aligned}$$

Since every radial weight with condition (L1) is essential (see Prop. 2 in [8]), we are done. \square

Corollary 7 *Let v and w be radial weights such that v is typical and satisfies condition (L1). Then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is compact if and only if*

$$\limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) = 0.$$

Proof. If $C_\phi - C_\psi$ is compact, then the conditions are satisfied by Theorem 6. Conversely, Theorem 6 implies the compactness of $C_\phi - C_\psi$ as soon as we know that $C_\phi - C_\psi$ is bounded. But by assumption we can choose $r < 1$ such that

$$\max \left\{ \sup_{|\phi(z)| > r} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{|\psi(z)| > r} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \leq 1.$$

Hence the boundedness follows from

$$\begin{aligned} & \max \left\{ \sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \sup_{z \in D} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \\ & \leq \max \left\{ 1, \sup_{z \in D} \frac{w(z)}{\tilde{v}(r)} \right\}. \end{aligned}$$

\square

Corollary 7 and the proof of the lower estimate in Theorem 6 permit us to obtain the following consequence.

Corollary 8 *Let v and w be radial weights such that v is typical and satisfies condition (L1). Then $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is completely continuous if and only if $C_\phi - C_\psi$ is compact.*

Theorem 9 *Let v and w be typical weights such that v satisfies condition (L1). There is a constant $C_v > 0$ such that, if $\phi, \psi : D \rightarrow D$ are analytic maps such that $\max\{\|\phi\|_\infty, \|\psi\|_\infty\} = 1$ and $C_\phi - C_\psi : H_v^0 \rightarrow H_w^0$ is bounded, then*

$$\begin{aligned} & \max \left\{ \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|z| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\} \leq \|C_\phi - C_\psi\|_e \\ & \leq C_v \max \left\{ \limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)), \limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) \right\}. \end{aligned}$$

Proof of Theorem 9. The difference with the proof of the lower bound of Theorem 6 is that now we get $b > c > d > 0$, a compact operator $K : H_v^0 \rightarrow H_w^0$ and a sequence $(z_n) \in D$ with $|z_n| \rightarrow 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))} \rho(\phi(z_n), \psi(z_n)) \geq b > c > d > \|C_\phi - C_\psi - K\| \quad \text{for all } n.$$

We can assume that $\phi(z_n) \rightarrow z_0$ for some z_0 with $|z_0| \leq 1$. If $|z_0| \neq 1$, then $0 = \lim_n w(z_n) \geq b v(z_0) > 0$, which is a contradiction. Therefore $|\phi(z_n)| \rightarrow 1$ and we can continue as in the proof of Theorem 6. Notice also that in the proof of the upper bound the operators $C_k : H_v^0 \rightarrow H_w^0$ are well-defined since v is typical.

Example 10 We select $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, where the real number t is so small that ψ is a self-map on D . Moreover we choose $w(z) = 1 - |z|$ and $v(z) = (1 - |z|)^3 = \tilde{v}(z)$. Now $C_\phi, C_\psi : H_v^\infty \rightarrow H_w^\infty$ are not bounded since for $r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = \frac{8}{(1-r)^2} \rightarrow \infty$ and $\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1-r}{(1-\frac{r+1}{2}+t(r-1)^3)^3} \rightarrow \infty$ if $r \rightarrow 1$. It follows from Proposition 2 that the operator $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded (see Example 4). But it is not compact, since

$$\frac{w(r)}{\tilde{v}(\phi(r))} \rho(\phi(r), \psi(r)) = \frac{8}{(1-r)^2} \left| \frac{-t(r-1)^3}{1 - \frac{r+1}{2}(\frac{r+1}{2} + t(r-1)^3)} \right| \rightarrow 8|t| \quad \text{if } r \rightarrow 1.$$

For examples of compact and non-compact differences of composition operators $C_\phi - C_\psi : H^\infty \rightarrow H^\infty$, see [19] Example 1. The change of the behaviour of the operator $C_\phi - C_\psi$ depending on the weights v and w is emphasized in our last example.

Example 11 We consider $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z-1}{2}$, $z \in D$, which are both analytic self maps of the unit disk. By definition we obtain $\rho(\phi(z), \psi(z)) = \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right|$. Hence $\lim_{|\phi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \rightarrow 1} \rho(\phi(z), \psi(z)) = 1$.

- (a) Select $w(z) = 1 - |z| = v(z) = \tilde{v}(z)$. Obviously v is typical and satisfies (L1). By Theorem 6 we get

$$\limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\phi(z)| \rightarrow 1} \frac{1 - |z|}{1 - |\frac{z+1}{2}|} \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right| = 1$$

and

$$\limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \rightarrow 1} \frac{1 - |z|}{1 - |\frac{z-1}{2}|} \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right| = 1$$

Hence

$$1 \leq \|C_\phi - C_\psi\|_e \leq C_v.$$

We conclude that $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is bounded, but not compact.

- (b) Choose $w(z) = 1$ and $v(z) = 1 - |z|$. We get

$$\sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \sup_{z \in D} \frac{1}{1 - |\frac{z+1}{2}|} \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right| = \infty.$$

Hence $C_\phi - C_\psi : H_v^\infty \rightarrow H_w^\infty$ is not bounded.

(c) Consider $w(z) = 1 - |z|$, $v(z) = 1$ to obtain

$$\limsup_{|\phi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\phi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\phi(z)| \rightarrow 1} (1 - |z|) \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right| = 0$$

and

$$\limsup_{|\psi(z)| \rightarrow 1} \frac{w(z)}{\tilde{v}(\psi(z))} \rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \rightarrow 1} (1 - |z|) \left| \frac{1}{1 - \frac{z+1}{2} \frac{z-1}{2}} \right| = 0.$$

Since the upper estimate in Theorem 6 is valid without the assumption that v is typical, we conclude that $\|C_\phi - C_\psi\|_e = 0$, and the operator is compact.

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