Czechoslovak Mathematical Journal

Ze-Hua Zhou; Yu-Xia Liang

Differences of weighted composition operators from Hardy space to weighted-type spaces on the unit ball

Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 3, 695–708

Persistent URL: http://dml.cz/dmlcz/143020

Czechoslovak Mathematical Journal, 62 (137) (2012), 695–708

DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS FROM HARDY SPACE TO WEIGHTED-TYPE SPACES ON THE UNIT BALL

Ze-Hua Zhou, Yu-Xia Liang, Tianjin

(Received April 21, 2011)

Abstract. In this paper, we limit our analysis to the difference of the weighted composition operators acting from the Hardy space to weighted-type space in the unit ball of $\mathbb{C}^N$, and give some necessary and sufficient conditions for their boundedness or compactness. The results generalize the corresponding results on the single weighted composition operators and on the differences of composition operators, for example, M. Lindström and E. Wolf: Essential norm of the difference of weighted composition operators. Monatsh. Math. 153 (2008), 133–143.

Keywords: weighted composition operator, Hardy space, weighted Bergman space, essential norm, compact, difference

MSC 2010: 47B38, 32A37, 32H02, 47G10, 47B33

1. Introduction

Let N be a fixed positive integer and B_N be the open unit ball of the complex vector space $\mathbb{C}^N$. Denote by $H(B_N)$ the space of all holomorphic functions on B_N and $S(B_N)$ the collection of all the holomorphic self-maps of B_N. Let $\mathrm{d}\sigma$ be the normalized Lebesgue measure on the boundary ∂B_N of B_N. For $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in $\mathbb{C}^N$, the inner product of z and w is denoted by

$$\langle z, w\rangle = z_1\overline{w}_1 + \ldots + z_N\overline{w}_N,$$

and we write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \ldots + |z_N|^2}$.

Supported in part by the National Natural Science Foundation of China (Grant Nos. 10971153, 10671141).

For $\varphi \in S(B_N), u \in H(B_N)$, we define the weighted composition operator $W_{\varphi,u}$ by

$$W_{\varphi,u}(f) = u \cdot (f \circ \varphi)$$

for $f \in H(B_N)$. For $u \equiv 1$, the weighted composition operator $W_{\varphi,1}$ is the usual composition operator, denoted by C_φ. When φ is the identity mapping I, the operator $W_{I,u}$ is also called the multiplication operator.

Despite the simplicity of the definitions of $W_{\varphi,u}$ and C_φ, there are uncommon problems containing this type of operators, which require profound and interesting analytical machinery; moreover, the study of composition operators has arguably become a major driving force in the development of modern complex analysis. Readers interested in the development of the theory of composition operators can refer to the excellent books [5], [18], as well as some recent papers [1], [17], [24], [25], [23] and the related references therein, which have been expended on characterizing those holomorphic maps which induce bounded or compact (weighted) composition operators acting on different spaces of holomorphic functions.

Recently, there have been an increasing interest in studying the compact difference of composition operators acting on different spaces of holomorphic functions. In 2005, Moorhouse [16] answered the question of compact difference for composition operators acting on the standard weighted Bergman spaces and necessary conditions were given on a larger scale of weighted Dirichlet spaces. Most papers in this area have focused on the classical reflexive spaces, however, some classical non-reflexive spaces have also been discussed lately in the unit disc D in the complex plane. Hosokawa and Ohno [11] in 2006, and [12] in 2007, discussed the topological structures of the sets of composition operators and gave a characterization of compact difference on Bloch space in the unit disc, see also Yang and Zhou's paper in [22]. In 2008, Fang and Zhou [8] also gave a characterization of compact difference between Bloch space and the set of all bounded analytic functions on the unit polydisc. In 2001, MacCluer and co-workers [15] used the pseudo-hyperbolic metric to discuss the topological components of the set of composition operators acting on $H^\infty(D)$. They provided a geometric condition when two composition operators with non-compact difference lie in the same component. In 2005, Hosokawa and co-workers [10] continued this investigation. They studied properties of the topological space of weighted composition operators on the space of bounded analytic functions on the open unit disk in the uniform operator topology. These results were extended to the setting of $H^\infty(B_N)$ by Toews [20] in 2004, and independently by Gorkin and co-workers [9] in 2003, and the setting of $H^\infty(D^N)$ by Fang and Zhou [7] in 2008, where B_N is the unit ball, D^N is the unit polydisk. In 2008, Bonet and co-workers [4] discussed the same problem for the composition operator on the weighted Banach spaces of holomorphic functions in the unit disk, which was also extended to the unit polydisc by Wolf in [21] in 2008. The case of weighted composition operators on the above spaces was treated by Lindström and Wolf in [13] in 2008.

For the purpose of this paper, we limit our analysis to the difference of the weighted composition operators from the Hardy space to weighted-type space in the unit ball. The paper is organized as follows. Definitions, along with some necessary background material and lemmas, follow in Section 2. Sections 3 and 4 are devoted to characterizing the boundedness and compactness of differences of weighted composition operators from Hardy space to weighted-type spaces on the unit ball of $\mathbb{C}^N$. The results generalize the corresponding results on the single weighted composition operators and on the differences of composition operators.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leqslant A \leqslant CB$.

2. Background and some lemmas

Recall that the classical Hardy space, denoted $H^p = H^p(B_N)$, $0 < p < \infty$, is the space of all $f \in H(B_N)$, satisfying the norm condition (see, e.g. [26])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial B_N} |f(r\zeta)|^p \, \mathrm{d}\sigma(\zeta) < \infty.$$

This space is the most well-known and widely studied space of holomorphic functions. When $1 \leqslant p < \infty$, H^p is a Banach space with norm $\|\cdot\|_{H^p}$. If $0 < p < 1$, H^p is a Fréchet space with the metric $\|\cdot\|_{H^p}$.

For $0 < \alpha < \infty$, let H_α^∞ be the weighted space, consisting of all $f \in H(B_N)$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in B_N} (1 - |z|^2)^\alpha |f(z)|$$

is finite. As we all know, H_α^∞ is a Banach space with the norm $\|\cdot\|$.

Let φ be a positive continuous function on B_N (weight). The weighted-type space $H_\varphi^\infty(B_N) = H_\varphi^\infty$ consists of all $f \in H(B_N)$ such that

$$\|f\|_{H_\varphi^\infty} = \sup_{z \in B_N} \varphi(z)|f(z)| < \infty.$$

It is known that H_φ^∞ is a Banach space. For some related results on weighted-type spaces see [2], [3], [4], [14], [19], as well as the related references therein.

For any point $a \in B_N \setminus \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in B_N,$$

where $s_a = \sqrt{1-|a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^N$ onto the one-dimensional subspace $[a]$ generated by a, and $Q_a = I - P_a$ is the projection onto the orthogonal complement of $[a]$, that is

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a; \quad Q_a(z) = z - P_a(z), \quad z \in B_N.$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is well known that each φ_a is a homeomorphism of the closed unit ball $\overline{B_N}$ onto $\overline{B_N}$. The pseudohyperbolic metric on B_N defined by

$$\varrho(a, z) = |\varphi_a(z)|.$$

We know that $\varrho(a, z)$ is invariant under automorphisms (see, e.g. [26]).

For any two points z and w in B_N, let $\gamma(t) = (r_1(t), \ldots, \gamma_N(t))\colon [0,1] \to B_N$ be a smooth curve to connect z and w. Define

$$l(\gamma) = \int_0^1 \sqrt{\langle B(\gamma(t))\gamma'(t), \gamma'(t)\rangle}\, \mathrm{d}t.$$

The infimum of the set consisting of all $l(\gamma)$ is denoted by $\beta(z, w)$, where γ is a smooth curve in B_N from z and w. We call β the Bergman metric (see, e.g. [26]) on B_N. It is known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \varrho(z, w)}{1 - \varrho(z, w)}.$$

Now let us state a couple of lemmas, which are used in the proof of the main results in the next sections.

Lemma 1. *For z and w in B_N,*

$$\frac{1 - \varrho(z, w)}{1 + \varrho(z, w)} \leqslant \frac{1 - |z|^2}{1 - |w|^2} \leqslant \frac{1 + \varrho(z, w)}{1 - \varrho(z, w)}.$$

Proof. The proof of this Lemma can be found in Lemma 3.1 of [6]. □

Lemma 2 (Theorem 4.17, [26]). *Suppose $0 < p < \infty$. If $f \in H^p(B_N)$, then*

$$|f(z)| \leqslant \frac{\|f\|_{H^p}}{(1 - |z|^2)^{n/p}}$$

for all $z \in B_N$.

Lemma 3. *For $1 < p < \infty$, if $f \in H^p$, then*

$$|(1-|z|^2)^{n/p} f(z) - (1-|w|^2)^{n/p} f(w)| \leqslant C\|f\|_{H^p}\varrho(z,w)$$

for all $z, w \in B_N$.

Proof. By Lemma 2 it follows that if $f \in H^p$ then $f \in H^\infty_{(1-|z|^2)^{n/p}}$, and moreover $\|f\|_{H^\infty_{(1-|z|^2)^{n/p}}} \leqslant \|f\|_{H^p}$. By Lemma 3.2 in [6], there is a $C > 0$ such that

$$\begin{aligned}|(1-|z|^2)^{n/p} f(z) - (1-|w|^2)^{n/p} f(w)| &\leqslant C\|f\|_{H^\infty_{(1-|z|^2)^{n/p}}}\varrho(z,w) \\ &\leqslant C\|f\|_{H^p}\varrho(z,w)\end{aligned}$$

for each $f \in H^\infty_{(1-|z|^2)^{n/p}}$ and $z, w \in B_N$. This completes the proof of this Lemma. □

Remark 1. From the Remark 3.3 of [6], for any $z, w \in rB_N = \{z \in B_N \colon |z| < r < 1\}$,

$$|(1-|z|^2)^{n/p} f(z) - (1-|w|^2)^{n/p} f(w)| \leqslant C\|f_r\|_{H^\infty_{(1-|z|^2)^{n/p}}}\varrho(z,w)$$

for any $f \in H^\infty_{(1-|z|^2)^{n/p}}$, where $\|f_r\|_{H^\infty_{(1-|z|^2)^{n/p}}} = \sup\limits_{z \in rB_N} (1-|z|^2)^{n/p}|f(z)|$. Thus by the above arguments and Lemma 2, one has

$$|(1-|z|^2)^{n/p} f(z) - (1-|w|^2)^{n/p} f(w)| \leqslant C\|f_r\|_{H^\infty_{(1-|z|^2)^{n/p}}}\varrho(z,w) \tag{1}$$

for any $f \in H^p$.

The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 of [5].

Lemma 4. *Suppose that $1 \leqslant p < \infty$. Let $u, v \in H(B_N)$ and $\varphi, \psi \in S(B_N)$, then the operator $W_{\varphi,u} - W_{\psi,v} \colon H^p \to H^\infty_\varphi$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in H^p with $f_n \to 0$, $n \to \infty$ uniformly on compact subsets of B_N, then $\|(W_{\varphi,u} - W_{\psi,v})f_n\|_{H^\infty_\varphi} \to 0$, $n \to \infty$.*

3. The boundedness of $W_{\varphi,u} - W_{\psi,v}$

In this section we will characterize the boundedness of $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$. For that purpose, we consider the following three conditions:

$$\sup_{z\in B_N} \frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) < \infty; \tag{2}$$

$$\sup_{z\in B_N} \frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) < \infty; \tag{3}$$

$$\sup_{z\in B_N} \Bigl|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}\Bigr| < \infty. \tag{4}$$

Theorem 1. *Suppose that $1 \leqslant p < \infty$. Let $u, v \in H(B_N)$ and $\varphi, \psi \in S(B_N)$, then the following statements are equivalent.*

(i) $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$ *is bounded.*
(ii) *The conditions* (2) *and* (4) *hold.*
(iii) *The conditions* (3) *and* (4) *hold.*

Proof. First, we prove the implication (i) $\Rightarrow$ (ii). Assume that $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$ is bounded. Fix $w \in B_N$, we consider the function f_w defined by

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^\alpha}{(1-\langle z,\varphi(w)\rangle)^{n/p+\alpha}} \cdot \frac{\langle \varphi_{\psi(w)}(z), \varphi_{\psi(w)}(\varphi(w))\rangle}{|\varphi_{\psi(w)}(\varphi(w))|} \tag{5}$$

for $z \in B_N$ and $\alpha > 0$.

Then using Theorem 1.12 in [26], one gets $f_w \in H^p(B_N)$. In fact

$$\begin{aligned} \|f_w\|_{H^p} = {} & \sup_{0<r<1} \biggl(\int_{\partial B_N} \frac{(1-|\varphi(w)|^2)^{p\alpha}}{|1-\langle r\zeta,\varphi(w)\rangle|^{n+p\alpha}} \\ & \times \Bigl|\frac{\langle \varphi_{\psi(w)}(r\zeta), \varphi_{\psi(w)}(\varphi(w))\rangle}{\varphi_{\psi(w)}(\varphi(w))}\Bigr|^p \,\mathrm{d}\sigma(\zeta)\biggr)^{1/p} \\ \leqslant {} & \sup_{0<r<1} \biggl(\int_{\partial B_N} \frac{(1-|\varphi(w)|^2)^{p\alpha}}{|1-\langle r\zeta,\varphi(w)\rangle|^{n+p\alpha}} \,\mathrm{d}\sigma(\zeta)\biggr)^{1/p} \\ \leqslant {} & \sup_{0<r<1} \frac{(1-|\varphi(w)|^2)^\alpha}{(1-r|\varphi(w)|^2)^\alpha} \leqslant 1. \end{aligned} \tag{6}$$

Thus $f_w \in H^p$ and $\|f_w\|_{H^p} \leqslant 1$. Note that

$$f_w(\varphi(w)) = \frac{\varrho(\varphi(w),\psi(w))}{(1-|\varphi(w)|^2)^{n/p}}, \quad f_w(\psi(w)) = 0. \tag{7}$$

By the boundeness of $W_{\varphi,u} - W_{\psi,v}\colon\ H^p \to H^\infty_\varphi$ and using (7), it then follows that

$$\begin{aligned}(8)\qquad \infty > \|(W_{\varphi,u} - W_{\psi,v})f_w\|_{H^\infty_\varphi} &= \sup_{z\in B_N} \varphi(z)|f_w(\varphi(z))u(z) - f_w(\psi(z))v(z)| \\ &\geqslant \varphi(w)|f_w(\varphi(w))u(w) - f_w(\psi(w))v(w)| = \frac{\varphi(w)|u(w)|}{(1-|\varphi(w)|^2)^{n/p}}\varrho(\varphi(w),\psi(w))\end{aligned}$$

for any $w \in B_N$. Since $w \in B_N$ is an arbitrary element, then from (8) we can obtain (2).

Next we prove (4). For given $w \in B_N$, we consider the function

$$(9)\qquad g_w(z) = \frac{(1-|\psi(w)|^2)^\alpha}{(1-\langle z,\psi(w)\rangle)^{n/p+\alpha}}$$

where $\alpha > 0$. By a similar argument as for (6), one can obtain that $g_w \in H^p(B_N)$ with $\|g_w\|_{H^p} \leqslant 1$. Note that

$$(10)\qquad g_w(\psi(w)) = \frac{1}{(1-|\psi(w)|^2)^{n/p}},$$

and by the boundeness of $W_{\varphi,u} - W_{\psi,v} : H^p \to H^\infty_\varphi$, one sees that

$$\begin{aligned}(11)\qquad \infty > \|(W_{\varphi,u} - W_{\psi,v})g_w\|_{H^\infty_\varphi} &\geqslant \varphi(w)|g_w(\varphi(w))u(w) - g_w(\psi(w))v(w)| \\ &= |I(w) + J(w)|,\end{aligned}$$

where

$$\begin{aligned}I(w) &= (1-|\psi(w)|^2)^{n/p} g_w(\psi(w))\Big[\frac{\varphi(w)u(w)}{(1-|\varphi(w)|^2)^{n/p}} - \frac{\varphi(w)v(w)}{(1-|\psi(w)|^2)^{n/p}}\Big] \\ &= \frac{\varphi(w)u(w)}{(1-|\varphi(w)|^2)^{n/p}} - \frac{\varphi(w)v(w)}{(1-|\psi(w)|^2)^{n/p}}, \\ J(w) &= \frac{\varphi(w)u(w)}{(1-|\varphi(w)|^2)^{n/p}}[(1-|\varphi(w)|^2)^{n/p} g_w(\varphi(w)) - (1-|\psi(w)|^2)^{n/p} g_w(\psi(w))].\end{aligned}$$

By (2) and Lemma 3, we conclude that

$$\begin{aligned}|J(w)| &\leqslant C\frac{\varphi(w)|u(w)|\varrho(\varphi(w),\psi(w))}{(1-|\varphi(w)|^2)^{n/p}}\|g_w\|_{H^p} \\ &\leqslant C\frac{\varphi(w)|u(w)|}{(1-|\varphi(w)|^2)^{n/p}}\varrho(\varphi(w),\psi(w)) < \infty\end{aligned}$$

for all $w \in B_N$, which by (11), shows that $|I(w)| < \infty$ for all $w \in B_N$. Thus (4) holds.

(ii) $\Rightarrow$ (iii). Assume that (2) and (4) hold. We need only to show that (3) holds. Note that the pseudohyperbolic metric ϱ is less than 1. Then we have that

$$\begin{aligned}(12)\qquad \frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) &\leqslant \frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) \\ &+ \Big|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}}-\frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}\Big|\varrho(\varphi(z),\psi(z)).\end{aligned}$$

From which, using (2) and (4), the desired condition (3) follows.

(iii) $\Rightarrow$ (i). Assume that (3) and (4) hold. By Lemma 3 and Lemma 2, for any $f\in H^p$, we have

$$\begin{aligned}\varphi(z)|(W_{\varphi,u}-W_{\psi,v})f(z)| &= \varphi(z)|f(\varphi(z))u(z)-f(\psi(z))v(z)| \\ &= \Big|(1-|\varphi(z)|^2)^{n/p}f(\varphi(z))\Big[\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}}-\frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}\Big] \\ &\quad+\frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}[(1-|\varphi(z)|^2)^{n/p}f(\varphi(z))-(1-|\psi(z)|^2)^{n/p}f(\psi(z))]\Big| \\ &\leqslant \|f\|_{H^p}\Big|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}}-\frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}\Big| \\ &\quad+C\|f\|_{H^p}\varrho(\varphi(z),\psi(z))\frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}} \\ &\leqslant C\|f\|_{H^p}.\end{aligned}$$

From which it follows that $W_{\varphi,u}-W_{\psi,v}\colon H^p\to H^\infty_\varphi$ is bounded. The whole proof is completed. $\square$

The following corollary follows easily from the simple case $v=0$ of Theorem 1.

Corollary 1. *Suppose that* $1\leqslant p<\infty$. *Let* $\varphi\in S(B_N)$ *and* $u\in H(B_N)$, *then* $W_{\varphi,u}\colon H^p\to H^\infty_\varphi$ *is bounded if and only if*

$$(13)\qquad \sup_{z\in B_N}\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}<\infty.$$

Corollary 2. *Suppose that* $1\leqslant p<\infty$. *Let* $\varphi,\psi\in S(B_N)$ *and* $u\in H(B_N)$, *then* $uC_\varphi-uC_\psi\colon H^p\to H^\infty_\varphi$ *is bounded if and only if the following two conditions hold:*

$$(14)\qquad \sup_{z\in B_N}\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z))<\infty$$

and

$$\sup_{z\in B_N} \frac{\varphi(z)|u(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) < \infty. \tag{15}$$

P r o o f. The "if" direction is accomplished by letting $v = u$ in Theorem 1.

For the converse direction, by Theorem 1, it suffices to show that if the conditions (14) and (15) hold, then

$$\Big|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)u(z)}{(1-|\psi(z)|^2)^{n/p}}\Big| < \infty \tag{16}$$

for all $z \in B_N$.

To prove our claim, we divide it into two cases as follows:

The first case, when $\varrho(\varphi(z),\psi(z)) > 1/2$: by (14) and (15) we obtain that

$$\begin{aligned}
&\Big|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)u(z)}{(1-|\psi(z)|^2)^{n/p}}\Big| \\
&\leqslant 2\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) + 2\frac{\varphi(z)|u(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) < \infty.
\end{aligned} \tag{17}$$

The second case, when $\varrho(\varphi(z),\psi(z)) \leqslant 1/2$: by Lemma 1 and (14) we have

$$\begin{aligned}
&\Big|\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)u(z)}{(1-|\psi(z)|^2)^{n/p}}\Big| \\
&\quad = \frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\Big|1 - \Big(\frac{1-|\varphi(z)|^2}{1-|\psi(z)|^2}\Big)^{n/p}\Big| \\
&\quad \leqslant \frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\Big|1 - \Big(\frac{1+\varrho(\varphi(z),\psi(z))}{1-\varrho(\varphi(z),\psi(z))}\Big)^{n/p}\Big| \\
&\quad \leqslant C\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) < \infty.
\end{aligned} \tag{18}$$

Combing (17) with (18), and using Theorem 1, we obtain the boundedness of $uC_\varphi - uC_\psi\colon H^p \to H^\infty_\varphi$. The proof of the Corollary is complete. $\square$

4. The compactness of $W_{\varphi,u} - W_{\psi,v}$

In this section, we turn our attention to the question of compact difference. Here we consider the following conditions:

$$\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) \to 0 \quad \text{as } |\varphi(z)| \to 1; \tag{19}$$

$$\frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) \to 0 \quad \text{as } |\psi(z)| \to 1; \tag{20}$$

$$\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}} \to 0 \quad \text{as } |\varphi(z)| \to 1, |\psi(z)| \to 1. \tag{21}$$

Theorem 2. *Suppose that $1 \leqslant p < \infty$. Let $u, v \in H(B_N)$ and $\varphi, \psi \in S(B_N)$, then $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$ is compact if and only if $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$ is bounded and the conditions* (19)–(21) *hold.*

Proof. First we suppose that $W_{\varphi,u} - W_{\psi,v}\colon H^p \to H^\infty_\varphi$ is bounded and the conditions (19)–(21) hold. Then the conditions (2)–(4) hold by Theorem 1.

From (19)–(21), it follows that for any $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\frac{\varphi(z)|u(z)|}{(1-|\varphi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) \leqslant \varepsilon \quad \text{for } |\varphi(z)| > r, \tag{22}$$

$$\frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}}\varrho(\varphi(z),\psi(z)) \leqslant \varepsilon \quad \text{for } |\psi(z)| > r, \tag{23}$$

$$\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}} \leqslant \varepsilon \quad \text{for } |\varphi(z)| > r, |\psi(z)| > r. \tag{24}$$

Now, let $\{f_n\}$ be a sequence in H^p such that $\|f_n\|_{H^p} \leqslant 1$ and $f_n \to 0$ uniformly on compact subsets of B_N. By using Lemma 4 we only need to show that $\|(W_{\varphi,u} - W_{\psi,v})f_n\|_{H^\infty_\varphi} \to 0$ as $n \to \infty$.

A direct calculation shows that

$$\begin{aligned} \varphi(z)|(W_{\varphi,u} - W_{\psi,v})f_n(z)| &= \varphi(z)|f_n(\varphi(z))u(z) - f_n(\psi(z))v(z)| \\ &= |I_n(z) + J_n(z)|, \end{aligned} \tag{25}$$

where

$$I_n(z) = (1-|\varphi(z)|^2)^{n/p} f_n(\varphi(z))\Big[\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}\Big]$$

and

$$J_n(z) = \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}}[(1-|\varphi(z)|^2)^{n/p} f_n(\varphi(z)) - (1-|\psi(z)|^2)^{n/p} f_n(\psi(z))].$$

We divide the argument into a few cases.

Case 1. $|\varphi(z)| \leqslant r$ and $|\psi(z)| \leqslant r$.

By the assumption, note that $\{f_n\}$ converges to zero uniformly on $E = \{w\colon\ |w| \leqslant r\}$ as $n \to \infty$, and using (4), it is easy to check that $I_n(z) \to 0$, $n \to \infty$ uniformly for all z with $|\varphi(z)| \leqslant r$.

On the other hand, it follows from (1) that

$$\begin{aligned}&|(1-|\varphi(z)|^2)^{n/p} f_n(\varphi(z)) - (1-|\psi(z)|^2)^{n/p} f_n(\psi(z))|\\ &\quad\leqslant C\varrho(\varphi(z),\psi(z)) \sup_{|\varphi(z)|\leqslant r} (1-|\varphi(z)|^2)^{n/p} |f_n(\varphi(z))|.\end{aligned}$$

From which, together with the condition (3) and the fact that $f_n \to 0$, uniformly on $E = \{w\colon\ |w| \leqslant r\}$ as $n \to \infty$, we have

$$|J_n(z)| \leqslant C \frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}} \varrho(\varphi(z),\psi(z)) \sup_{|w|\leqslant r} (1-|w|^2)^{n/p} |f_n(w)| \leqslant C\varepsilon.$$

Case 2. $|\psi(z)| > r$ and $|\varphi(z)| \leqslant r$.

As in the proof of Case 1, $I_n(z) \to 0$ uniformly as $n \to \infty$. On the other hand, using Lemma 3 and (23) we obtain

$$|J_n(z)| \leqslant C\|f_n\|_{H^p} \frac{\varphi(z)|v(z)|}{(1-|\psi(z)|^2)^{n/p}} \varrho(\varphi(z),\psi(z)) \leqslant C\varepsilon.$$

Case 3. $|\psi(z)| > r$ and $|\varphi(z)| > r$.

By (24) we obtain that

$$|I_n(z)| \leqslant \|f_n\|_{H^p} \left| \frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}} \right| \leqslant \varepsilon$$

for n sufficiently large. Meanwhile, $J_n(z) \to 0$ uniformly as $n \to \infty$ uniformly by the same proof as in Case 2.

Case 4. $|\psi(z)| \leqslant r$ and $|\varphi(z)| > r$. We rewrite

$$\varphi(z)|f_n(\varphi(z))u(z) - f_n(\psi(z))v(z)| = |P_n(z) + Q_n(z)|,$$

where

$$P_n(z) = (1-|\psi(z)|^2)^{n/p} f_n(\psi(z)) \Big[\frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} - \frac{\varphi(z)v(z)}{(1-|\psi(z)|^2)^{n/p}} \Big],$$

and

$$Q_n(z) = \frac{\varphi(z)u(z)}{(1-|\varphi(z)|^2)^{n/p}} [(1-|\varphi(z)|^2)^{n/p} f_n(\varphi(z)) - (1-|\psi(z)|^2)^{n/p} f_n(\psi(z))].$$

The desired result follows by an argument analogous to that in the proof of Case 2. Thus, together with the above cases, we conclude that

$$\|(W_{\varphi,u}-W_{\psi,v})f_n\|_{H^\infty_\varphi} = \sup_{z\in B_N}\varphi(z)|f_n(\varphi(z))u(z)-f_n(\psi(z))v(z)| \leqslant C\varepsilon \tag{26}$$

for sufficiently large n. Employing Lemma 4 we obtain the compactness of $W_{\varphi,u}-W_{\psi,v}\colon H^p\to H^\infty_\varphi$.

For the converse direction, we suppose that $W_{\varphi,u}-W_{\psi,v}\colon H^p\to H^\infty_\varphi$ is compact. From that we can easily obtain the boundedness of $W_{\varphi,u}-W_{\psi,v}\colon H^p\to H^\infty_\varphi$. Next we only need to show that (19)–(21) hold. Let $\{z_n\}$ be a sequence of points in B_N such that $|\varphi(z_n)|\to 1$ as $n\to\infty$. Define the functions

$$f_n(z)=\frac{(1-|\varphi(z_n)|^2)^\alpha}{(1-\langle z,\varphi(z_n)\rangle)^{n/p+\alpha}}\cdot\frac{\langle\varphi_{\psi(z_n)}(z),\varphi_{\psi(z_n)}(\varphi(z_n))\rangle}{|\varphi_{\psi(z_n)}(\varphi(z_n))|} \tag{27}$$

where $\alpha>0$. Clearly, f_n converges to 0 uniformly on compact subsets of B_N as $n\to\infty$ and $f_n\in H^p$ with $\|f_n\|_{H^p}\leqslant 1$ for all n. Moreover,

$$f_n(\varphi(z_n))=\frac{\varrho(\varphi(z_n),\psi(z_n))}{(1-|\varphi(z_n)|^2)^{n/p}},\quad f_n(\psi(z_n))=0. \tag{28}$$

By the compactness of $W_{\varphi,u}-W_{\psi,v}\colon H^p\to H^\infty_\varphi$ and Lemma 4, it follows that $\|(W_{\varphi,u}-W_{\psi,v})f_n\|_{H^\infty_\varphi}\to 0$, $n\to\infty$. On the other hand, using (28) we have

$$\begin{aligned}\|(W_{\varphi,u}-W_{\psi,v})f_n\|_{H^\infty_\varphi} &= \sup_{z\in B_N}\varphi(z)|f_n(\varphi(z))u(z)-f_n(\psi(z))v(z)|\\ &\geqslant \varphi(z_n)|f_n(\varphi(z_n))u(z_n)-f_n(\psi(z_n))v(z_n)|\\ &=\frac{\varphi(z_n)|u(z_n)|}{(1-|\varphi(z_n)|^2)^{n/p}}\varrho(\varphi(z_n),\psi(z_n)).\end{aligned} \tag{29}$$

Letting $n\to\infty$ in (29), it follows that (19) holds. The condition (20) holds by similar arguments.

Now we need only to show that the condition (21) holds. Assume that $\{z_n\}$ is a sequence of points in B_N such that $|\varphi(z_n)|\to 1$ and $|\psi(z_n)|\to 1$ as $n\to\infty$. Define the function

$$g_n(z)=\frac{(1-|\psi(z_n)|^2)^\alpha}{(1-\langle z,\psi(z_n)\rangle)^{n/p+\alpha}}$$

where $\alpha>0$. It is easy to check that g_n converges to 0 uniformly on compact subsets of B_N as $n\to\infty$ and $g_n\in H^p$ with $\|g_n\|_{H^p}\leqslant 1$ for all $n\in\mathbb{N}$. Note that

$$g_n(\psi(z_n))=\frac{1}{(1-|\psi(z_n)|^2)^{n/p}}. \tag{30}$$

By Lemma 4 we obtain $\|(W_{\varphi,u}-W_{\psi,v})g_n\|_{H^\infty_\varphi} \to 0$, $n\to\infty$. On the other hand we obtain that

$$\begin{aligned}\tag{31} \|(W_{\varphi,u}-W_{\psi,v})g_n\|_{H^\infty_\varphi} &\geqslant \varphi(z_n)|g_n(\varphi(z_n))u(z_n)-g_n(\psi(z_n))v(z_n)| \\ &= |I(z_n)+J(z_n)|\end{aligned}$$

where

$$\begin{aligned} I(z_n) &= (1-|\psi(z_n)|^2)^{n/p} g_n(\psi(z_n))\Big[\frac{\varphi(z_n)u(z_n)}{(1-|\varphi(z_n)|^2)^{n/p}} - \frac{\varphi(z_n)v(z_n)}{(1-|\psi(z_n)|^2)^{n/p}}\Big] \\ &= \frac{\varphi(z_n)u(z_n)}{(1-|\varphi(z_n)|^2)^{n/p}} - \frac{\varphi(z_n)v(z_n)}{(1-|\psi(z_n)|^2)^{n/p}} \\ J(z_n) &= \frac{\varphi(z_n)u(z_n)}{(1-|\varphi(z_n)|^2)^{n/p}}[(1-|\varphi(z_n)|^2)^{n/p}g_n(\varphi(z_n)) - (1-|\psi(z_n)|^2)^{n/p}g_n(\psi(z_n))].\end{aligned}$$

By Lemma 3 and the condition (19) that has been proved, we get

$$\tag{32} |J(z_n)| \leqslant C\|g_n\|_{H^p}\frac{\varphi(z_n)|u(z_n)|}{(1-|\varphi(z_n)|^2)^{n/p}}\varrho(\varphi(z_n),\psi(z_n)) \to 0,\ |\varphi(z_n)|\to 1.$$

Combing (31) and (32), we obtain that $I(z_n)\to 0$ as $n\to\infty$. This shows that (21) is true. The whole proof is complete. □

References

[1] *R. F. Allen, F. Colonna*: Weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain. Integral Equations Oper. Theory *66* (2010), 21–40.

[2] *J. Bonet, P. Domański, M. Lindström*: Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions. Can. Math. Bull. *42* (1999), 139–148.

[3] *J. Bonet, P. Domański, M. Lindström, J. Taskinen*: Composition operators between weighted Banach spaces of analytic functions. J. Aust. Math. Soc., Ser. A *64* (1998), 101–118.

[4] *J. Bonet, M. Lindström, E. Wolf*: Differences of composition operators between weighted Banach spaces of holomorphic functions. J. Aust. Math. Soc. *84* (2008), 9–20.

[5] *C. C. Cowen, B. D. MacCluer*: Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics, Boca Raton, FL, CRC Press, 1995.

[6] *J. N. Dai, C. H. Ouyang*: Differences of weighted composition operators on $H^\infty_\alpha(B_N)$. J. Inequal. Appl. Article ID 127431 (2009), 19 pp.

[7] *Z. S. Fang, Z. H. Zhou*: Differences of composition operators on the space of bounded analytic functions in the polydisc. Abstr. Appl. Anal. Article ID 983132 (2008), 10 pp.

[8] *Z. S. Fang, Z. H. Zhou*: Differences of composition operators on the Bloch space in the polydisc. Bull. Aust. Math. Soc. *79* (2009), 465–471.

[9] *P. Gorkin, R. Mortini, D. Suarez*: Homotopic composition operators on $H^\infty(B^n)$. Jarosz, Krzysztof (ed.), Function spaces. Proceedings of the 4th conference, Edwardsville, IL, USA. 2002; Providence, RI: American Mathematical Society (AMS), Contemp. Math. vol. 328, 2003, pp. 177–188.

[10] *T. Hosokawa, K. Izuchi, S. Ohno*: Topological structure of the space of weighted composition operators on H^∞. Integral Equations Oper. Theory *53* (2005), 509–526.

[11] *T. Hosokawa, S. Ohno*: Topological structures of the sets of composition operators on the Bloch spaces,. J. Math. Anal. Appl. *314* (2006), 736–748.

[12] *T. Hosokawa, S. Ohno*: Differences of composition operators on the Bloch spaces. J. Oper. Theory *57* (2007), 229–242.

[13] *M. Lindström, E. Wolf*: Essential norm of the difference of weighted composition operators. Monatsh. Math. *153* (2008), 133–143.

[14] *B. D. MacCluer*: Compact composition operators on $H^p(B_N)$. Mich. Math. J. *32* (1985), 237–248.

[15] *B. D. MacCluer, S. Ohno, R. Zhao*: Topological structure of the space of composition operators on H^∞. Integral Equations Oper. Theory *40* (2001), 481–494.

[16] *J. Moorhouse*: Compact differences of composition operators. J. Funct. Anal. *219* (2005), 70–92.

[17] *S. Ohno, K. Stroethoff, R. Zhao*: Weighted composition operators between Bloch-type spaces. Rocky Mt. J. Math. *33* (2003), 191–215.

[18] *J. H. Shapiro*: Composition Operators and Classical Function Theory. Universitext: Tracts in Mathematics, New York, Springer, 1993.

[19] *S. Stević, E. Wolf*: Differences of composition operators between weighted-type spaces of holomorphic functions on the unit ball of C^n. Appl. Math. Comput. *215* (2009), 1752–1760.

[20] *C. Toews*: Topological components of the set of composition operators on $H^\infty(B_N)$. Integral Equations Oper. Theory *48* (2004), 265–280.

[21] *E. Wolf*: Differences of composition operators between weighted Banach spaces of holomorphic functions on the unit polydisk. Result. Math. *51* (2008), 361–372.

[22] *K. B. Yang, Z. H. Zhou*: Essential norm of the difference of composition operators on Bloch space. Czech. Math. J. *60* (2010), 1139–1152.

[23] *H. G. Zeng, Z. H. Zhou*: An estimate of the essential norm of a composition operator from $F(p,q,s)$ to $\mathcal{B}^\alpha$ in the unit ball. J. Inequal. Appl. Article ID 132970 (2010), 22 pp.

[24] *Z. H. Zhou, R. Y. Chen*: Weighted composition operators from $F(p,q,s)$ to Bloch type spaces. Int. J. Math. *19* (2008), 899–926.

[25] *Z. H. Zhou, J. H. Shi*: Compactness of composition operators on the Bloch space in classical bounded symmetric domains. Mich. Math. J. *50* (2002), 381–405.

[26] *K. H. Zhu*: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Mathematics 226, Springer, New York, 2005.

Authors' address: Ze-Hua Zhou (corresponding author), Yu-Xia Liang, Department of Mathematics, Tianjin University, Tianjin 300072, P. R. China, email: `zehuazhou2003@yahoo.com.cn`, `liangyx1986@126.com`.